TAIWANESE JOURNAL OF MATHEMATICS
Vol. 3, No. 1, pp. 83-88, March 1999

WIDE DIAMETERS OF BUTTERFLY NETWORKS

Sheng-Chyang Liaw and Gerard J. Chang*

Abstract. Reliability and efficiency are important criteria in the design of interconnection networks. Recently, the w-wide diameter $d_w(G)$, the $(w-1)$-fault diameter $D_w(G)$, and the w-Rabin number $r_w(G)$ have been used to measure network reliability and efficiency. In this paper, we study wide diameters for an important class of parallel networks—butterfly networks. The main result of this paper is to determine their wide diameters.

1. Introduction

Reliability and efficiency are important criteria in the design of interconnection networks. Connectivity is widely used to measure network fault-tolerance capacity, while diameter determines routing efficiency along individual paths. In practice, we are interested in high-connectivity, small-diameter networks.

The *distance* $d_G(x, y)$ from a vertex x to another vertex y in a network G is the minimum number of edges of a path from x to y. The *diameter* $d(G)$ of a network G is the maximum distance from one vertex to another. The *connectivity* $k(G)$ of a network G is the minimum number of vertices whose removal results in a disconnected or trivial network. According to Menger's theorem, there are k (internally) vertex-disjoint paths from a vertex x to another vertex y in a network of connectivity k. Throughout this paper, "vertex-disjoint" always means "internally vertex-disjoint."

For a network G with connectivity $k(G)$ and $w \leq k(G)$, the three parameters $d_w(G)$, $D_w(G)$, and $r_w(G)$ (defined below) arise from the study of,

Received May 12, 1997; revised January 12, 1998.
Communicated by S.-Y. Shaw.
1991 *Mathematics Subject Classification*: 05C12, 05C40.
Key words and phrases: Diameter, connectivity, wide diameter, butterfly network, banyan network, level.
*Supported in part by the National Science Council under grant NSC86-2115-M009-002.

respectively, parallel routing, fault-tolerant systems, and randomized routing (see [3, 6, 9, 10]). Due to widespread use of (and demand for) reliable, efficient, and fault-tolerant networks, these three parameters have been the subjects of extensive study over the past decade (see [3]).

The *w-wide diameter* $d_w(G)$ of a network G is the minimum l such that for any two distinct vertices x and y there exist w vertex-disjoint paths of length at most l from x to y. The notion of w-wide diameter was introduced by Hsu [3] to unify the concepts of diameter and connectivity.

The $(w-1)$-*fault diameter* of G is $D_w(G) = \max\{d(G-S) : |S| \leq w-1\}$. This notion was defined by Hsu [3], and the special case in which $w = k(G)$ was first defined by Krishnamoorthy and Krishnamurthy [6] who studied the fault-tolerant properties of graphs and networks.

The *w-Rabin number* $r_w(G)$ of a network G is the minimum l such that for any $w+1$ distinct vertices $x, y_1, \cdots, y_w$ there exist w vertex-disjoint paths of length at most l from x to $y_1, y_2, \cdots, y_w$. This concept was first defined by Hsu [3], and the special case in which $w = k(G)$ was studied by Rabin [10] in conjunction with a randomized routing algorithm.

It is clear that when $w = 1$, $d_1(G) = D_1(G) = r_1(G) = d(G)$ for any network G. On the other hand, these parameters can be very large, as in the case in which $w = k(G)$. For example, Hsu and Luczak [4] showed that $d_k(G) = \frac{n}{2}$ for some regular graphs G having connectivity and degree k and n vertices. The following are basic properties and relationships among $d_w(G)$, $D_w(G)$, and $r_w(G)$.

Proposition 1. [8] *The following statements hold for any network G of connectivity k.*

(1) $D_1(G) \leq D_2(G) \leq \cdots \leq D_k(G)$.

(2) $d_1(G) \leq d_2(G) \leq \cdots \leq d_k(G)$.

(3) $r_1(G) \leq r_2(G) \leq \cdots \leq r_k(G)$.

(4) $D_w(G) \leq d_w(G)$ *and* $D_w(G) \leq r_w(G)$ *for* $1 \leq w \leq k$.

This paper examines the above parameters for butterfly networks, which are banyan networks in the literature. The *butterfly network* B_n is the graph whose vertices are $x = (x_0, x_1, \cdots, x_n)$ with $0 \leq x_0 \leq n$ and $x_i \in \{0, 1\}$ for $1 \leq i \leq n$, and two vertices x and y are adjacent if and only if $y_0 = x_0 + 1$ and $x_i = y_i$ for $1 \leq i \leq n$ with $i \neq y_0$. Note that B_1 is a 4-cycle. For a vertex $x = (x_0, x_1, \cdots, x_n)$ in B_n, we say that x is in *level* x_0 of B_n and call x_i the *ith coordinate* of x. FIG. 1 shows an example of B_3, in which the top line indicates the level numbers and the left column indicates the names $(x_1, x_2, \cdots, x_n)$.

FIG. 1. The butterfly network B_3.

Cao, Du, Hsu and Wan [1] gave the connectivity, the diameter, the fault diameter, and bounds of the wide diameter and the Rabin number of the butterfly network B_n as follows.

Theoremm 2. **[1]** *If $n \geq 2$, then $k(B_n) = 2$, $d(B_n) = 2n$, $D_2(B_n) = 2n + 2$, $2n + 2 \leq d_2(B_n) \leq 2n + 4$, and $2n + 2 \leq r_2(B_n) \leq 2n + 4$.*

In this paper, we prove that $d_2(B_n) = 2n + 2$ for $n \geq 2$.

2. The Wide Diameter $d_2(B_n)$

For any $a \in \{0, 1\}$, $\overline{a}$ is defined to be $1-a$. Suppose y and x are two vertices with $y_0 = i \leq j = x_0$ and $y_k = x_k$ for $k \in \{1, 2, \cdots, i\} \cup \{j+1, j+2, \cdots, n\}$. Denote as $P_{i,j}(y, x)$, or $P_{i,j}$ with y and x specified, the following path of length $j - i$ from y to x:

$$\begin{aligned}
& (i,\ y_1,\ \cdots,\ y_i,\ y_{i+1},\ y_{i+2},\ y_{i+3},\ \cdots,\ y_j,\ y_{j+1},\ \cdots,\ y_n) \\
\rightarrow\ & (i+1,\ y_1,\ \cdots,\ y_i,\ x_{i+1},\ y_{i+2},\ y_{i+3},\ \cdots,\ y_j,\ y_{j+1},\ \cdots,\ y_n) \\
\rightarrow\ & (i+2,\ y_1,\ \cdots,\ y_i,\ x_{i+1},\ x_{i+2},\ y_{i+3},\ \cdots,\ y_j,\ y_{j+1},\ \cdots,\ y_n) \\
\rightarrow\ & \cdots\cdots \\
\rightarrow\ & (j,\ y_1,\ \cdots,\ y_i,\ x_{i+1},\ x_{i+2},\ x_{i+3},\ \cdots,\ x_j,\ y_{j+1},\ \cdots,\ y_n).
\end{aligned}$$

Similarly, if y and x are two vertices with $y_0 = i \geq j = x_0$ and $y_k = x_k$ for $k \in \{1, 2, \cdots, j\} \cup \{i+1, i+2, \cdots, n\}$. Denote as $Q_{i,j}(y, x)$, or $Q_{i,j}$ with y and x specified, the following path of length $i - j$ from y to x:

$$\begin{array}{rl} & (i,\ y_1,\ \cdots,\ y_j,\ y_{j+1},\ \cdots,\ y_{i-2},\ y_{i-1},\ y_i,\ y_{i+1},\ \cdots,\ y_n) \\ \rightarrow & (i-1,\ y_1,\ \cdots,\ y_j,\ y_{j+1},\ \cdots,\ y_{i-2},\ y_{i-1},\ x_i,\ y_{i+1},\ \cdots,\ y_n) \\ \rightarrow & (i-2,\ y_1,\ \cdots,\ y_j,\ y_{j+1},\ \cdots,\ y_{i-2},\ x_{i-1},\ x_i,\ y_{i+1},\ \cdots,\ y_n) \\ \rightarrow & \cdots\cdots \\ \rightarrow & (j,\ y_1,\ \cdots,\ y_j,\ x_{j+1},\ \cdots,\ x_{i-2},\ x_{i-1},\ x_i,\ y_{i+1},\ \cdots,\ y_n). \end{array}$$

We are now ready to prove the main result.

Theorem 3. *If $n \geq 2$, then $d_2(B_n) = 2n + 2$.*

Proof. According to Proposition 1 (4) and the fact that $D_2(B_n) = 2n + 2$ (see [1]), it suffices to show that for any two vertices $y = (y_0, y_1, \cdots, y_n)$ and $x = (x_0, x_1, \cdots, x_n)$, there exist two vertex-disjoint y-x paths of lengths at most $2n + 2$. We, in fact, will construct two vertex-disjoint y-x walks based on the following three cases. Without loss of generality, we may assume that $y_0 \geq x_0$. Let $a = \lceil \frac{y_0 + x_0 - 2}{2} \rceil$.

Case 1. $y_0 \geq x_0 + 2$. In this case, we have $y_0 > a + 1 > x_0$. The first y-x walk is $W = P_{y_0,n}(y, u^1) Q_{n,a}(u^1, u^2) Q_{a,0}(u^2, u^3) P_{0,a+1}(u^3, u^4) Q_{a+1,x_0}(u^4, x)$, where

$$\begin{aligned} y &= (y_0, y_1, \cdots, y_{x_0}, y_{x_0+1}, y_{x_0+2}, \cdots, y_a, y_{a+1}, y_{a+2}, \cdots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \cdots, y_n), \\ u^1 &= (n, y_1, \cdots, y_{x_0}, y_{x_0+1}, y_{x_0+2}, \cdots, y_a, y_{a+1}, y_{a+2}, \cdots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \cdots, y_n), \\ u^2 &= (a, y_1, \cdots, y_{x_0}, y_{x_0+1}, y_{x_0+2}, \cdots, y_a, \overline{x_{a+1}}, x_{a+2}, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n), \\ u^3 &= (0, x_1, \cdots, x_{x_0}, x_{x_0+1}, x_{x_0+2}, \cdots, x_a, \overline{x_{a+1}}, x_{a+2}, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n), \\ u^4 &= (a+1, x_1, \cdots, x_{x_0}, \overline{y_{x_0+1}}, x_{x_0+2}, \cdots, x_a, \overline{x_{a+1}}, x_{a+2}, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n), \\ x &= (x_0, x_1, \cdots, x_{x_0}, x_{x_0+1}, x_{x_0+2}, \cdots, x_a, x_{a+1}, x_{a+2}, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n), \end{aligned}$$

and W has a length of $(n - y_0) + (n - a) + a + (a + 1) + (a + 1 - x_0) = 2n + 2 + 2a - y_0 - x_0 \leq 2n + 1$. The second y-x walk is $W' = Q_{y_0,a}(y, v^1)$ $P_{a,n}(v^1, v^2) Q_{n,0}(v^2, v^3)$ $P_{0,x_0}(v^3, x)$, where

$$\begin{aligned} y &= (y_0, y_1, \cdots, y_{x_0}, y_{x_0+1}, y_{x_0+2}, \cdots, y_a, y_{a+1}, y_{a+2}, \cdots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \cdots, y_n), \\ v^1 &= (a, y_1, \cdots, y_{x_0}, y_{x_0+1}, y_{x_0+2}, \cdots, y_a, \overline{y_{a+1}}, y_{a+2}, \cdots, y_{y_0-1}, \overline{x_{y_0}}, y_{y_0+1}, \cdots, y_n), \\ v^2 &= (n, y_1, \cdots, y_{x_0}, y_{x_0+1}, y_{x_0+2}, \cdots, y_a, \overline{y_{a+1}}, y_{a+2}, \cdots, y_{y_0-1}, \overline{x_{y_0}}, y_{y_0+1}, \cdots, y_n), \\ v^3 &= (0, x_1, \cdots, x_{x_0}, x_{x_0+1}, x_{x_0+2}, \cdots, x_a, x_{a+1}, x_{a+2}, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n), \\ x &= (x_0, x_1, \cdots, x_{x_0}, x_{x_0+1}, x_{x_0+2}, \cdots, x_a, x_{a+1}, x_{a+2}, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n), \end{aligned}$$

and W' has a length of $(y_0 - a) + (n - a) + n + x_0 = 2n - 2a + y_0 + x_0 \leq 2n + 2$. Moreover, between levels n and y_0, vertices in W and W' differ at $(a + 1)$th coordinate; between levels y_0 and 0, vertices in W and W' differ at y_0th, $(a+1)$th, or (x_0+1)th coordinate. So, W and W' are vertex-disjoint. From W and W' we can find two vertex-disjoint y-x paths as desired.

Case 2. $y_0 = x_0+1$ or $y_0 = x_0 \neq 0$. In this case, we have $y_0 = a+1 \geq x_0$. For $y_0 = x_0+1$, the first y-x walk is $W = Q_{y_0,0}(y, u^1)\ P_{0,n}(u^1, u^2)\ Q_{n,y_0}(u^2, u^3)\ Q_{y_0,x_0}(u^3, x)$, where

$$\begin{aligned} y &= (y_0, y_1, \cdots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \cdots, y_n), \\ u^1 &= (0, y_1, \cdots, y_{y_0-1}, \overline{x_{y_0}}, y_{y_0+1}, \cdots, y_n), \\ u^2 &= (n, x_1, \cdots, x_{y_0-1}, \overline{x_{y_0}}, x_{y_0+1}, \cdots, x_n), \\ u^3 &= (y_0, x_1, \cdots, x_{y_0-1}, \overline{x_{y_0}}, x_{y_0+1}, \cdots, x_n), \\ x &= (x_0, x_1, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n). \end{aligned}$$

Note that the length of W is $y_0+n+(n-y_0)+(y_0-x_0) = 2n+y_0-x_0 \leq 2n+1$. For $y_0 = x_0$, replace $Q_{y_0,x_0}(u^3, x)$ with $Q_{y_0,x_0-1}(u^3, u^4)P_{x_0-1,x_0}(u^4, x)$, where $u^4 = (x_0-1, x_1, \cdots, x_{y_0-1}, \overline{x_{y_0}}, x_{y_0+1}, \cdots, x_n)$, to obtain the first y-x walk W of length $2n+2$. The second y-x walk is $W' = Q_{y_0,0}(y, v^1)P_{0,n}(v^1, v^2)Q_{n,x_0}(v^2, x)$, where

$$\begin{aligned} y &= (y_0, y_1, \cdots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \cdots, y_n), \\ v^1 &= (0, y_1, \cdots, y_{y_0-1}, x_{y_0}, y_{y_0+1}, \cdots, y_n), \\ v^2 &= (n, x_1, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n), \\ x &= (x_0, x_1, \cdots, x_{y_0-1}, x_{y_0}, x_{y_0+1}, \cdots, x_n). \end{aligned}$$

Note that the length of W' is $y_0 + n + (n - x_0) = 2n + y_0 - x_0 \leq 2n + 1$. Moreover, vertices in W and W' differ at the y_0th coordinate and hence are disjoint.

Case 3. $y_0 = x_0 = 0$. Consider y-x walks $W^j = P^j_{0,n}Q^j_{n,0}$ for $j = 0$ or 1, where $P^j_{0,n}$ is from $y = (0, y_1, \cdots, y_n)$ to $(n, j, x_2, \cdots, x_n)$ and $Q^j_{n,0}$ is from $(n, j, x_2, \cdots, x_n)$ to $x = (0, x_1, \cdots, x_n)$. It is clear that vertices in W^0 and W^1 differ at the 1st coordinate and hence are disjoint. Moreover, the length of W^0 or W^1 is $2n$. ■

The referee provides the information that Chen and Li [2] extended the study of wide diameter to k-ary butterfly networks. In particular, they proved that the wide diameter of the k-ary butterfly network is bounded above by $D+4$ if $n \geq 4$, by $D+2$ if $n \geq 8$, and by $D+3$ if $4 \leq n \leq 7$, where n is the dimension and D the diameter of the network.

Determining the exact values of $r_2(B_n)$ remains open. Although there is no strong indication, after checking many special cases, we do believe the following conjecture should be true.

Conjecture. $r_2(B_n) = 2n + 2$.

Acknowledgments

We thank Dingzhu Du and Frank Hsu for bringing to our attention the paper [1] about the wide diameters of butterfly networks. We also thank a referee for many useful suggestions.

References

1. F. Cao, D. Z. Du, D. F. Hsu and P. Wan, Fault-tolerant routing in butterfly networks, Technical Report TR 95-073, Department of Computer Science, University of Minnesota (1995).
2. Y. Chen and Q. Li, The wide-diameter of the n-dimensional k-ary butterfly graph, preprint.
3. D. F. Hsu, On container width and length in graphs, groups, and networks, *IEICE Trans. Fund. Electron. Comm. Comput. Sci.* **E77-A** (1994), 668-680.
4. D. F. Hsu and T. Luczak, Note on the k-diameter of k-regular k-connected graphs, *Discrete Math.* **133** (1994), 291-296.
5. D. F. Hsu and Y. D. Lyuu, A graph theoretical study of transmission delay and fault tolerance, *Internat. J. Mini Microcomput.* **16** (1994), 35-42.
6. M. S. Krishnamoorthy and B. Krishnamurthy, Fault diameter of interconnection networks, *Comput. Math. Appl.* **13** (1987), 577-582.
7. Q. Li, D. Sotteau and J. Xu, 2-Diameter of de Bruijn graphs, *Networks* **28** (1996), 7-14.
8. S. C. Liaw, F. Cao, G. J. Chang and D. F. Hsu, Fault-tolerant routing in circulant networks and cycle prefix networks, submitted for publication.
9. S. C. Liaw and G. J. Chang, Generalized diameters and Rabin numbers of networks, submitted for publication.
10. M. O. Rabin, Efficient dispersal of information for security, load balancing, and fault tolerance, *J. Assoc. Comput. Mach.* **36** (1989), 335-348.

Department of Applied Mathematics
National Chiao Tung University
Hsinchu 30050, Taiwan
E-mail address: gjchang@math.nctu.edu.tw