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ANALYSIS OF LARGE DEFORMATION OF A NONPRISMATIC BEAM

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Abstract. This paper studies the mathematical model that describes the deformation of a nonprismatic beam by its own weight. The nonprismatic beam is considered to be with circular or rectangular cross-section. We fix the density and hold an angle α at one end but free at the other end. The shape of the beam depends on the angle α, the density and the length to that of flexural rigidity. We analyze the bifurcation phenomena for the vertical case, $\alpha = \pi$. Several numerical results are presented.

1. Introduction

The deformation of a cantilever by its own weight is of interest both practically due to its engineering significance and theoretically its nonlinearity. We assume that a tapered cantilever (nonprismatic beam) of circular or rectangular cross-section and density is held fixed with an angle α at one end and free at the other end. If the tapered cantilever is thin enough, then its deformed shape can be described by the elastic theory. For the prismatic beam (cross-section shape is fixed), Wang [7] studied the bifurcation phenomena numerically, and Hsu and Hwang [3] gave the complete global bifurcation results for the prismatic vertical cantilever mathematically. In this paper, we consider two types of nonprismatic beam with circular cross-section and rectangular cross-section. We first give the uniqueness for the solutions of the governing equation, and then give the complete bifurcation results for the vertical case, $\alpha = \pi$, in the spirit of [3], [5]. Finally, for $\alpha \neq \pi$, we give the global bifurcation phenomenon of numerical results.

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2. Formulation

We assume a tapered beam with length L, uniform density ρ and contracted circular cross-section, the circular cross-section area given by $[\delta(L - s')]^2\pi$, which is held fixed at an angle α at one end, say, the origin, and is free at the other end. Let us consider a small segment of the tapered beam. A moment balance gives (see Fig. 1)

$$m - g\rho \int_{s'}^{L} \pi \left[\delta(L-\xi)\right]^2 d\xi \, \sin\theta \, ds' = m + dm, \tag{2.1}$$

where $m = m(s')$ is the local moment, s' is the arc length from the origin, $\theta = \theta(s')$ is the local angle of inclination, and δ is a small constant such that the cantilever is thin enough so that its deformed shape can be described by the elastic theory. According to Euler, the local moment is proportional to the curvature $d\theta/ds'$, i.e.,

$$m = -EI\frac{d\theta}{ds'}, \tag{2.2}$$

where E is the modulus of elasticity, and for the circular cross-section I is the moment of inertia $\pi\delta^4(L - s')^4/4$. From (2.1), (2.2), we obtain

$$\frac{1}{3}g\rho\pi\delta^2(L-s')^3 \sin\theta = \frac{1}{4}\pi E\delta^4(L-s')^4 \frac{d^2\theta}{ds'^2} - \pi E\delta^4(L-s')^3\frac{d\theta}{ds'} \tag{2.3}$$

and the boundary condition is

$$\theta(0) = \alpha, \qquad \frac{d\theta}{ds'}(L) = 0. \tag{2.4}$$

Let $s = s'/L$. Then (2.3), (2.4) become

$$\begin{cases} (1-s)^4\dfrac{d^2\theta}{ds^2} = 4(1-s)^3\dfrac{d\theta}{ds} + K\,(1-s)^3\sin\theta, \quad K>0, \quad 0\le s\le 1; \\ \theta(0) = \alpha, \qquad \theta'(1) = 0, \qquad -\pi\le\alpha\le\pi. \end{cases} \tag{2.5}$$

The parameter $K = 4Lg\rho/3E\delta^2$ represents the relative importance of density and length to that of flexural rigidity. The main concern of this paper is to determine the multiplicities of solutions of (2.5) provided that $K > 0$, $-\pi \le \alpha \le \pi$ are given. The reformulated mathematical problem is as follows:

Let

$$\psi(s) = \theta(1-s), \qquad 0 \le s \le 1.$$

FIG. 1.

Then (2.5) becomes

$$(P_c)_\alpha) \qquad \begin{cases} (s^4\psi'(s))' = K\, s^3 \sin\psi(s), & K > 0, \quad 0 \le s \le 1; \\ \psi(1) = \alpha, \quad \psi'(0) = 0, & -\pi \le \alpha \le \pi. \end{cases}$$

Since $\psi(s), 0 \le s \le 1$, is a solution of $(P_c)_\alpha$ if and only if $-\psi(s)$ is a solution of $(P_c)_{-\alpha}$, we only consider the problem with $0 \le \alpha \le \pi$. We may also reduce the problem $(P_c)_\alpha, 0 \le \alpha \le \pi$, by the following rescaling:

$$\phi(s) = \psi(s/K).$$

Then $\phi(s)$ satisfies

$$\text{(2.6)} \qquad \begin{cases} (s^4\phi(s))' = s^3 \sin\phi(s), & 0 \le s \le K; \\ \phi'(0) = 0, \quad \phi(K) = \alpha, & 0 \le \alpha \le \pi. \end{cases}$$

For a tapered cantilever of the rectangular cross-section, the cross-section area is given by $\delta b(L - s')$, with width b and height $\delta(L - s')$, and the moment of inertia $I = b\delta^3(L - s')^3/12$, where the fixed b and δ are small constants such that the cantilever is thin enough so that its deformed shape can be described by the elastic theory. Then we have

$$(P_r)_\alpha) \qquad \begin{cases} (s^3\psi'(s))' = K\, s^2 \sin\psi(s), & K > 0, 0 \le s \le 1; \\ \psi(1) = \alpha, \quad \psi'(0) = 0, & -\pi \le \alpha \le \pi, \end{cases}$$

where $K = 6Lg\rho/E\delta^2$. Since $\psi(s),\ 0 \le s \le 1$, is a solution of $(P_r)_\alpha$ if and only if $-\psi(s)$ is a solution of $(P_r)_{-\alpha}$, we only consider the problem with $0 \le \alpha \le \pi$. We may also reduce the problem $(P_r)_\alpha,\ 0 \le \alpha \le \pi$, by the following scaling:

$$\phi(s) = \psi(s/K).$$

Then $\phi(s)$ satisfies

$$\text{(2.7)} \qquad \begin{cases} (s^3\phi(s))' = s^2 \sin\phi(s), & 0 \le s \le K; \\ \phi'(0) = 0, \quad \phi(K) = \alpha, & 0 \le \alpha \le \pi. \end{cases}$$

3. Uniqueness of Solutions of $(P_c)_\alpha$ and $(P_r)_\alpha$

In this section, we present some results concerning the uniqueness of solutions of the boundary value problems $(P_c)_\alpha$ and $(P_r)_\alpha$:

$$(P_c)_\alpha) \qquad \begin{cases} (s^4\psi'(s))' = K\, s^3 \sin\psi(s), & K > 0, \quad 0 \le s \le 1; \\ \psi(1) = \alpha, \quad \psi'(0) = 0, & -\pi \le \alpha \le \pi, \end{cases}$$

and

$$(P_r)_\alpha) \qquad \begin{cases} (s^3\psi'(s))' = K\, s^2 \sin\psi(s), & K > 0, \quad 0 \le s \le 1; \\ \psi(1) = \alpha, \quad \psi'(0) = 0, & -\pi \le \alpha \le \pi. \end{cases}$$

Theorem 3.1. *The solution of* $(P_c)_0$ *(resp.* $(P_r)_0$*) is unique, namely,*

$$\psi(s) = 0, \quad 0 \le s \le 1, \qquad \text{for any} \quad K > 0.$$

Proof. Obviously $\psi(s) = 0$ is a solution of $(P_c)_0$. Multiplying the equation in $(P_c)_0$ by ψ'/s^2 and integrating the resulting equation from 0 to 1, we obtain

$$\int_0^1 s^2\, \psi''(s)\psi'(s)ds + \int_0^1 4s(\psi'(s))^2 ds = \int_0^1 Ks(\sin\, \psi(s))\psi'(s)ds.$$

Then we obtain

$$\frac{1}{2}(\psi'(1))^2 + \int_0^1 3s(\psi'(s))^2 ds = K\left[\int_0^1 \cos\, \psi(s)ds - 1\right] \ge 0.$$

However,

$$\left[\int_0^1 cos\, \psi(s)ds - 1\right] \le 0.$$

Hence we have $\psi'(1) = 0$. Since $\psi(1) = 0$ and $\psi'(1) = 0$, the conclusion $\psi(s) = 0$ follows directly from the uniqueness of solutions of ordinary differential equations. The proof is similar for the $(P_r)_0$ case, if we multiply the equation in $(P_r)_0$ by ψ'/s with the same discussion. So we omit it. ∎

The existence of solutions of problems $(P_c)_\alpha$ and $(P_r)_\alpha$ follows directly from the results in [4] since the right-hand side of $(P_c)_\alpha$ and $(P_r)_\alpha$, $Ks^3 \sin\psi$ and $Ks^2 \sin\psi$, are bounded for $0 \le s \le 1$. Hence, one can present a uniqueness property of $(P_c)_\alpha$ and $(P_r)_\alpha$.

Theorem 3.2. *If* $K < \sqrt{35}$*, then* $(P_c)_\alpha$ *has a unique solution for every* $\alpha \in [0, \pi]$.

Proof. Let $\psi(s)$ be a solution of $(P_c)_\alpha$. Then

$$\psi(s) = \alpha - \int_0^1 K\xi^3 \sin\, \psi(\xi)G(s, \xi)\, d\xi,$$

where

$$G(s, \xi) = ((\max(s, \xi))^{-3} - 1)/3.$$

Let $\psi_1(s), \psi_2(s)$ be solutions of $(P_c)_\alpha$. Then

$$\begin{aligned}|\psi_1(s) - \psi_2(s)| &\le K \int_0^1 G(s, \xi)\, \xi^3 |\psi_1(\xi) - \psi_2(\xi)|\, d\xi \\ &\le K\left[\int_0^1 G^2(s, \xi)\, \xi^6\, d\xi\right]^{1/2} \|\psi_1 - \psi_2\|_2\end{aligned}$$

or

$$\begin{aligned}\|\psi_1-\psi_2\|_2^2 &= \int_0^1 |\psi_1(s)-\psi_2(s)|^2\,ds \\ &\le K^2\left(\int_0^1\int_0^1 G^2(s,\xi)\,\xi^6\,d\xi\,ds\right)\|\psi_1-\psi_2\|_2^2,\end{aligned}$$

since

$$\int_0^1\int_0^1 G^2(s,\xi)\,\xi^6\,d\xi\,ds = 1/35.$$

If $K<\sqrt{35}$, then we must have

$$\psi_1=\psi_2. \qquad \blacksquare$$

Theorem 3.3. *If $K<\sqrt{20}$, then $(P_r)_\alpha$ has a unique solution for every $\alpha\in[0,\pi]$.*

The proof of Theorem 3.3 is similar to that of Theorem 3.2 by considering the Green's function $G(s,\xi)=((\max(s,\xi))^{-2}-1)/2$.

4. The Multiplicities of the Solutions of $(P_c)_\alpha$ and $(P_r)_\alpha$ for $\alpha=\pi$

In this section, we shall present the analytic results for the vertical case, $\alpha=\pi$. The discussions for circular cross-section and rectangular cross-section are similar; we only consider the tapered cantilever of circular cross-section. The analytic results for this special case will help us to understand the bifurcation phenomena for the general problem $(P_c)_\alpha$, $0<\alpha<\pi$. In the remainder of this section, we shall restrict our attention to the vertical case $\alpha=\pi$. So by $(P_c)_\alpha$ we have

$$\frac{d}{ds}\left(s^4\frac{d\psi(s)}{ds}\right)=Ks^3\sin\psi(s),\quad \psi'(0)=0,\quad \psi(1)=\pi. \tag{4.1}$$

Let $s=x$ and $v(x)=\psi(x/K)-\pi$. Then (4.1) takes the form

$$(x^4v'(x))'+x^3\sin v(x)=0,\quad v'(0)=0,\quad v(K)=0,\quad {}'=d/dx. \tag{4.2}$$

We shall study the boundary value problem (4.2) by the shooting method and consider the following initial value problem

$$(x^4v'(x))'+x^3\,sin\,v(x)=0,\quad v'(0)=0,\quad v(0)=a,\quad a\in\mathbb{R}. \tag{4.3}$$

We denote the solution of (4.3) by $v(x, a)$. From the uniqueness of solutions of ordinary differential equations, it follows that

$$\begin{aligned} &v(x, 2\pi + a) = 2\pi + v(x, a), \\ &v(x, 2\pi - a) = 2\pi - v(x, a), \\ &v(x, a) = -v(x, -a), \\ &v(x, 0) = 0, \qquad v(x, \pi) = \pi. \end{aligned} \tag{4.4}$$

From (4.4), we shall consider $v(x, a)$ only for $0 < a < \pi$.

Lemma 4.1. *Let $0 < a < \pi$. Then*

(i) $-\pi/2 < v(x, a) < \pi/2$ *for* $0 < a < \pi/2, \quad 0 \le x$.

(ii) $-\pi < v(x, a) < \pi$ *for* $\pi/2 \le a < \pi, \quad 0 \le x$.

(iii) $v(x, a)$ *is oscillatory over* $[0, \infty)$ *for all* $0 < a < \pi$.

Proof. Multiply (4.3) by $v'(x)/x^2$ and integrate the resulting equation from 0 to x. We obtain

$$\int_0^x \xi^2 v''(\xi) v'(\xi) d\xi + \int_0^x 4\xi (v'(\xi))^2 d\xi = \int_0^x -\xi v'(\xi) \sin v(\xi)\, d\xi.$$

Then, we have

$$\frac{1}{2} x^2 (v'(x))^2 + \int_0^x 3\xi (v'(\xi))^2 d\xi = x \cos v(x) - \int_0^x \cos v(\xi)\, d\xi \ge 0. \tag{4.5}$$

If $0 < a < \pi/2$, then $\cos a = \cos v(0) > 0$. We claim that $\cos v(x) > 0$ for all $x \ge 0$. If not, then there exists $x_0 > 0$ such that $\cos v(x) > 0$ for all $0 \le x < x_0$ and $\cos v(x_0) = 0$. This contradicts (4.5) with $x = x_0$ and we complete the proof for (i).

If $\pi/2 \le a < \pi$, then $\cos a = \cos v(0) \in (-1, 0]$. We claim that $\cos v(x) \ne -1$ for all $x \ge 0$. If not, then there exists $x_0 > 0$ such that $\cos v(x_0) = -1$ and $\cos v(x) > -1$ for $0 \le x < x_0$. Again from (4.5), we obtain a contradiction. Hence $-\pi < v(x, a) < \pi$ for all $x \ge 0$ and we have established (ii).

We next show that $v(x, a)$ is oscillatory over $[0, \infty)$ for any $0 < a < \pi$. Let

$$V(x) = (1 - \cos v(x)) + \frac{x\,(v'(x))^2}{2}.$$

It is easy to verify that

$$V'(x) = -\frac{7}{2}\,(v'(x))^2 \le 0.$$

Then we have

$$1 - \cos v(x) \leq V(x) \leq V(0) = 1 - \cos a.$$

Since $-\pi < v(x) < \pi$, we then have $|v(x)| \leq a$ for all $x \geq 0$. We rewrite the equation in (4.3) as

$$(x^4 v'(x))' + x^3 \frac{\sin v(x)}{v(x)} v(x) = 0. \tag{4.6}$$

Let $0 < \delta < \min_{0 \leq v \leq a}(\sin v/v)$. Using Sturm's comparison theorem, we compare (4.6) with

$$(x^4 v'(x))' + \delta\, x^3\, v(x) = 0. \tag{4.7}$$

Since the Bessel function $x^{-3/2} J_3(2\sqrt{\delta x})$ is the solution of (4.7), which is oscillatory over $[0, \infty)$, $v(x)$ is oscillatory over $[0, \infty)$, too. Thus we complete the proof for (iii). ■

Next, let us define:

$$\Delta(x, a) = \frac{dv}{da}(x, a), \qquad \phi(x) = \Delta(x, 0).$$

Differentiating (4.3) with respect to a yields

$$(x^4 \Delta'(x))' + x^3\, \Delta(x)\, cos\, v(x, a) = 0, \quad \Delta(0) = 1, \quad \Delta'(0) = 0. \tag{4.8}$$

Setting $a = 0$ in (4.8) yields

$$(x^4 \phi'(x))' + x^3 \phi(x) = 0, \quad \phi(0) = 1, \quad \phi'(0) = 0. \tag{4.9}$$

The equation in (4.9) is oscillatory over $[0, \infty)$. Let λ_n and γ_n be the nth zero of $\phi(x)$ and $\phi'(x)$, respectively, for $n = 1, 2, \ldots$. We note that

$$\begin{array}{llll} \lambda_1 \approx 10.1765, & \lambda_2 \approx 23.8194, & \lambda_3 \approx 42.3488, \\ \lambda_4 \approx 65.8002, & \lambda_5 \approx 94.1813, & \quad \text{etc.} \end{array} \tag{4.10}$$

From Lemma 4.1 (iii), $v(x, a)$ is oscillatory over $[0, \infty)$ for any $0 < a < \pi$. Let $y_n(a)$ and $z_n(a)$ be the nth zero of $v(x, a)$ and $v'(x, a)$, respectively, for $n = 1, 2, \ldots, 0 < a < \pi$.

Lemma 4.2. (i) $\lim_{a \to 0^+} y_n(a) = \lambda_n, \lim_{a \to 0^+} z_n(a) = \gamma_n$ *for* $n = 1, 2, \ldots$. (ii) $\lim_{a \to \pi^-} y_n(a) = +\infty$ *for* $n = 1, 2, \ldots$.

The proof of Lemma 4.2 follows directly from [3].

In addition to the properties (i), (ii) in Lemma 4.2, we shall show that, for all $0 < a < \pi$, $y_n(a)$ satisfies

$$\frac{dy_n(a)}{da} > 0 \quad for\ all\ n = 1, 2, \ldots \quad \text{and} \quad 0 < a < \pi. \tag{4.11}$$

Assume that (4.11) holds, then we may plot the following graph for $y_n(a),\ n = 1, 2, \ldots$. (See Fig. 2.) Then we conclude from (4.2) and (4.4) the following:

If $0 < K < \lambda_1$, then (4.2) has the unique solution $v(x) = 0$.
If $\lambda_1 < K < \lambda_2$, then (4.2) has three distinct solutions.
If $\lambda_n < K < \lambda_{n+1}$, then (4.2) has $2n + 1$ distinct solutions.

Since

$$v(y_n(a), a) = 0, \qquad 0 < a < \pi, \tag{4.12}$$

FIG. 2.

differentiating (4.12) with respect to a yields

$$v'(y_n(a), a)\frac{dy_n(a)}{da} + \frac{dv}{da}(y_n(a), a) = 0$$

or

$$\frac{dy_n(a)}{da} = -\frac{\Delta(y_n(a), a)}{v'(y_n(a), a)}. \tag{4.13}$$

We now state our main result.

Theorem 4.1. *Let* $0 < a < \pi$.

(i) *The solution* $v(x, a)$ *of* (4.3) *has an infinite number of isolated zeros* $y_n(a)$, *with* $y_1 < y_2 < \cdots < y_n$ *and* $y_n \to \infty$ *as* $n \to \infty$; *likewise* $v'(x, a)$ *has an infinite number of isolated zeros,* $z_n(a)$, *with* $z_1 < z_2 < \cdots < z_n$, *interlacing the* y_n; *furthermore*

$$\lim_{a\to 0^+} y_n(a) = \lambda_n, \qquad \lim_{a\to 0^+} z_n(a) = \gamma_n,$$

and

$$\lim_{a\to \pi^-} y_n(a) = \infty \qquad for\ n = 1, 2, \ldots.$$

(ii) $y_n(a)$ *is a differentiable function of* a *and*

$$\frac{dy_n}{da} > 0 \qquad \text{for } n = 1, 2, \ldots.$$

We have shown part (i) in the above lemmas. The proof of (ii) follows directly from (4.13) and Lemma 4.3 below.

Lemma 4.3. *Let* $0 < a < \pi$. *Then* $\Delta(x, a)$ *has an infinite number of isolated zeros* α_n, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$. $\Delta'(x, a)$ *satisfies the following:*

(i) *If* $0 < a < \pi/2$, *then* $\Delta'(x, a)$ *has an infinite number of isolated zeros* $\beta_n(a), 0 = \beta_1 < \beta_2 < \cdots < \beta_n < \cdots$. *Furthermore* $\beta_1 = z_1 = 0 < y_1 < \alpha_1 < z_2 < \beta_2 < y_2 < \alpha_2 < \cdots < y_n < \alpha_n < z_{n+1} < \beta_{n+1} < y_{n+1} < \cdots$.

(ii) *If* $\pi/2 \le a < \pi$, *then* $\Delta'(x, a)$ *has an infinite number of isolated zeros* $\beta_n(a)$, $0 = \beta_0 < \beta_1 < \cdots < \beta_n < \cdots$. *Furthermore,* $\beta_0 = z_1 = 0 < \beta_1 < y_1 < \alpha_1 < z_2 < \beta_2 < y_2 < \cdots < y_n < \alpha_n < z_{n+1} < \beta_{n+1} < y_{n+1} < \cdots$.

Before we prove Lemma 4.3, we consider (4.3) and (4.8). Let

$$\text{(A):} \quad (x^4 v')' + x^3 \sin v = 0, \quad v(0) = a, \quad v'(0) = 1,$$

$$\text{(B):}\quad (x^4\Delta')' + x^3\,\Delta\,\cos v = 0,\quad \Delta(0) = 1,\quad \Delta'(0) = 0.$$

In addition to (A) and (B), we form the following equations satisfied by Δ' and $w = (x^2 - 3xy_n)v'$, respectively:

$$\text{(C):}\quad (x^4\Delta'')' + (4x^3\Delta')' + (x^3\,\Delta\,\cos v)' = 0,$$

$$\text{(D):}\quad (x^4 w')' + x^3\,w\,\cos v = -3x^3(x - y_n)\,\sin v - 2x^4 v'.$$

Multiplying (A) by Δ and multiplying (B) by v, subtracting the resulting equations from each other and integrating the final expression from α to β, we obtain

$$\text{(a):}\quad x^4(v'\Delta - v\Delta')|_\alpha^\beta = \int_\alpha^\beta x^3\Delta\,v(\cos v - v^{-1}\sin v)\,dx.$$

Multiplying (A) by Δ' and multiplying (C) by v, subtracting the resulting equations from each other and integrating the final expression from α to β, we obtain

$$\begin{aligned}\text{(b):}\quad x^4(v'\Delta' - v\Delta'')|_\alpha^\beta &= 4x^3\Delta' v|_\alpha^\beta - \int_\alpha^\beta 4x^3\Delta'\,v'\,dx \\ &\quad + x^3\Delta\,v\,\cos v|_\alpha^\beta - x^3\Delta\,\sin v|_\alpha^\beta \\ &\quad + \int_\alpha^\beta 3x^2\Delta\,\sin v\,dx.\end{aligned}$$

Multiplying (D) by Δ and multiplying (B) by w, subtracting the resulting equations from each other and integrating the final expression from α to β, we obtain

$$\text{(c):}\quad x^4(w'\Delta - w\Delta')|_\alpha^\beta = -\int_\alpha^\beta 3x^3(x - y_n)\,\Delta\,\sin v\,dx - \int_\alpha^\beta 2x^4 v'\,\Delta\,dx.$$

Finally, since $v'(0) = 0,\ v(0) = a,\ \Delta(0) = 1,\ \Delta'(0) = 0,\ 0 < a < \pi$, we have

$$\begin{aligned}&\operatorname{sg} v = (-1)^n \text{ for } y_n < x < y_{n+1},\\ &\operatorname{sg} v' = (-1)^n \text{ for } z_n < x < z_{n+1},\\ &\operatorname{sg} \Delta = (-1)^n \text{ for } \alpha_n < x < \alpha_{n+1},\\ &\operatorname{sg} \Delta' = (-1)^n \text{ for } \beta_n < x < \beta_{n+1},\end{aligned}$$

Proof of Lemma 4.3. We shall prove the lemma by induction on m.

If $\pi/2 \le a < \pi$, then we claim that $\beta_1 < y_1$. Otherwise if $\beta_1 \ge y_1$, then $\Delta'(x) > 0$ for all $0 \le x \le y_1$. We specialize (α,β) in (b) to $(0, y_1)$. Then we obtain

$$x^4(v'\Delta' - v\Delta'')|_0^{y_1} = 4x^3\Delta' v|_0^{y_1} - \int_0^{y_1} 4x^3\Delta'\, v'\, dx + x^3\Delta\, v\, \cos\, v|_0^{y_1} - x^3\Delta\, \sin\, v|_0^{y_1} + \int_0^{y_1} 3x^2\Delta\, \sin\, v\, dx. \tag{4.14}$$

Since $v(y_1) = 0$, (4.14) becomes

$$y_1^4 v'(y_1)\Delta'(y_1) = -\int_0^{y_1} 4x^3\Delta' v' dx + \int_0^{y_1} 3x^2\Delta \sin\, v\, dx.$$

It is easy to verify $y_1^4 v'(y_1)\Delta'(y_1) < 0$, $\int_0^{y_1} 4x^3\Delta' v' dx < 0$, and $\int_0^{y_1} 3x^2\Delta \sin\, v\, dx > 0$. Thus we obtain a contradiction.

We shall now show that $\alpha_1 > y_1$ for $0 < a < \pi$. If not, then there exists $\alpha^* \in (0, y_1)$ such that $\Delta(\alpha^*) = 0, \Delta'(\alpha^*) < 0$ and $\Delta(x) > 0$ for $0 \le x < \alpha^*$. We specialize (α,β) in (a) to $(0,\alpha^*)$. That is

$$x^4(v'\Delta - \Delta' v)|_0^{\alpha^*} = \int_0^{\alpha^*} x^3\Delta\, v(\cos\, v - v^{-1}\, \sin\, v)\, dx. \tag{4.15}$$

Since $\Delta(\alpha^*) = 0$, (4.15) becomes

$$-(\alpha^*)^4\, v(\alpha^*)\Delta'(\alpha^*) = \int_0^{\alpha^*} x^3\Delta\, v(\cos\, v - v^{-1}\, \sin\, v)\, dx. \tag{4.16}$$

Since $\cos\, v \le (\sin\, v)/v$ for $-\pi < v < \pi$ and $\Delta(x) > 0, v(x) > 0$ for $0 \le x < \alpha^*$, it follows that the right-hand side of (4.16) is negative. However, the left-hand side of (4.16) is positive. This leads to a contradiction.

Now, we want to complete the induction by assuming the truth of the statement us to m. For $0 < a < \pi$, we want to show the following:

(i) $y_m < \alpha_m < z_{m+1}$. By the induction hypothesis $y_m < \alpha_m$, we want to show that $\alpha_m < z_{m+1}$. For $a = 0$, it is obvious that $\alpha_m(0) = \lambda_m$. From Lemma 4.2, we have

$$\lim_{a\to 0^+} z_{m+1}(a) = \gamma_{m+1} > \lambda_m.$$

By continuous dependence on parameter a, we have that $\alpha_m(a) < z_{m+1}(a)$ for $a > 0$ sufficiently small. We claim that $\alpha_m(a) < z_{m+1}(a)$ for all $0 < a < \pi$. If not, then there exists $a^* \in (0,\pi)$ such that $\alpha_m(a^*) = z_{m+1}(a^*)$. We now specialize (α,β) in (c) to $(z_m(a^*), z_{m+1}(a^*))$ and $n = m$. Then we obtain

$$x^4(w'\Delta - w\Delta')|_{z_m}^{z_{m+1}} = -\int_{z_m}^{z_{m+1}} 3x^3(x-y_m)\Delta\, \sin\, v\, dx - \int_{z_m}^{z_{m+1}} 2x^4\Delta\, v'\, dx. \tag{4.17}$$

Since $w(z_{m+1}) = w(z_m) = 0, z_{m+1} = \alpha_m, w'(z_m) = (z_m^2 - 3z_m y_m)v''(z_m)$, (4.17) becomes

$$-z_m^5(z_m - 3y_m)v''(z_m)\Delta(z_m) = -\int_{z_m}^{z_{m+1}} 3x^3(x - y_m) \sin v \,\Delta\, dx - \int_{z_m}^{z_{m+1}} 2x^4 \Delta\, v'\, dx. \tag{4.18}$$

Since $(z_m - 3y_m)v''(z_m)\Delta(z_m) > 0$, $\int_{z_m}^{z_{m+1}} 3x^3(x-y_m)\Delta \sin v < 0$ and $\int_{z_m}^{z_{m+1}} 2x^4 \Delta\, v' dx < 0$, the left-hand side of (4.18) is negative while the right-hand side of (4.18) is positive. This is the desired contradiction. Hence, we have $\alpha_m(a) < z_{m+1}(a)$ for all $0 < a < \pi$.

(ii) $z_{m+1} < \beta_{m+1} < y_{m+1} < \alpha_{m+1}$. First we show that $z_{m+1} < \beta_{m+1}$. If not, then $\alpha_m < \beta_{m+1} \le z_{m+1}$. We specialize (α, β) in (a) to (α_m, β_{m+1}). Then we obtain

$$x^4(v'\Delta - v\Delta')\Big|_{\alpha_m}^{\beta_{m+1}} = \int_{\alpha_m}^{\beta_{m+1}} x^3 \Delta\, v(\cos v - v^{-1} \sin v)\, dx. \tag{4.19}$$

Since $\Delta(\alpha_m) = 0$, $\Delta'(\beta_{m+1}) = 0$, (4.19) becomes

$$\begin{aligned} &\beta_{m+1}^4 v'(\beta_{m+1})\Delta(\beta_{m+1}) + \alpha_m^4 v(\alpha_m)\Delta'(\alpha_m) \\ &= \int_{\alpha_m}^{\beta_{m+1}} x^3 \Delta\, v(\cos v - v^{-1} \sin v)\, dx. \end{aligned} \tag{4.20}$$

It is easy to verify that the left-hand side of (4.20) is positive while the right-hand side is negative. This is the desired contradiction.

Next we show that $\beta_{m+1} < y_{m+1}$. If not, then $\beta_{m+1} \ge y_{m+1}$. We specialize (α, β) in (b) to (z_{m+1}, y_{m+1}). Follow similar arguments in the case $\beta_1 < y_1$. Since $v(y_{m+1}) = 0$, $v'(z_{m+1}) = 0$, we deduce that

$$\begin{aligned} &y_{m+1}^4 v'(y_{m+1})\Delta'(y_{m+1}) + z_{m+1}^4 \Delta''(z_{m+1})v(z_{m+1}) \\ &= -4z_{m+1}^3 \Delta'(z_{m+1})v(z_{m+1}) - \int_{z_{m+1}}^{y_{m+1}} 4x^3 \Delta' v' dx \\ &\quad + \int_{z_{m+1}}^{y_{m+1}} 3x^2 \Delta \sin v\, dx - z_{m+1}^3 \cos v(z_{m+1})\Delta(z_{m+1})v(z_{m+1}) \\ &\quad + z_{m+1}^3 \Delta(z_{m+1}) \sin v(z_{m+1}). \end{aligned} \tag{4.21}$$

Because $z_{m+1}^4 \Delta''(z_{m+1}) + 4z_{m+1}^3 \Delta'(z_{m+1}) + z_{m+1}^3 \cos v(z_{m+1})\Delta(z_{m+1}) = 0$, (4.21) becomes

$$\begin{aligned} y_{m+1}^4 v'(y_{m+1})\Delta'(y_{m+1}) &= z_{m+1}^3 \Delta(z_{m+1}) \sin v(z_{m+1}) \\ &\quad + \int_{z_{m+1}}^{y_{m+1}} 3x^2 \Delta \sin v\, dx - \int_{z_{m+1}}^{y_{m+1}} 4x^3 \Delta' v' dx. \end{aligned} \tag{4.22}$$

It is easy to verify $y_{m+1}^4 v'(y_{m+1})\Delta'(y_{m+1}) \le 0$, $z_{m+1}^3\Delta(z_{m+1})\sin v(z_{m+1}) > 0$, $\int_{z_{m+1}}^{y_{m+1}} 4x^3\Delta' v' dx < 0$, and $\int_{z_{m+1}}^{y_{m+1}} 3x^2\Delta\, sin\, v\, dx > 0$. Thus we obtain a contradiction.

Finally, we want to show that $y_{m+1} < \alpha_{m+1}$. If not, then $y_{m+1} \ge \alpha_{m+1}$. We specialize (α, β) in (a) to $(\beta_{m+1}, \alpha_{m+1})$. Then we have

$$\begin{aligned} &-\alpha_{m+1}^4 v(\alpha_{m+1})\Delta'(\alpha_{m+1}) - \beta_{m+1}^4 v'(\beta_{m+1})\Delta(\beta_{m+1}) \\ &= \int_{\beta_{m+1}}^{\alpha_{m+1}} x^3\Delta\, v(\cos\, v - v^{-1}\,\sin\, v)\, dx. \end{aligned} \tag{4.23}$$

It is easy to verify that the left-hand side of (4.23) is positive while the right-hand side is negative. This is a contradiction. ∎

For the tapered cantilever of rectangular cross-section, in the vertical case, we study the following boundary value problem $(P_r)_\pi$:

$$\begin{cases} \dfrac{d}{ds}\left(s^3\dfrac{d\psi(s)}{ds}\right) = Ks^2\,\sin\,\psi(s), \\ \psi'(0) = 0, \qquad \psi(1) = \pi. \end{cases} \tag{4.24}$$

Let $x = s, v(x) = \psi(s/K) - \pi$. Then (4.24) takes the form

$$\begin{cases} (x^3v'(x))' + x^2\,\sin\, v(x) = 0, \qquad ' = d/dx, \\ v'(0) = 0,\ v(K) = 0. \end{cases} \tag{4.25}$$

If we take the same function $V(x) = (1 - \cos\, v(x)) + x(v'(x))^2/2$, the initial value problem

$$\begin{cases} (x^3v'(x))' + x^2\,\sin\, v(x) = 0, \qquad ' = d/dx, \\ v(0) = a,\ v'(0) = 0,\ a \in \mathbb{R}, \end{cases} \tag{4.26}$$

with the same results as Lemma 4.1 and Lemma 4.2, then Lemma 4.3 also holds if we introduce the following equations,

$$\begin{aligned} (A):&\quad (x^3v')' + x^2\,\sin\, v = 0,\quad v(0) = a,\quad v'(0) = 1, \\ (B):&\quad (x^3\Delta')' + x^2\,\Delta\,\cos\, v = 0,\quad \Delta(0) = 1,\quad \Delta'(0) = 0, \\ (C):&\quad (x^3\Delta'')' + (3x^2\Delta')' + (x^2\,\Delta\,\cos\, v)' = 0, \\ (D):&\quad (x^3w')' + x^2\, w\,\cos\, v = -3x^2(x - y_n)\,\sin\, v - x^3v', \end{aligned}$$

where $w = (x^2 - 3xy_n)v'$. Hence the bifurcation phenomena of (4.24) is the same as the bifurcation phenomena of (4.1).

5. Numerical Studies for $\alpha \neq \pi$ and Discussions

In this section, we present our numerical studies for the multiplicities of the solutions for $0 < \alpha < \pi$. $(P_c)_\alpha$ and $(P_r)_\alpha$ will have the same bifurcation phenomena, so we only consider the problem of $(P_c)_\alpha$. From (2.6), we consider the following bifurcation problem,

$$
(5.1) \qquad \begin{cases} (s^4\phi(s))' = s^3 \sin\phi(s), & 0 \le s \le K, \\ \phi'(0) = 0, \quad \phi(K) = \alpha, & \qquad 0 < \alpha < \pi. \end{cases}
$$

Let $\phi(s, a)$ be the solution of the following initial value problem:

$$
(5.2) \qquad (s^4\phi(s))' = s^3 \sin\phi(s), \quad \phi(0) = a, \quad \phi'(0) = 0.
$$

It is easy to verify the following relations:

$$
\begin{aligned}
\phi(s, a + 2\pi) &= \phi(s, a) + 2\pi, \\
\phi(s, a - 2\pi) &= \phi(s, a) - 2\pi,
\end{aligned}
$$

so we only consider $-\pi < a < \pi$. For any $0 < \alpha < \pi$, the maps

$$
\begin{array}{lll}
a & \mapsto & y_1^u(a;\alpha), \\
a & \mapsto & y_n^l(a;\alpha), \quad n = 2, 3, 4, \cdots, \\
a & \mapsto & y_n^u(a;\alpha), \quad n = 2, 3, 4, \cdots,
\end{array}
$$

satisfy $\phi(y_1^u(a;\alpha)) = \pi - \alpha = \phi(y_n^l(a;\alpha)) = \phi(y_n^u(a;\alpha))$. We have $y_1^u(a;\alpha) < y_3^l(a;\alpha) < y_3^u(a;\alpha) < y_5^l(a;\alpha) < y_5^u(a;\alpha) < \cdots < y_{2n+1}^l(a;\alpha) < y_{2n+1}^u(a;\alpha)$ and $y_2^l(a;\alpha) < y_2^u(a;\alpha) < y_4^l(a;\alpha) < y_4^u(a;\alpha) < \cdots < y_{2n}^l(a;\alpha) < y_{2n}^u(a;\alpha)$ for $0 < a < \pi$, and $-\pi < a < 0$, respectively. Moreover $y_1^u(a_1;\alpha) = y_1^u(\pi - \alpha;\alpha) = 0$, $y_n^u(a_n;\alpha) = y_n^l(a_n;\alpha)$ with $0 < a_1 = \pi - \alpha < a_3 < \cdots < a_{2n+1} < \cdots < \pi$, and $-\pi < \cdots < a_{2n} < \cdots < a_4 < a_2 < 0$. (See Fig. 3.)

From the numerical computation, we conjecture that the following hold:

$$
\begin{array}{ll}
\dfrac{d}{da} y_{2n+1}^u(a;\alpha) > 0 & for\ a \in (a_{2n+1}, \pi), \\[2ex]
\dfrac{d}{da} y_{2n}^u(a;\alpha) < 0 & for\ a \in (-\pi, a_{2n}), \\[2ex]
\dfrac{d^2}{da^2} y_{2n+1}^l(a;\alpha) > 0 & for\ a \in (a_{2n+1}, \pi), \\[2ex]
\dfrac{d^2}{da^2} y_{2n}^l(a;\alpha) > 0 & for\ a \in (-\pi, a_{2n});
\end{array}
$$

FIG. 3.

FIG. 4.

moreover, $y^l_{2n}(\eta_{2n};\alpha)$ and $y^l_{2n+1}(\eta_{2n+1};\alpha)$ take the global minimum value in the open interval $(-\pi, a_{2n})$ and (a_{2n+1}, π), $n = 1, 2, 3, \ldots$, respectively, where $0 < \eta_3 < \eta_5 < \cdots < \eta_{2n+1} < \cdots < \pi$ and $-\pi < \cdots < \eta_{2n} < \cdots < \eta_4 < \eta_2 < 0$. Let $\lambda_n = y^l_n(\eta_n;\alpha), n = 2, 3, 4, \ldots$. We have $0 < \lambda_2 < \lambda_3 < \lambda_4 < \cdots < \lambda_n < \cdots$.

Then we conclude from the above conjecture the following:

If $0 < K < \lambda_2$, then (5.1) has a unique solution.
If $K = \lambda_2$, then (5.1) has exactly two distinct solutions.
If $\lambda_2 < K < \lambda_3$, then (5.1) has three distinct solutions.
If $K{=}\lambda_n$, then (5.1) has exactly $2n{-}2$ distinct solutions, for $n{=}2, 3, \ldots$.
If $\lambda_n < K < \lambda_{n+1}$, then (5.1) has $2n - 1$ distinct solutions, for $n = 2, 3, \ldots$.

Here we used the ODE Solver DGEAR of the IMSL Library to compute the bifurcation phenomena and the values of $\lambda_n, n = 2, 3, \ldots$. For $\alpha = \pi - 0.01$, we have (see Fig.4)

$$\lambda_2 \approx 10.961,\ \lambda_3 \approx 26.857,\ \lambda_4 \approx 49.812,\ \lambda_5 \approx 80.311,\ \lambda_6 \approx 118.628,\ \text{etc.}$$

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