TAIWANESE JOURNAL OF MATHEMATICS
Vol. 3, No. 1, pp. 53-70, March 1999

SUBMANIFOLDS OF CONSTANT SCALAR CURVATURE IN A HYPERBOLIC SPACE FORM

Zhong Hua Hou

Abstract. Let M^n be a closed submanifold immersed into a real hyperbolic space form $\mathbb{H}^{n+p}$ of constant curvature -1. Denote by R the normalized scalar curvature of M^n and by H the mean curvature of M^n. Suppose that R is constant and bigger than or equal to -1. We first extend Cheng-Yau's technique to higher codimensional cases. Then, for M^n with parallel normalized mean curvature vector field, we show that, if H satisfies a certain inequality, then M^n is totally umbilical or the equality part holds. We describe all M^n whose H satisfies this equality.

0. Introduction

Let M^n be an oriented, connected submanifold immersed into a space form $\overline{M}_c^{n+p}$ of constant curvature c. We say that M^n is closed if it is compact and without boundary. Denote by σ the second fundamental form and by ξ the mean curvature vector field of M^n. Denote by H the length of ξ which we call the mean curvature of M^n. We denote by $\mathbb{H}^{n+p}(c)$ the real hyperbolic space form of constant curvature c (< 0). We simply denote $\mathbb{H}^{n+p}(-1)$ by $\mathbb{H}^{n+p}$.

As far as we know, there is a lot of results obtained on the rigidity problem for minimal submanifolds and for submanifolds with parallel mean curvature vector field immersed into a sphere or a Euclidean space, but less of that were obtained for submanifolds immersed into a hyperbolic space form, even for hypersurfaces. Walter [11] gave a classification for non-negatively curved compact hypersurfaces in a space form under the assumption that the rth mean curvature is constant, where the rth mean curvature is defined to be the rth elementary symmetric function of the principal curvatures. Morvan–Wu [7],

Received February 10, 1997.
Communicated by Jyh-Yang Wu.
1991 *Mathematics Subject Classification*: Primary 53C42, Secondary 53A10.
Key words and phrases: Normalized mean curvature vector field, normalized scalar curvature.

Wu [12] also proved some rigidity theorems for a complete hypersurface M_0^n in a hyperbolic space form $\mathbb{H}^{n+1}(c)$ under the assumption that the Ricci curvature tensor is positive semi-definite and the mean curvature is constant. Moreover, they proved that M_0^n is a geodesic distance sphere in $\mathbb{H}^{n+p}(c)$ provided that it is compact.

Cheng-Yau [2] constructed a second order differential operator and used it to classify compact hypersurfaces with non-negative sectional curvature and constant scalar curvature in a space form. They also classified complete non-compact convex hypersurfaces with constant scalar curvature in a Euclidean space.

In this paper, we first extend Cheng-Yau's technique to higher codimensional cases and then study the rigidity problem for closed submanifolds with constant scalar curvature in $\mathbb{H}^{n+p}$.

In Section 3, we will prove the following:

Proposition 3.1. *Let M^n be a connected submanifold immersed into $\overline{M}_c^{n+p}$. Suppose that the normalized scalar curvature R of M^n is constant and greater than or equal to c. Then*

$$|\nabla\sigma|^2 \geq n^2|\nabla H|^2 \tag{3.1}$$

and the symmetric tensor T defined by (1.14) *is negative semi-definite. Moreover, suppose that the equality in* (3.1) *holds everywhere on M^n. Then*

(i) *if $R - c > 0$, then H is constant and T is negative definite;*

(ii) *if $R - c = 0$, then either H is constant or M^n lies in a totally geodesic subspace $\overline{M}_c^{n+1}$ of $\overline{M}_c^{n+p}$. In the latter case, if H is not constant on M^n, then $r(L_{n+1}) \leq 1$ on M^n, where $r(L_{n+1})$ denotes the rank of L_{n+1}.*

Remark 0.1. It should be noted that, in Proposition 3.1, if we denote $\sigma_\xi = \langle\sigma, \xi\rangle$, then we have $T = \sigma_\xi - (nH)g$, where g denotes the induced metric of M^n. Moreover, L_{n+1} is nothing but the coefficient matrix of σ_ξ under an orthonormal frame field of $T(M^n)$.

In Section 4, we will prove the following:

Theorem 4.1. *Let M^n $(n \geq 3)$ be a closed submanifold in $\mathbb{H}^{n+p}$ with parallel normalized mean curvature vector field. Suppose that the normalized scalar curvature R is constant and greater than or equal to $-(n-2)/(n-1)$. Let h be defined by* (4.10)*. If the normal bundle of M^n is flat and $H \leq h$, then $R > 0$ and either*

(i) *$H = \sqrt{R+1}$ and M^n is a geodesic distance sphere $S^n(1/\sqrt{R})$ in $\mathbb{H}^{n+p}$; or*

(ii) *$H = h$ and one of the following cases occurs:*

(a) *$H = \sqrt{(n-1)R/(n-2)+1}$ and M^n is a Clifford hypersurface*

$$S^{n-1}(r_0\sqrt{(n-1)/n}) \times S^1(r_0\sqrt{1/n})$$

in a geodesic distance sphere $S^{n+1}(r_0)$ of $\mathbb{H}^{n+p}$, where $r_0 = \sqrt{(n-2)/(n-1)R}$;

(b) *$\sqrt{R+1+R/n(n-2)} < H < \sqrt{(n-1)R/(n-2)+1}$ and M^n is a pythagorean product of the form $S^{n-1}(r) \times S^1(\sqrt{r_1^2 - r^2})$ in a geodesic distance sphere $S^{n+1}(r_1)$ of $\mathbb{H}^{n+p}$, where $r_1^{-2} = nR/(n-2) - R^2/(n-2)^2(H^2 - R - 1)$ and $r = \sqrt{(n-2)/nR}$.*

Theorem 4.2. *Let M^n be a closed submanifold in $\mathbb{H}^{n+p}$ with parallel normalized mean curvature vector field. Suppose that the normalized scalar curvature R is constant and greater than or equal to $-(3n-5)/(3n-3)$. Let $\tilde{h}$ be defined by (4.18). If $H \le \tilde{h}$, then $R > 0$ and either*

(i) *$H = \sqrt{R+1}$ and M^n is a geodesic distance sphere $S^n(1/\sqrt{R})$ in $\mathbb{H}^{n+p}$; or*

(ii) *$H = \sqrt{3R+1}$ and M^2 is a Veronese surface in a totally geodesic sphere $S^4(1/\sqrt{3R})$ of a geodesic distance sphere $S^5(1/\sqrt{3R})$ in $\mathbb{H}^{2+p}$.*

Remark 0.2. It should be pointed out that the assumption that the normalized mean curvature vector field ξ/H of M^n is parallel is different from that the mean curvature vector field ξ of M^n is parallel. It is significant to consider the difference between these two assumptions. H. Li proved that, if M^n is a closed and oriented pseudo-umbilical submanifold in $\overline{M}_c^{n+p}$ and H is nowhere zero, then H is constant if and only if ξ/H is parallel.

Acknowledgements. The author would like to express the deep gratitude to the Okazaki Kaheita International Scholarship Foundation which afforded him such a good opportunity for study in Japan and gave him very precious support while he was staying in Tokyo.

1. Preliminaries

Let M^n be a connected submanifold immersed into a space form $\overline{M}_c^{n+p}$. We always assume that M^n is oriented and identify M^n with its immersed image in $\overline{M}_c^{n+p}$.

Choose a local orthonormal frame field $\{e_A\}_{A=1}^{n+p}$ of $T(\overline{M}_c^{n+p})$ over M^n such that $\{e_i\}_{i=1}^n$ lies in the tangent bundle $T(M^n)$ and $\{e_\alpha\}_{\alpha=n+1}^{n+p}$ in the normal bundle $N(M^n)$ of M^n. Let $\{\omega_A\}_{A=1}^{n+p}$ denote the dual coframe field and $(\omega_{AB})_{A,B=1}^{n+p}$ the Riemannian connection matrix associated with $\{e_A\}_{A=1}^{n+p}$. Then $(\omega_{ij})_{i,j=1}^{n}$ defines a Riemannian connection in $T(M^n)$ associated with $\{e_i\}_{i=1}^{n}$ and $(\omega_{\alpha\beta})_{\alpha,\beta=n+1}^{n+p}$ defines a normal connection in $N(M^n)$ associated with $\{e_\alpha\}_{\alpha=n+1}^{n+p}$.

Throughout this paper, we agree on the following index ranges:

$$1 \leq i, j, k, \ldots \leq n; \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p; \quad 1 \leq A, B, C, \ldots \leq n+p.$$

We know that the second fundamental form of M^n can be expressed as

$$\sigma = \sum_{(i,\alpha)} \omega_i \otimes \omega_{i\alpha} \otimes e_\alpha = \sum_{(i,j,\alpha)} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

where $\omega_{i\alpha} = \sum_{(j)} h_{ij}^\alpha \omega_j$ for all i and α. It is well-known that $h_{ij}^\alpha = h_{ji}^\alpha$ for all i, j and α.

Denote $L_\alpha = (h_{ij}^\alpha)_{n\times n}$ and $H_\alpha = (1/n)\sum_{(i)} h_{ii}^\alpha$ for every α. The mean curvature vector field ξ, the mean curvature H and the square of the length of the second fundamental form, say S, are expressed as

$$\xi = \sum_{(\alpha)} H_\alpha e_\alpha, \quad H = |\xi|, \quad S = \sum_{(\alpha,i,j)} (h_{ij}^\alpha)^2.$$

The Riemannian curvature tensor $\{R_{ijkl}\}$, the normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the normalized scalar curvature R are expressed as

$$\begin{aligned} R_{ijkl} &= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\, c + h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha, \\ R_{\alpha\beta kl} &= h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta, \\ R_{ik} &= (n-1)\, c\, \delta_{ik} + (nH_\alpha) h_{ik}^\alpha - h_{ij}^\alpha h_{jk}^\alpha, \\ R &= \frac{1}{n(n-1)} \sum_{(i)} R_{ii} = c + \frac{1}{n(n-1)}(n^2H^2 - S). \end{aligned} \tag{1.1}$$

We define the first and the second covariant derivatives of $\{h_{ij}^\alpha\}$ by

$$\nabla h_{ij}^\alpha = h_{ijk}^\alpha \omega_k \equiv dh_{ij}^\alpha + h_{mj}^\alpha \omega_{mi} + h_{im}^\alpha \omega_{mj} + h_{ij}^\beta \omega_{\beta\alpha}, \tag{1.2}$$

$$\nabla h_{ijk}^\alpha = h_{ijkl}^\alpha \omega_l \equiv dh_{ijk}^\alpha + h_{mjk}^\alpha \omega_{mi} + h_{imk}^\alpha \omega_{mj} + h_{ijm}^\alpha \omega_{mk} + h_{ijk}^\beta \omega_{\beta\alpha}. \tag{1.3}$$

It follows from Ricci's identity that

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \quad h_{ijkl}^\alpha - h_{ijlk}^\alpha = h_{mj}^\alpha R_{mikl} + h_{im}^\alpha R_{mjkl} + h_{ij}^\beta R_{\beta\alpha kl}. \tag{1.4}$$

The Laplacian of h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_{(k)} h_{ijkk}^\alpha$. Using (1.4), we obtain

$$\begin{aligned}\Delta h_{ij}^\alpha =& nH_{\alpha,ij} + n\,c\,h_{ij}^\alpha - n\,c\,H_\alpha\delta_{ij} + nH_\beta h_{im}^\alpha h_{mj}^\beta - S_{\alpha\beta}h_{ij}^\beta \\ &+2h_{ik}^\beta h_{km}^\alpha h_{mj}^\beta - h_{im}^\alpha h_{mk}^\beta h_{kj}^\beta - h_{ik}^\beta h_{km}^\beta h_{mj}^\alpha,\end{aligned}$$

where $S_{\alpha\beta} = \sum_{(i,j)} h_{ij}^\alpha h_{ij}^\beta$ for all α and β. For a real matrix $A = (a_{ij})_{n\times n}$, we define $N(A) = \sum_{(i,j)} a_{ij}^2$. Then we have

$$\begin{aligned}\sum_{(i,j)} h_{ij}^\alpha \Delta h_{ij}^\alpha =& n\sum_{(i,j)} H_{\alpha,ij}h_{ij}^\alpha + n\,c\,S_\alpha - c\,n^2H_\alpha^2 \\ &+nH\mathrm{Tr}(L_\alpha^2 L_{n+1}) - S_{n+1\alpha}^2 - N(L_\alpha L_{n+1} - L_{n+1}L_\alpha) \\ &- \sum_{(\beta>n+1)} S_{\alpha\beta}^2 - \sum_{(\beta>n+1)} N(L_\alpha L_\beta - L_\beta L_\alpha),\end{aligned} \tag{1.5}$$

where we denote $S_\alpha = S_{\alpha\alpha} = \sum_{(i,j)}(h_{ij}^\alpha)^2$, for all α.

Suppose that e_{n+1} has the same direction as ξ. Then $\xi = He_{n+1}$ and

$$H_{n+1} = H, \quad H_\alpha = 0, \quad \text{for} \quad \alpha > n+1. \tag{1.6}$$

It follows from (1.2) and (1.6) that

$$H_{n+1,k}\omega_k = dH = H_k\omega_k, \quad H_{\alpha,k}\omega_k = H\omega_{n+1\alpha}, \quad \text{for} \quad \alpha > n+1. \tag{1.7}$$

We define a real function ε_H on M^n as follows:

$$\varepsilon_H = \begin{cases} 1/H & \text{if } H \neq 0, \\ 0 & \text{if } H = 0. \end{cases}$$

It follows from (1.3), (1.6) and (1.7) that

$$H_{n+1,kl} = H_{kl} - \varepsilon_H \sum_{(\beta>n+1)} H_{\beta,k}H_{\beta,l}, \tag{1.8}$$

where we denote $\nabla H_k = H_{kl}\omega_l \equiv dH_k + H_l\omega_{lk}$ for all k.

From (1.5) and (1.8), we have

$$\begin{aligned}\sum_{(i,j)} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1} =& n\sum_{(i,j)} H_{ij}h_{ij}^{n+1} - n\,\varepsilon_H \sum_{(i,j)}\sum_{(\beta>n+1)} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} \\ &+n\,c\,S_{n+1} - c\,n^2H^2 + nHf_{n+1} - S_{n+1}^2 - \sum_{(\beta>n+1)} S_{n+1\beta}^2 \\ &- \sum_{(\beta>n+1)} N(L_{n+1}L_\beta - L_\beta L_{n+1}),\end{aligned} \tag{1.9}$$

where $f_{n+1} = \mathrm{Tr}(L_{n+1})^3$. We define $\tilde{L}_{n+1}$ and $\tilde{S}_{n+1}$ by

$$\tilde{L}_{n+1} = L_{n+1} - H\, I_n, \quad \tilde{S}_{n+1} = N(\tilde{L}_{n+1}) = S_{n+1} - nH^2, \tag{1.10}$$

where I_n denotes the identity matrix of degree n.

By using the same arguments as in [4, pp. 1194], we obtain

$$\begin{aligned} n\,c\,S_{n+1} - c\,n^2H^2 + nHf_{n+1} - S_{n+1}^2 \\ \geq \tilde{S}_{n+1}\left\{n\,c - (\tilde{S}_{n+1} - nH^2) - n(n-2)H\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}}\right\}. \end{aligned} \tag{1.11}$$

It follows from (1.2) that

$$\sum_{(\beta>n+1)} (S_{n+1\beta})^2 = \sum_{(\beta>n+1)} \left\{\sum_{(i,j)} (h_{ij}^{n+1} - H\delta_{ij})h_{ij}^{\beta}\right\}^2. \tag{1.12}$$

Denote $S_I = \sum_{(\beta>n+1)} S_\beta$. From (1.12), we have

$$\sum_{(\beta>n+1)} (S_{n+1\beta})^2 \leq \tilde{S}_{n+1} S_I, \tag{1.13}$$

and the equality in (1.13) holds if and only if there exists a real function c_β such that $L_\beta = c_\beta\, \tilde{L}_{n+1}$, for every $\beta > n+1$.

Let $T = \sum_{(i,j)} T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = h_{ij}^{n+1} - nH\delta_{ij}, \quad i,j = 1, \cdots, n. \tag{1.14}$$

Associated to T, we define a second order differential operator $\Box$ on $C^2(M)$ by

$$\Box f = \sum_{(i,j)} T_{ij} f_{ij} = \sum_{(i,j)} h_{ij}^{n+1} f_{ij} - (nH)\Delta f, \quad f \in C^2(M). \tag{1.15}$$

Since (T_{ij}) is divergence-free, it follows from [2, Proposition 1] that $\Box$ is a self-adjoint operator relative to the L^2-inner product of $C^2(M^n)$. Setting $f = H$ in (1.15), we have

$$\sum_{(i,j)} h_{ij}^{n+1} H_{ij} = \Box H + nH\Delta H = \Box H + \frac{n}{2}\Delta(H^2) - n|\nabla H|^2. \tag{1.16}$$

Denote $\tilde{S} = \tilde{S}_{n+1} + S_I$. Substituting (1.11), (1.13) and (1.16) into (1.9), we get

$$
\begin{aligned}
\sum_{(i,j)} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1} \geq n\Box H + \tfrac{1}{2}\Delta(n^2H^2) - n^2|\nabla H|^2 \\
-n\,\varepsilon_H \sum_{(\beta>n+1)}\sum_{(i,j)} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} \\
- \sum_{(\beta>n+1)} N(L_{n+1}L_\beta - L_\beta L_{n+1}) \\
+\tilde{S}_{n+1}\left\{n\,c + nH^2 - \tilde{S} - n(n-2)H\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}}\right\}.
\end{aligned}
\tag{1.17}
$$

2. Umbilical Hypersurfaces in a Hyperbolic Space Form

In this section, we consider some special hypersurfaces in a hyperbolic space form that will play an important role in our latter discussions.

We propose to give a description of the real hyperbolic space form $\mathbb{H}^{n+1}(c)$ of constant curvature c (<0). For any two vectors X and Y in $\mathbb{R}^{n+2}$, we set

$$g(X,Y) = \sum_{i=1}^{n+1} X^iY^i - X^{n+2}Y^{n+2}.$$

$(\mathbb{R}^{n+2}, g)$ is the so-called Minkowski space-time. Denote $R = \sqrt{-1/c}$. We define

$$\mathbb{H}^{n+1}(c) = \left\{x \in \mathbb{R}^{n+2} \mid x_{n+2} > 0, \quad g(x,x) = -R^2\right\}.$$

Then $\mathbb{H}^{n+1}(c)$ is a connected simply-connected hypersurface of $\mathbb{R}^{n+2}$. It is not hard to check that the restriction of g to the tangent space of $\mathbb{H}^{n+1}(c)$ yields a complete Riemanian metric of constant curvature c. Hence we obtain a model of a real hyperbolic space form.

We are interested in those complete hypersurfaces with at most two constant distinct principal curvatures in $\mathbb{H}^{n+1}(c)$. This kind of hypersurfaces was described by Lawson [5] and completely classified by Ryan [8].

Lemma 2.1(Ryan[8]). *Let M^n be a complete hypersurface in $\mathbb{H}^{n+1}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of $T(M^n)$, the shape operator over $T(M^n)$ is expressed as a matrix A. If M^n has at most two distinct constant principal curvatures, then it is congruent to one of the following:*

(1) $M_1 = \{x \in \mathbb{H}^{n+1}(c) \mid x_1 = 0\}$. *In this case, $A = 0$, and M_1 is totally geodesic. Hence M_1 is isometric to $\mathbb{H}^n(c)$;*

(2) $M_2 = \{x \in \mathbb{H}^{n+1}(c) \mid x_1 = r > 0\}$. *In this case,* $A = \dfrac{1/R^2}{\sqrt{1/R^2 + 1/r^2}}\, I_n$, *where* I_n *denotes the identity matrix of degree* n*, and* M_2 *is isometric to* $\mathbb{H}^n(-1/(r^2 + R^2))$*;*

(3) $M_3 = \{x \in \mathbb{H}^{n+1}(c) \mid x_{n+2} = x_{n+1} + R\}$. In this case, $A = (1/R)\, I_n$, and M_3 is isometric to a Euclidean space E^n;

(4) $M_4 = \left\{x \in \mathbb{H}^{n+1}(c) \mid \sum_{i=1}^{n+1} x_i^2 = r^2 > 0\right\}$. In this case, $A = \sqrt{1/R^2 + 1/r^2}$ I_n, and M_4 is isometric to a round sphere $S^n(r)$ of radius r;

(5) $M_5 = \left\{x \in \mathbb{H}^{n+1}(c) \mid \sum_{i=1}^{k+1} x_i^2 = r^2 > 0, \sum_{j=k+2}^{n+1} x_j^2 - x_{n+2}^2 = -R^2 - r^2\right\}$. In this case, $A = \lambda\, I_k \oplus \mu\, I_{n-k}$, where $\lambda = \sqrt{1/R^2 + 1/r^2}$ and $\mu = \dfrac{1/R^2}{\sqrt{1/R^2 + 1/r^2}}$, and M_5 is isometric to $S^k(r) \times \mathbb{H}^{n-k}(-1/(r^2 + R^2))$.

Remark 2.1. $M_1, \cdots, M_5$ are often called the standard examples of complete hypersurfaces in $\mathbb{H}^{n+1}(c)$ with at most two distinct constant principal curvatures. It is obvious that $M_1, \cdots, M_4$ are totally umbilical. In the sense of Chen [1], they are called the hypersphere of $\mathbb{H}^{n+1}(c)$. M_3 is called the horosphere and M_4 the geodesic distance sphere of $\mathbb{H}^{n+1}(c)$.

Remark 2.2. Ryan [8] stated that the shape operator of M_2 is $A = \sqrt{1/r^2 - 1/R^2} I_n$, and M_2 is isometric to $\mathbb{H}^n(-1/r^2)$, where $r \leq R$. This is incorrect and we have it corrected here.

After a similar discussion as in the proof of Proposition 4.2 in [1, pp. 133], we get

Lemma 2.2. *Let* M^n *be a complete submanifold immersed into a hyperbolic space form* $\mathbb{H}^{n+p}(c)$*. Assume that there is a globally defined unit normal vector field* ξ *on* M^n *such that* ξ *is parallel and the second fundamental form in the direction of* ξ *has constant equal eigenvalues everywhere on* M^n*. Then* M^n *lies in a hypersphere of* $\mathbb{H}^{n+p}(c)$*. Moreover, if* ξ *is the normalized mean curvature vector field of* M^n*, then the immersion is minimal.*

3. An Extension of Cheng-Yau's Technique

Cheng-Yau [2] gave a lower bound estimate for the square of the length of the covariant derivative $\nabla\sigma$ of σ, which plays an important role in their discussion. They proved that, for a hypersurface M^n in $\overline{M}_c^{n+1}$, if the normalized scalar curvature R of M^n is constant and greater than or equal to c, then $|\nabla\sigma|^2 \geq n^2|\nabla H|^2$. In this section, we propose to extend this inequality to higher codimensional cases. Namely, we want to prove the following:

Proposition 3.1. *Let M^n be a connected submanifold immersed into $\overline{M}_c^{n+p}$. Suppose that the normalized scalar curvature R is constant and greater than or equal to c. Then*

$$|\nabla\sigma|^2 \geq n^2|\nabla H|^2 \tag{3.1}$$

and the symmetric tensor T defined by (1.14) *is negative semi-definite. Moreover, suppose that the equality in* (3.1) *holds everywhere on M^n. Then*

(i) *if $R-c>0$, then H is constant and T is negative definite;*

(ii) *if $R-c=0$, then either H is constant or M^n lies in a totally geodesic subspace $\overline{M}_c^{n+1}$ of $\overline{M}_c^{n+p}$. In the latter case, if H is not constant on M^n, then $r(L_{n+1}) \leq 1$ on M^n, where $r(L_{n+1})$ denotes the rank of L_{n+1}.*

Proof. It is known that $n^2H^2 - S = n(n-1)(R-c) \geq 0$. Taking covariant derivative on both sides of this equality, we get $n^2H\,H_k = \sum_{(i,j,\alpha)} h_{ij}^\alpha\, h_{ijk}^\alpha$ for every k. It follows from Cauchy-Schwarz's inequality that

$$n^4H^2H_k^2 \leq S \sum_{(i,j,\alpha)} (h_{ijk}^\alpha)^2 \tag{3.2}$$

for all k. Moreover, the equality in (3.2) holds if and only if there exists a real function c_k on M^n such that

$$h_{ijk}^\alpha = c_k\, h_{ij}^\alpha, \quad i,j=1,\cdots,n, \quad \alpha=n+1,\cdots,n+p, \tag{3.3}$$

for every k.

Taking sum with respect to k on both sides of (3.2), we obtain $n^4H^2|\nabla H|^2 \leq S|\nabla\sigma|^2$, where $|\nabla\sigma|^2 = \sum_{(i,j,k,\alpha)}(h_{ijk}^\alpha)^2$. It follows that

$$0 \leq n^3(n-1)(R-c)|\nabla H|^2 \leq S\left(|\nabla\sigma|^2 - n^2|\nabla H|^2\right). \tag{3.4}$$

Therefore (3.1) holds on M^n since S is continuous on M^n.

Denote the eigenvalues of L_{n+1} by $\{\lambda_i^{n+1}\}_{i=1}^n$. Then $(\lambda_i^{n+1})^2 \leq S_{n+1} \leq S \leq n^2H^2$ for all i. Thus $|\lambda_i^{n+1}| \leq nH$ for all i. Therefore $T=(T_{ij}) = L_{n+1} - nHI_n$ is negative semi-definite.

Let $|\nabla\sigma|^2 = n^2|\nabla H|^2$ on M^n. It follows from (3.4) that $(R-c)|\nabla H|^2=0$ on M^n. If $R-c>0$, then $|\nabla H|^2=0$ on M^n. In this case, none of $|\lambda_i^{n+1}|$'s is equal to nH. Hence T is negative definite. Thus (i) follows.

Suppose $R-c=0$. Then $S=n^2H^2$ on M^n. In this case, the equality in (3.2) holds for all k. From (3.3), we have

$$h_{ijk}^{n+1} = c_k\, h_{ij}^{n+1}; \quad h_{ijk}^\alpha = c_k\, h_{ij}^\alpha, \quad \alpha > n+1; \quad i,j,k=1,\cdots,n. \tag{3.5}$$

Taking sum on both-sides of equations in (3.5) with respect to $i = j$, we get

$$H_k = c_k H; \quad H_{\alpha,k} = 0, \quad \alpha > n+1; \quad k = 1, \cdots, n. \tag{3.6}$$

From (1.7) and the second equation in (3.6) we can see that e_{n+1} is parallel.

Multiplying both-sides of the first equation in (3.5) by H and using (3.6), we have

$$Hh_{ijk}^{n+1} = H_k h_{ij}^{n+1}, \quad i, j, k = 1, \cdots, n. \tag{3.7}$$

Taking sum on both-sides of (3.7) with respect to $j = k$, we have

$$(nH)H_i = H_j h_{ij}^{n+1}, \quad i = 1, \cdots, n. \tag{3.8}$$

From (3.7) and the fact $|\nabla\sigma|^2 = n^2|\nabla H|^2$, we have $H^2|\nabla S_I|^2 = 0$ on M^n. From (3.8) and the fact $S = n^2H^2$, we have $S_I|\nabla H|^2 = 0$ on M^n.

Denote $M_1 = \{x \in M^n \mid S_I(x) > 0\}$ and $M_2 = \{x \in M^n \mid |\nabla H|(x) > 0\}$. Then M_1 and M_2 are open in M^n. It follows from the equality $S_I|\nabla H|^2 = 0$ that $M_1 \cap M_2 = \emptyset$.

We assert that at least one of M_1 and M_2 is empty. In fact, if both of them are not empty, then $|\nabla H|^2 = 0$ in M_1 and $S_I = 0$ in M_2. Hence H is constant in M_1. It follows from the equality $S = n^2H^2$ that $H > 0$ in M_1. Thus we have from the equality $H^2|\nabla S_I|^2 = 0$ that $|\nabla S_I|^2 = 0$ in M_1. Therefore S_I is constant in M_1. This is contradictory to the fact $S_I = 0$ in M_2 since S_I is continuous on M^n. Our assertion follows.

If $M_1 \neq \emptyset$, then $M_2 = \emptyset$. Hence H is constant on M^n. On the other hand, if $M_2 \neq \emptyset$, then $M_1 = \emptyset$. So $S_I = 0$ on M^n. Therefore we proved the first part of (ii).

Suppose that $S_I = 0$ on M^n and $M_2 \neq \emptyset$. Then it follows from (3.8) that $\lambda_{i_0}^{n+1} = nH$ for some i_0 in M_2. Using the fact that $S_{n+1} = S = n^2H^2$, we can see that $\lambda_i^{n+1} = 0$ for all $i \neq i_0$ in M_2. From the continuity of λ_i^{n+1}'s, we have that $\lambda_i^{n+1} = 0$ for all $i \neq i_0$ on the closure $\mathrm{cl}(M_2)$ of M_2. If $M \setminus \mathrm{cl}(M_2) \neq \emptyset$, it follows from (3.7) that all of λ_i^{n+1}'s are constant in $M \setminus \mathrm{cl}(M_2)$. Since all of λ_i^{n+1}'s are continuous, we have that $\lambda_i^{n+1} = 0$ for all $i \neq i_0$ on M^n. Hence $r(L_{n+1}) \leq 1$ on M^n. The second part of (ii) follows. □

Remark 3.1. It is significant to classify all of M^n on which the equality in (3.1) holds.

4. The Rigidity of Submanifolds in a Hyperbolic Space Form

In this section, we propose to study the rigidity problem for submanifolds in $\mathbb{H}^{n+p}$ with constant scalar curvature. We continue to use the same notation as in Section 1.

Let M^n be a closed submanifold with parallel normalized mean curvature vector field immersed into $\mathbb{H}^{n+p}$. Assume that R is constant and greater than and equal to $-(n-2)/(n-1)$.

Choose e_{n+1} to have the same direction as ξ. Then $\xi = He_{n+1}$. From the fact that e_{n+1} is parallel, it follows that $\omega_{n+1\alpha} = 0$, which implies

$$L_{n+1}L_\alpha - L_\alpha L_{n+1} = 0 \tag{4.1}$$

for all α. From (1.2) and (1.7), we have

$$H_{\alpha,k} = 0, \quad H_{\alpha,kl} = 0 \tag{4.2}$$

for all k, l and $\alpha > n+1$. Substituting (4.1) and (4.2) into (1.17), we have

$$\begin{aligned} \sum_{(i,j)} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \geq\ & n\Box H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 \\ & + \tilde{S}_{n+1}\left\{-n + nH^2 - \tilde{S} - n(n-2)H\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}}\right\}. \end{aligned} \tag{4.3}$$

Taking sum on both-sides of (1.5) with respect to $\alpha > n+1$ and using (4.1) and (4.2), we have

$$\begin{aligned} \sum_{(i,j,\alpha>n+1)} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} =\ & (-n + n\,H^2)S_I + (nH)\sum_{(\alpha>n+1)} \mathrm{Tr}(L_\alpha^2 \tilde{L}_{n+1}) \\ & - \sum_{(\alpha>n+1)} S_{n+1\alpha}^2 - \sum_{(\alpha,\beta>n+1)}\left\{S_{\alpha\beta}^2 + N(L_\beta L_\alpha - L_\alpha L_\beta)\right\}. \end{aligned} \tag{4.4}$$

To estimate the right hand-side of (4.4), we need the following

Lemma 4.1 (Santos[9]). *Let A and B be $n \times n$ symmetric matrices satisfying* $\mathrm{Tr}\,A = 0$, $\mathrm{Tr}\,B = 0$ *and* $A\,B - B\,A = 0$. *Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{Tr}A^2)(\mathrm{Tr}B^2)^{1/2} \leq \mathrm{Tr}A^2B \leq \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{Tr}A^2)(\mathrm{Tr}B^2)^{1/2}, \tag{4.5}$$

and the equality on the right (resp. left) hand side of (4.5) *holds if and only if $n-1$ many eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy*

$$|x_i| = \frac{(\mathrm{Tr}A^2)^{1/2}}{\sqrt{n(n-1)}}, \quad x_i\,x_j \geq 0; \quad y_i = -\frac{(\mathrm{Tr}B^2)^{1/2}}{\sqrt{n(n-1)}}\left(\textit{resp.} \quad y_i = \frac{(\mathrm{Tr}B^2)^{1/2}}{\sqrt{n(n-1)}}\right).$$

Using the left hand side of (4.5) to every $\mathrm{Tr}(L_\alpha^2\tilde{L}_{n+1})$, we have

$$\mathrm{Tr}(L_\alpha^2\tilde{L}_{n+1}) \geq -(n-2)S_\alpha\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}}.$$

Substituting this and (1.13) into (4.4), we have

$$\sum_{(i,j,\alpha>n+1)} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geq S_I\left\{n(-1+H^2)-n(n-2)H\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}}-\tilde{S}_{n+1}\right\} - \sum_{(\alpha,\beta>n+1)}\left\{S_{\alpha\beta}^2+N(L_\beta L_\alpha-L_\alpha L_\beta)\right\}. \tag{4.6}$$

Note that $\Delta S=\Delta(n^2H^2)$ and

$$\frac{1}{2}\Delta S=|\nabla\sigma|^2+\sum_{(i,j)}h_{ij}^{n+1}\Delta h_{ij}^{n+1}+\sum_{(i,j,\alpha>n+1)}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}.$$

It follows from (4.3) and (4.6) that

$$\begin{aligned}0 \geq{} & n\Box H+|\nabla\sigma|^2-n^2|\nabla H|^2+S_I^2\\ & +\tilde{S}\left\{(-n+nH^2)-n(n-2)H\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}}-\tilde{S}\right\}\\ & -\sum_{(\alpha,\beta>n+1)}\left\{S_{\alpha\beta}^2+N(L_\beta L_\alpha-L_\alpha L_\beta)\right\}.\end{aligned} \tag{4.7}$$

Let us now prove the main results of this section. We will reduce our consideration to the following two cases. Each of them is interesting in its own right.

Case 1. *Submanifold with flat normal bundle.* Suppose in addition that M^n is with flat normal bundle. Then $\Omega_{\alpha\beta}=0$ for all α and β, which is equivalent to

$$L_\alpha L_\beta - L_\beta L_\alpha = 0 \tag{4.8}$$

for all α and β.

Since the matrix $(S_{\alpha\beta})_{\alpha,\beta>n+1}$ is symmetric, under a suitable choice of $\{e_\beta\}_{\beta=n+2}^{n+p}$, we can assume that $S_{\alpha\beta}=0$ for all $\alpha,\beta>n+1$ and $\alpha\neq\beta$. Hence

$$\sum_{(\alpha,\beta>n+1)}S_{\alpha\beta}^2=\sum_{(\beta>n+1)}S_\beta^2\leq S_I^2, \tag{4.9}$$

where the equality holds if and only if at most one of S_α's is not zero.

Denote $\phi_{n+1}=\sqrt{\tilde{S}_{n+1}/n(n-1)}$ and define

$$h=2C\left(\phi_{n+1}+\sqrt{\phi_{n+1}^2+4C}\right)^{-1}, \tag{4.10}$$

where $C = (n-1)R/(n-2)+1 > 0$. Note that $\tilde{S} = n(n-1)(H^2 - R - 1)$. Then

$$\tilde{S}\left\{(-n+nH^2) - n(n-2)H\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}} - \tilde{S}\right\} = \Phi\,(H - \sqrt{R+1})(h-H),$$

where $\Phi = n^2(n-1)(n-2)(H+\sqrt{R+1})(H+Ch^{-1})$. Hence (4.7) turns into

$$\begin{aligned} 0 &\geq n\Box H + |\nabla\sigma|^2 - n^2|\nabla H|^2 + S_I^2 \\ &- \sum_{(\beta>n+1)} S_\beta^2 + \Phi\,(H-\sqrt{R+1})(h-H). \end{aligned} \tag{4.11}$$

If $H \leq h$, then $\Phi > 0$ and $\Phi\,(H-\sqrt{R+1})(h-H) \geq 0$. Integrating both-sides of (4.11) on M^n, we have from (3.1) and (4.9) that

$$|\nabla\sigma|^2 = n^2|\nabla H|^2, \quad \sum_{(\beta>n+1)} S_\beta^2 = S_I^2, \ (H-\sqrt{R+1})(h-H) = 0 \tag{4.12}$$

on M^n. So we can prove the following:

Theorem 4.1. *Let M^n $(n \geq 3)$ be a closed submanifold in $\mathbb{H}^{n+p}$ with parallel normalized mean curvature vector field. Suppose that the normalized scalar curvature R is constant and greater than or equal to $-(n-2)/(n-1)$. Let h be defined by* (4.10). *If the normal bundle of M^n is flat and $H \leq h$, then $R > 0$ and either*

(i) *$H = \sqrt{R+1}$ and M^n is a geodesic distance sphere $S^n(1/\sqrt{R})$ in $\mathbb{H}^{n+p}$; or*

(ii) *$H = h$ and one of the following cases occurs:*

 (a) *$H = \sqrt{(n-1)R/(n-2)+1}$ and M^n is a Clifford hypersurface*

$$S^{n-1}(r_0\sqrt{(n-1)/n}) \times S^1(r_0\sqrt{1/n})$$

 in a geodesic distance sphere $S^{n+1}(r_0)$ of $\mathbb{H}^{n+p}$, where $r_0 = \sqrt{(n-2)/(n-1)R}$;

 (b) *$\sqrt{R+1+R/n(n-2)} < H < \sqrt{(n-1)R/(n-2)+1}$ and M^n is a pythagorean product of the form $S^{n-1}(r) \times S^1(\sqrt{r_1^2 - r^2})$ in a geodesic distance sphere $S^{n+1}(r_1)$ of $\mathbb{H}^{n+p}$, where $r_1^{-2} = nR/(n-2) - R^2/(n-2)^2(H^2-R-1)$ and $r = \sqrt{(n-2)/nR}$.*

Proof. From the first equality in (4.12) and Proposition 3.1, we have that H is constant. It follows from the third equality in (4.12) that $H = \sqrt{R+1}$ or $H = h$.

If $H = \sqrt{R+1}$, then $S = n(R+1)$. So M^n is umbilically immersed into a totally geodesic subspace $\mathbb{H}^{n+1}$ of $\mathbb{H}^{n+p}$. Since M^n is closed, it follows from Lemma 2.1 that $R > 0$ and M^n is a geodesic distance sphere $S^n(1/\sqrt{R})$ of $\mathbb{H}^{n+p}$.

If $H = h$, then $\tilde{S} = n(-1+H^2) - n(n-2)H\sqrt{\tilde{S}_{n+1}/n(n-1)}$. In this case, all of the inequalities concerned become equalities. It follows from (4.9) and the second equality in (4.12) that M^n lies in a totally geodesic subspace $\mathbb{H}^{n+2}$ of $\mathbb{H}^{n+p}$. Without loss of generality, we assume that $S_\beta \equiv 0$ for all $\alpha \geq n+3$. From the equality on the left hand side of (4.5), we can assume that, under a suitable choice of $\{e_i\}_{i=1}^n$, $\tilde{L}_{n+1}$ and L_{n+2} are of the form:

$$\begin{cases} \tilde{L}_{n+1} = a_1 U_n, \quad a_1 = \sqrt{\dfrac{\tilde{S}_{n+1}}{n(n-1)}} \\ L_{n+2} = a_2 U_n, \quad a_2 = \sqrt{\dfrac{S_{n+2}}{n(n-1)}} \end{cases}, \quad \text{where} \quad U_n = \left(\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \hline 0 & \cdots & 0 & 1-n \end{array}\right).$$

It is easy to see that $a_1 = \phi_{n+1}$ and

$$a_1^2 + a_2^2 = \frac{\tilde{S}}{n(n-1)} = H^2 - (R+1). \tag{4.13}$$

If $\phi_{n+1} = 0$, then $H = \sqrt{(n-1)R/(n-2)+1}$ and $L_{n+1} = HI_n$. Therefore M^n is pseudo-umbilical and hence can be minimally immersed into a hypersphere $S^{n+1}(r_0)$ of $\mathbb{H}^{n+2}$, where $r_0^{-2} = (n-1)R/(n-2)$. Since M^n is closed with two distinct constant principal curvatures, it follows from Lemma 2.1 that $r_0 < +\infty$. Therefore $R > 0$ and M^n is a Clifford hypersurface $S^{n-1}(r_0\sqrt{(n-1)/n}) \times S^1(r_0\sqrt{1/n})$ in a geodesic distance sphere$S^{n+1}(r_0)$.

Suppose that $\phi_{n+1} > 0$ on M^n. It is easy to see that $e_{n+1} = \xi/H$ and $e_{n+2}(\perp e_{n+1})$ are globally defined and parallel on M^n. Take the transformation of normal frame fields as follows:

$$\hat{e}_{n+1} = \frac{a_2}{\sqrt{a_1^2+a_2^2}} e_{n+1} - \frac{a_1}{\sqrt{a_1^2+a_2^2}} e_{n+2},$$
$$\hat{e}_{n+2} = \frac{a_1}{\sqrt{a_1^2+a_2^2}} e_{n+1} + \frac{a_2}{\sqrt{a_1^2+a_2^2}} e_{n+2}.$$

Then the second fundamental forms with respect to $\hat{e}_{n+1}$ and $\hat{e}_{n+2}$ turn into

$$\hat{L}_{n+1} = \frac{H\,a_2}{\sqrt{a_1^2+a_2^2}} I_n, \quad \hat{L}_{n+2} = \sqrt{a_1^2+a_2^2}\, U_n + \frac{H\,a_1}{\sqrt{a_1^2+a_2^2}} I_n. \tag{4.14}$$

From (4.14), we have that $\hat{e}_{n+1}$ is an umblical unit normal vector of M^n. From Lemma 2.2 we have that M^n lies in a hypersphere $S^{n+1}(r_1)$ of $\mathbb{H}^{n+2}$, where $r_1^{-2} = -1 + H^2 a_2^2/(a_1^2 + a_2^2)$. It is easy to check that

$$r_1^{-2} = \frac{nR}{n-2} - \frac{1}{H^2 - (R+1)}\left(\frac{R}{n-2}\right)^2. \tag{4.15}$$

Since M^n is closed with two distinct constant pricipal curvatures, it follows from Lemma 2.1 that $r_1 < +\infty$. Hence $R > 0$ and M^n lies in a geodesic distance sphere $S^{n+1}(r_1)$ of $\mathbb{H}^{n+2}$. It follows from (4.15) that $H = h > \sqrt{R + 1 + R/n(n-2)}$.

It is obvious that the second fundamental form of M^n in $S^{n+1}(r_1)$ is just $\hat{L}_{n+2}$. Thus $M^n = S^{n-1}(r) \times S^1(\sqrt{r_1^2 - r^2}) \subset S^{n+1}(r_1)$, where

$$r^{-2} = r_1^{-2} + \left(\sqrt{a_1^2 + a_2^2} + \frac{Ha_1}{\sqrt{a_1^2 + a_2^2}}\right)^2 = \frac{nR}{n-2}.$$

Therefore we complete the proof. □

Remark 4.1. It is very interesting that in the case (b) of (ii) of Theorem 4.1, the radius r of the $(n-1)$-dimension leaf of $M^n = S^{n-1}(r) \times S^1(\sqrt{r_1^2 - r^2})$ depends only on n and the normalized scalar curvature R. Moreover, from (ii) of Theorem 4.1 we can see that if $\sqrt{R+1} < H \le h$, then $H > \sqrt{R + 1 + R/n(n-2)}$.

Case 2. *Submanifold with parallel normalized mean curvature vector field.* Suppose that M^n is one with parallel normalized mean curvature vector field. When $p = 2$, M^n is in fact with flat normal bundle which has been discussed in *Case 1.* We assume $p \ge 3$ in the following discussion.

We propose to estimate the last term in (4.7). At first, we need the following:

Lemma 4.2 (Li's [6]). *Let $A_1, A_2, \cdots, A_q$ be symmetric $n \times n$ matrices, where $q \ge 2$. Denote $S_{\alpha\beta} = \operatorname{Tr} A_\alpha^T A_\beta$, $S_\alpha = S_{\alpha\alpha} = \mathrm{N}(A_\alpha)$ and $S = S_1 + \cdots + S_q$. Then*

$$\sum_{(\alpha,\beta)} S_{\alpha\beta}^2 + \sum_{(\alpha,\beta)} N(A_\beta A_\alpha - A_\alpha A_\beta) \le \frac{3}{2}S^2, \tag{4.16}$$

and the equality holds if and only if one of the following conditions holds:

(i) $A_1 = \cdots = A_q = 0$;

(ii) *Only two of the matrices* $A_1, A_2, \cdots, A_q$ *are different from zero. Moreover, if we assume* $A_1 \neq 0, A_2 \neq 0$ *and* $A_3 = \cdots = A_q = 0$*, then* $S_1 = S_2$ *and there exists an orthogonal* $n \times n$ *matrix* U *such that*

$$U\,A_1\,U^T = \sqrt{\frac{S_1}{2}}\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \quad U\,A_2\,U^T = \sqrt{\frac{S_1}{2}}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Setting $A_{\alpha-(n+1)} = L_\alpha$ in (4.16) for $\alpha = n+2, \cdots, n+p$, we have

$$\sum_{(\alpha,\beta>n+1)} \left\{ S_{\alpha\beta}^2 + N(L_\beta L_\alpha - L_\alpha L_\beta) \right\} \leq \frac{3}{2} S_I^2.$$

Substituting this inequality into (4.7), we obtain

$$\begin{aligned} 0 &\geq n\Box H + |\nabla\sigma|^2 - n^2|\nabla H|^2 + \frac{1}{2}\tilde{S}_{n+1}(\tilde{S} + S_I) \\ &+\tilde{S}\left\{ n(-1+H^2) - n(n-2)H\sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}} - \frac{3}{2}\tilde{S} \right\}. \end{aligned} \tag{4.17}$$

Denote $\phi_{n+1} = \sqrt{\tilde{S}_{n+1}/n(n-1)}$ and define

$$\tilde{h} = (C_1 R + 1)\left(C_2\phi_{n+1} + \sqrt{C_2^2\phi_{n+1}^2 + C_1 R + 1} \right)^{-1}, \tag{4.18}$$

where $C_1 = (3n-3)/(3n-5)$ and $C_2 = (n-2)/(3n-5)$. Then

$$\tilde{S}\left\{ n(-1+H^2) - n(n-2)H\phi_{n+1} - (3/2)\tilde{S} \right\} = \Psi(H - \sqrt{R+1})(\tilde{h} - H),$$

where $\Psi = n^2(n-1)(3n-5)(H + \sqrt{R+1})[H + (C_1 R + 1)\tilde{h}^{-1}]/2$. So (4.17) turns into

$$0 \geq n\Box H + |\nabla\sigma|^2 - n^2|\nabla H|^2 + \frac{1}{2}\tilde{S}_{n+1}(\tilde{S} + S_I) + \Psi(H - \sqrt{R+1})(\tilde{h} - H). \tag{4.19}$$

Suppose $(3n-3)R/(3n-5) + 1 \geq 0$. Then $\tilde{h} \geq 0$. If $H \leq \tilde{h}$, then $\Psi > 0$. Integrating both-sides of (4.19) on M^n and using (3.1), we obtain

$$\tilde{S}_{n+1}(\tilde{S} + S_I) = 0, \quad (H - \sqrt{R+1})(\tilde{h} - H) = 0, \quad |\nabla\sigma|^2 = n^2|\nabla H|^2 \tag{4.20}$$

on M^n. Then we can prove the following theorem:

Theorem 4.2. *Let* M^n *be a closed submanifold in* $\mathbb{H}^{n+p}$ *with parallel normalized mean curvature vector field. Suppose that the normalized scalar curvature* R *is constant and greater than or equal to* $-(3n-5)/(3n-3)$*. Let* $\tilde{h}$ *be defined by* (4.18)*. If* $H \leq \tilde{h}$*, then* $R > 0$ *and either*

(i) *$H = \sqrt{R+1}$ and M^n is a geodesic distance sphere $S^n(1/\sqrt{R})$ in $\mathbb{H}^{n+p}$; or*

(ii) *$H = \sqrt{3R+1}$ and M^2 is a Veronese surface in a totally geodesic sphere $S^4(1/\sqrt{3R})$ of a geodesic distance sphere $S^5(1/\sqrt{3R})$ in $\mathbb{H}^{2+p}$.*

Proof. It follows from the third equality in (4.20) and Lemma 3.1 that H is constant. From the second equality in (4.20), we have $H = \sqrt{R+1}$ or $\tilde{h} = H$.

If $H = \sqrt{R+1}$, then $\tilde{S} = 0$. Thus M^n is umbilically immersed into a subspace $\mathbb{H}^{n+1}$ of $\mathbb{H}^{n+p}$. Since M^n is closed, it follows that $R > 0$ and $M^n = S^n(1/\sqrt{R}) \subset \mathbb{H}^{n+p}$. Hence (i) of Theorem 4.2 follows.

If $H = \tilde{h} > \sqrt{R+1}$, then

$$\tilde{S} = (2/3)\left\{n(-1+H^2) - n(n-2)H\sqrt{\tilde{S}_{n+1}/n(n-1)}\right\}.$$

In this case, all of the inequalities concerned become equalities. From the first equality of (4.20), we have $\tilde{S}_{n+1} = 0$. Hence M^n is pseudo-umbilical. It follows from Lemma 2.2 that M^n lies minimally in a hypersphere $S^{n+p-1}(r_2)$ of $\mathbb{H}^{n+p}$, where $r_2^{-2} = 3(n-1)R/(3n-5)$. The equalities in (4.5) and (4.16) imply that $n = 2$. Under a suitable choice of tangent frame fields, we can assume that

$$L_4 = \sqrt{\frac{S_4}{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_5 = \sqrt{\frac{S_5}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad L_\beta = 0, \quad \text{for all} \quad \beta > 5,$$

where $S_4 = S_5$. As $S_I = \tilde{S} = (4/3)(-1+H^2) = 4R$ and M^n is not a sphere, we have $R > 0$. Therefore M^n lies minimally in a geodesic distance sphere $S^4(1/\sqrt{3R})$ of $\mathbb{H}^{2+p}$. From the same arguments as in Chern-do Carmo-Kobayashi [3], we have that M^2 is a Veronese surface in $S^4(1/\sqrt{3R})$ of $\mathbb{H}^{p+2}$. Therefore (ii) of Theorem 4.2 follows. □

References

1. B. Y. Chen, Total mean curvature and submanifolds of finite type, World Scientific, Singapore 1984.

2. S. Y. Cheng and S. T. Yau, Hypersurfaces with constant scalar curvature, *Math. Ann.* **225** (1977), 195-204.

3. S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, *Functional Analysis and Related Fields* (1970), 59-75.

4. Z. H. Hou, Hypersurfaces in sphere with constant mean curvature, *Proc. Amer. Math. Soc.* **125** (1997), 1193-1196.

5. H. B. Lawson, Jr., Local rigidity theorems for minimal hypersurfaces, *Ann. of Math.* **(2) 89** (1969), 187-197.

6. A. M. Li and J. M. Li, An intrinsic rigidity theorem for minimalsubmanifolds in a sphere, *Arch. Math.* **58** (1992), 582-594.

7. J.-M. Morvan and B.-Q. Wu, Hypersurfaces with Constant mean Curvature in Hyperbolic Space Form, *Geom. Dedicata* **59** (1996), 197-222.

8. P.-J. Ryan, Hypersurfaces with Parallel Ricci Tensor, *Osaka J. Math.* **8** (1971), 251-259.

9. W. Santos, Submanifolds with parallel mean curvature vector in spheres, *Tôhoku Math. J.* **46** (1994), 403-415.

10. K. Shiohama and H. W. Xu, Rigidity and sphere theorems for submanifolds, *Kyushu J. Math.* **48** (1994), 291-306.

11. R. Walter, Compact Hypersurfaces with a Constant Higher Mean Curvature Function, *Math. Ann.* **270** (1985), 125-145.

12. B. Q. Wu, The Math. Heritage of C. F. Gauss, World Scientific, Singapore.

Department of Mathematics, Tokyo Institute of Technology, Tokyo, Japan
e-mail address: hou@math.titech.ac.jp
Present address: Department of Applied Mathematics
Dalian University of Technology
Dalian 116024, LiaoNing, China