ON THE OPTIMALITY OF SOME FAC AND AFAC METHODS FOR ELLIPTIC FINITE ELEMENT PROBLEMS

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Abstract. We consider some solution methods for large sparse linear systems of equations which arise from second-order elliptic finite element problems defined on composite meshes. Historically these methods were called FAC and AFAC methods. Optimal bounds of the condition number for certain AFAC iterative operator are established by proving a strengthened Cauchy-Schwarz inequality using an interpolation theorem for Hilbert scales. This work completes earlier work by Dryja and Widlund. We also apply an extension theorem for finite element functions to get a weaker bound under some more general assumptions. The optimality of the FAC methods, with exact solvers or spectrally equivalent inexact solvers being used, is also proved by using similar techniques and some ideas from multigrid theory.

1. Introduction

In this paper, we consider some solution methods of second-order elliptic boundary value problems which come from finite element discretizations on composite meshes.

When we solve some second-order elliptic boundary value problems by finite element methods, we sometimes need to refine the elements in some subregions in order to get more accuracy of the solution in these subregions because the coefficients of the differential equation or the boundary of the whole domain are not smooth enough. This results in finite element models on composite meshes. We usually use conforming finite element space in order to be able to employ a number of well-developed technical tools; cf. Ciarlet

[8]. This refinement process can be continued for many times. However, if we solve the corresponding linear system, there should be radical changes of our original data structure and old program for each individual case. It is because we cannot know in advance the number of such refinement process and these selected subregions which need further mesh refinement. It motivates us to develop some methods to solve the original finite element problem by only using some finite element solvers on uniform mesh. FAC and AFAC are some methods of such kind, whose convergence rate is independent of the number of refinement levels and mesh size parameters of all elements unlike the standard multigrid methods. In short, we will transform the original linear system into another equivalent linear system, which will be solved by a linear iteration with the identity iterator chosen or by a conjugate gradient method w.r.t. an appropriate inner product. In each iteration step, we will only solve so-called standard subproblems with uniform mesh sizes, which correspond to the finite element problem on the entire region and those on its selected subregions which need further mesh refinement.

Thomas and McCormick began the systematic study of the Fast Adaptive Composite (FAC) method for the variational case; cf. [20] and [21]. In order to implement FAC on parallel computers, Hart and McCormick [13] introduced the Asynchronous FAC (AFAC) methods a few years later. Bramble, Ewing, Pasciak, and Schatz [3] considered the so-called BEPS preconditioner for elliptic problems with refinement, which is actually a symmetrized version of basic two-level FAC algorithm. In Mandel and McCormick [15], a theory for two-level FAC and AFAC of general elliptic finite element problems was given. In their another paper [16], they proved the optimality of a multi-level AFAC algorithm for a model problem coming from the finite element discretization. Then Ewing, Lazarov and Vassilevski [10] developed two algebraic multilevel BEPS preconditioners and estimated their convergence bounds with some restriction on the mesh sizes. Bjørstad, Moe and Skogen [2], and Moe [22] have also carried out experiments that illustrate the convergence behavior and have described parallel implementations on both shared memory computers and on local memory systems. In [28], Widlund formulated the multilevel FAC and AFAC methods for elliptic finite element problems under the framework of multiplicative and additive Schwarz methods and established the optimality of FAC method only using the extension theorem for finite element functions. However, his proof about the optimal upper bound of certain AFAC methods was not complete. Then Dryja and Widlund in [9] proved that there is an optimal lower bound for certain AFAC methods by using the same extension theorem. The use of FAC and AFAC methods for mixed finite element problems has been studied by Mathew in [17] [18]. Recently, McCormick and Rüde established a convergence theory of a two-level FAC for elliptic finite volume

problems in [23]. The results of this paper generalize and clarify the work by
Dryja and Widlund, and provide a complete proof of one of the results given
in [28].

In Section 2, we introduce the finite element problems on composite meshes,
certain projections and describe basic FAC and AFAC algorithms.

In Section 3, we give a complete proof of the AFAC optimality which
is not complete in Widlund [28] by proving a strengthened Cauchy-Schwarz
inequality using an interpolation theorem for Hilbert scales. We also apply an
extension theorem for finite element functions to get a weaker bound under
more general assumptions. As a minor, we also close a gap about the proof of
the extension theorem for finite element functions described in [28]. We remark
that all convergence estimates developed in this section are independent of all
mesh-sizes unlike the standard multigrid method.

In Section 4, we consider the FAC algorithm with inexact solvers and
prove its optimality by using theoretical results developed in Section 3 and
some ideas from multigrid methods. We note that we must have the same
restriction on mesh-sizes in the general case as that in multigrid methods.
However, we can apply some trick such that we don't need any restriction of
mesh-sizes in this special case of exact solvers being used. This part of our
work generalizes the result in Widlund [28].

2. Composite Finite Element Problems and Basic Iterative Refinement Methods

We consider the following second-order elliptic boundary value problem
with homogeneous Dirichlet boundary data on a bounded Lipschitz polyhedral
region Ω in $\mathbb{R}^n$.

$$(1) \qquad -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\, a_{ij}(x) \frac{\partial u}{\partial x_j} = f \quad \text{in } \Omega, \ \ u = 0 \quad \text{on } \partial\Omega.$$

We use continuous, Lagrange finite element of type p, p being fixed, to approx-
imate the solution of (1). The variational formulation of (1) and its discrete
counterpart have the form

$$(2) \qquad a(u, v) = f(v), \qquad \forall v \in V \equiv H_0^1(\Omega),$$

and

$$(3) \qquad a(u_h, v_h) = f(v_h), \qquad \forall v_h \in V^h,$$

respectively. The space V^h which is a finite element subspace of V will be defined later. Here

$$a(u, v) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \quad \text{and} \quad f(v) = \int_\Omega f v dx.$$

We assume that the coefficient matrix $\{a_{ij}(x)\}$ in (1) varies moderately in Ω and is symmetric with its eigenvalues bounded from below by a positive constant uniformly for almost all x in Ω. We also assume that each $a_{ij}(x)$ is in $W^{1,\infty}(\Omega)$. Therefore the norm $(a(u, u))^{1/2}$ is equivalent to the seminorm $|u|_{H^1(\Omega)}$ in $H^1(\Omega)$.

We remark that all of our results can be generalized immediately to the case of mixed type boundary conditions. The space V^h is defined on a composite triangulation, which is based on a series of refinements on some selected elements. The triangulation of Ω is described as follows.

We first start with a coarse triangulation of Ω, also denoted by Ω_1. We assume that the elements in Ω_1, which are called level-1 elements, have quasi-uniform mesh size h_1 and denote the corresponding finite element space by V^{h_1}. Then we choose a subregion Ω_2 of Ω_1 where we need to improve the accuracy of the solution of (2) due to the solution singularity. Because we hope that a triangulation of Ω_1 is also a triangulation of Ω_2 when it is restricted to Ω_2, we assume $\partial\Omega_2$ aligns with boundaries of level-1 elements. We then subdivide those level-1 elements in Ω_2 to get the level-2 elements in Ω_2. Let us assume that those level-2 elements in Ω_2 have quasi-uniform mesh size h_2 and denote the corresponding finite element space by V^{h_2}. In order to assure the composite finite element space $V^{h_1} + V^{h_2}$ is a subspace of $V = H_0^1(\Omega)$, the functions of V^{h_2} must be zero on $\partial\Omega_2$. Such a process can be repeated for arbitrarily many times. In general, if we are given subregion Ω_{l-1} and the finite element space $V^{h_{l-1}}$ corresponding to these level-$(l-1)$ elements in Ω_{l-1}, we choose $\Omega_l \subset \Omega_{l-1}$ such that $\partial\Omega_l$ aligns with boundaries of level-$(l-1)$ elements and refine those level-$(l-1)$ elements in Ω_l to get the level-l elements, which have quasi-uniform mesh size h_l. Therefore we have a corresponding finite element space V^{h_l} which satisfies $V^{h_{l-1}} \cap H_0^1(\Omega_l) \subset V^{h_l} \subset H_0^1(\Omega_l)$, $l = 2, \cdots, k$. The composite finite element space V^h in (3) is then defined by

$$V^h = V^{h_1} + V^{h_2} + \cdots + V^{h_k}.$$

It is obvious that irregular nodes on the boundary of the refined region are not unknowns. We choose the basis functions in the following way. We just cut off the supports of the coarse grid basis functions by redefining them using fine grid basis functions. Then they would extend just one element into the fine grid space. We remark that our theoretical bounds developed in this paper

depend not only on the shape regularity of elements, but also on the shapes of the subregions Ω_l.

In order to get the stiffness matrix corresponding to (3), we can compute the bilinear form $a(\cdot, \cdot)$ for two chosen composite finite element functions in V^h by using a process of subassembly. We can decompose the whole domain Ω into some disjoint subregions such that those elements in each subregion have almost the same mesh size. Then we can apply the old codes for calculating the bilinear form of uniform mesh size finite element functions to get the bilinear form corresponding to each disjoint subregion. Finally we only need to add up those contributions of bilinear form from each disjoint subregion.

We now can describe FAC as follows.

Algorithm 1 (FAC). *Let $u_h^n \in V^h$ be the nth approximation to the solution of (3). Then an iteration step of computing u_h^{n+1} consists of k fractional steps: compute $w_i \in V^{h_i}$ from*

$$(4) \qquad a(u_h^{n+(i-1)/k} + w_i, v_i) = f(v_i), \quad \forall v_i \in V^{h_i},$$

and set

$$u_h^{n+i/k} = u_h^{n+(i-1)/k} + w_i \qquad for \ i = 1, 2, \cdots, k.$$

We remark that the above w_i also can be regarded as the solution of an inhomogeneous Dirichlet boundary value problem in Ω_i with the boundary data $u_h^{n+(i-1)/k}$ on $\partial\Omega_i$.

Let us define P_j^i, $i \leq j$, as the projections onto the spaces $V^{h_i} \cap H_0^1(\Omega_j)$. We note that if $j > i$, then we solve a problem on Ω_j with a coarser mesh than the mesh h_j of the space V^{h_j}. The function $P_j^i v_h$, $i \leq j$, is the unique element in $V^{h_i} \cap H_0^1(\Omega_j)$, which satisfies

$$(5) \qquad a(P_j^i v_h, \phi_h) = a(v_h, \phi_h), \quad \forall \ \phi_h \in V^{h_i} \cap H_0^1(\Omega_j).$$

If $u_h \in V^h$ is the solution of the discrete problem (3), then we can show that the error $e_h^n = u_h^n - u_h$ propagates as

$$e_h^{n+1} = (I - P_k^k)(I - P_{k-1}^{k-1})\cdots(I - P_1^1)e_h^n.$$

Therefore we can regard FAC as a multiplicative Schwarz method based on the subspaces V^{h_i} for $i = 1, 2, \cdots, k$.

The FAC multiplicative operator is a nonsymmetric operator that cannot be accelerated by standard conjugate gradient methods. However, GMRES could be used; cf. [25]. In order to get a symmetric operator, we have to use additional steps such that the symmetrized FAC has the error propagating as

$$e_h^{n+1} = (I - P_1^1)\cdots(I - P_{k-1}^{k-1})(I - P_k^k)(I - P_k^k)(I - P_{k-1}^{k-1})\cdots(I - P_1^1)e_h^n.$$

In order to adapt the FAC method to multiprocessor systems, several different asynchronous FAC algorithms have been introduced; cf. [13] and [28]. These variants of FAC belong to the additive Schwarz methods, which are more easily decomposed into independent processes, at the cost of somewhat slower convergence.

Algorithm 2 (AFAC1). *Apply an iterative method, e.g., the conjugate gradient method, to the symmetric and positive definite system*

$$P_a^{(1)} u_h \equiv (P_1^1 + P_2^2 + \cdots + P_k^k) u_h = g_h$$

for an appropriate g_h such that the solution u_h is the same as that of (3).

It is well-known that the number of steps required to decrease an appropriate norm of the error of a conjugate gradient iteration by a fixed factor is proportional to $\sqrt{\kappa}$, where κ is the condition number of the corresponding operator; cf. Golub and Van Loan [11]. We therefore need establish that the corresponding FAC or AFAC operator is not only invertible but that satisfactory upper and lower bounds on its eigenvalues can be obtained.

The condition number of the AFAC1 operator grows linearly with k as shown by Widlund in [28]. The eigenvalues of $P_a^{(1)}$ are always bounded from above by k. This bound is attained if $V^{h_1} \cap H_0^1(\Omega_k)$ is not empty, i.e., the coarsest mesh size h_1 is fine enough.

Improvements of the basic AFAC method were described by Mandel and McCormick in [15], [16] and by Widlund [28]. However, the original idea was known even earlier. In terms of orthogonal projections, the operator can be represented as

$$P_a^{(2)} = P_1^1 + (P_2^2 - P_2^1) + \cdots + (P_k^k - P_k^{k-1}).$$

All of individual terms, except the first, represent the difference between two solutions on the same subregion, using two different mesh sizes.

Algorithm 3 (AFAC2). *Apply an iterative method, e.g., the conjugate gradient method, to the symmetric and positive definite system*

$$P_a^{(2)} u_h \equiv (P_1^1 + (P_2^2 - P_2^1) + \cdots + (P_k^k - P_k^{k-1})) u_h = g_h$$

for an appropriate g_h such that the solution u_h is the same as that of (3).

We end this section by discussing several variants of these algorithms. We can use other energy-symmetric operators T_i^i to replace the projections P_i^i. Practically we should choose the T_i^i such that the $T_i^i u_h$ are much easier to

compute than the $P_i^i u_h$ and T_i^i is spectrally equivalent to P_i^i. The corresponding operator for the symmetrized FAC with k subspaces involves $2k$ fractional steps and has the form

(6) $I - E_k^* E_k$, where $E_k = (I - T_k^k)(I - T_{k-1}^{k-1}) \cdots (I - T_1^1)$.

We note that the error propagation operators for the basic FAC algorithm is E_k. However, if we apply standard conjugate gradient method to accelerate the symmetrized FAC algorithm, the convergence rate can be bounded in terms of the condition number of $I - E_k^* E_k$.

3. Optimal Convergence Rate Estimates for AFAC

In this section, we will prove that there is an optimal bound for AFAC by using an extension theorem for finite element functions and an interpolation theorem of Hilbert scales. We have already known an estimate of the condition number for AFAC1 from last section. Therefore it is sufficient to analyze the convergence rate for AFAC2.

Our first tool is an extension theorem for finite element functions. We need to make the following assumptions; cf. Fig. 1.

Assumption 1. *For each j, there exists a bounded Lipschitz polyhedral region $\tilde{\Omega}_j$ such that $\Omega_j \subset \tilde{\Omega}_j$, $(\tilde{\Omega}_j \setminus \Omega_j) \cap \Omega = \emptyset$, $\partial \tilde{\Omega}_j \cap \partial \Omega_{j+1} = \emptyset$ and the Lipschitz constants of $\tilde{\Omega}_j \setminus \Omega_{j+1}$ are uniformly bounded.*

We remark that in the above assumption $\tilde{\Omega}_j$ is not necessarily a subset of Ω; cf. Fig. 1. Assumption 1 is quite general. In the case of $\Omega_j \subset\subset \Omega$, the only choice of $\tilde{\Omega}_j$ is Ω_j and therefore we can only select Ω_{j+1} such that $\Omega_{j+1} \subset\subset \Omega_j$. The significance of Assumption 1 is that we can then get an extension theorem for finite element functions with the extra constraint due to the homogeneous Dirichlet boundary condition w.r.t. the norm induced by $a(\cdot, \cdot)$, which is stated but not proved in Widlund [28]; cf. Lemma 1.

As in [28], we need to define the operators H_{j+1}^j, $1 \leq j \leq k-1$, by

$$
\begin{aligned}
H_{j+1}^j u_h(x) &\in & V^{h_1} + V^{h_2} + \cdots + V^{h_j}, \\
H_{j+1}^j u_h(x) &=& u_h(x), \quad \forall x \in \Omega \setminus \Omega_{j+1}, \\
a(H_{j+1}^j u_h, v_h) &=& 0, \quad \forall v_h \in V^{h_j} \cap H_0^1(\Omega_{j+1}).
\end{aligned}
$$

We can call $H_{j+1}^j u_h$ the h_j-harmonic extension of u_h from $\Omega \setminus \Omega_{j+1}$ to Ω_{j+1} because it is the solution in V^{h_j} of a discrete Dirichlet problem with zero right hand side and with boundary data on $\partial \Omega_{j+1}$ given by u_h.
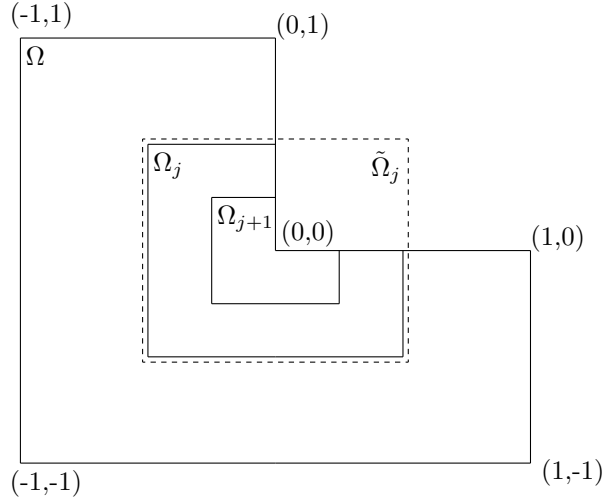
Figure 1: One choice of $\tilde{\Omega}_j$ in Assumption 1 for L-shaped subdomains

The following lemma is stated in Widlund [28] and he used it to show that the basic FAC algorithm has a uniformly bounded convergence rate, but he didn't give the proof. However, it is not a direct consequence of the standard extension theorem for finite element functions because we now deal with the norm induced by $a(\cdot, \cdot)$ inner product but not the standard Sobolev norm $\|\cdot\|_{H^1(\Omega)}$ and because we need to impose the extra constraint values due to the Dirichlet boundary condition on the extended finite element functions. This is the reason why we need Assumption 1. For completeness of this work, we also include the proof.

Lemma 1. *There exists a constant $C(\tilde{\Omega}_j \backslash \Omega_{j+1})$, which only depends on the Lipschitz constant of $\tilde{\Omega}_j \backslash \Omega_{j+1}$, where $\tilde{\Omega}_j$ is given in Assumption 1, such that*

$$a_{\Omega_{j+1}}(H_{j+1}^j u_h, H_{j+1}^j u_h) \le C(\tilde{\Omega}_j \backslash \Omega_{j+1}) a_{\Omega_j \backslash \Omega_{j+1}}(u_h, u_h), \quad \forall u_h \in V^h.$$

Here $a_{\Omega_{j+1}}(\cdot, \cdot)$ is the bilinear form obtained from replacing the integration domain Ω by its subdomain Ω_{j+1} in the definition of $a(\cdot, \cdot)$.

Before proving this lemma, we need to state and use the following two technical lemmas.

Lemma 2. *Let Ω be an open, bounded, and Lipschitz polyhedral region in $\mathbb{R}^n$, $n \geq 2$. A triangulation on Ω with a quasi-uniform mesh size h is given and $\mathbb{R}^n \setminus \Omega$ is also triangulated in an equally benign way as Ω. There exists a constant $C(\Omega)$, which depends only on the Lipschitz constant of $\partial\Omega$ and the shape regularity of the triangulation, such that for all $u_h \in V^h(\Omega)$ we can extend u_h to $\tilde{u}_h \in V^h(\mathbb{R}^n)$ and $\tilde{u}_h$ satisfies*

$$\|\tilde{u}_h\|_{H^1(\mathbb{R}^n)} \leq C(\Omega)\|u_h\|_{H^1(\Omega)}.$$

The result is also true if we only extend u_h to some domain $\tilde{\Omega}$ which contains Ω and we do not impose any extra constraint on the extended function $\tilde{u}_h$.

Lemma 3. *Let*

$$\{u\}_\Omega = \frac{1}{|\Omega|} \int_\Omega u.$$

Then there exists a constant $C(\Omega)$, which only depends upon the Lipschitz constant of $\partial\Omega$, such that

$$\|u - \{u\}_\Omega\|_{L^2(\Omega)} \leq C(\Omega)H_\Omega|u|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Here H_Ω is the diameter of Ω.

The first lemma is the standard extension theorem for finite element functions given in Widlund [27]. The second lemma is called Poincaré inequality; cf. [24].

Proof of Lemma 1. Without loss of generality, we may assume that the differential operator is the Laplacian. To prove this lemma, we need to use Lemmas 2 and 3. Using the condition $(\tilde{\Omega}_j \setminus \Omega_j) \cap \Omega = \emptyset$, we can make a constant extension of any function in V^h, which has a constant value on $\partial\Omega$, to the larger region $\Omega \cup \tilde{\Omega}_j$. This extended function is still a composite finite element function. We now apply Lemma 2 to the region $\tilde{\Omega}_j \setminus \Omega_{j+1}$. Given any function $u_h \in V^h$, we define $c = \{u_h\}_{\tilde{\Omega}_j \setminus \Omega_{j+1}}$ as in Lemma 3. Obviously $u_h - c$ is in $V^{h_j}(\tilde{\Omega}_j \setminus \Omega_{j+1})$ when it is restricted to this smaller subdomain. Let $\tilde{u}_{h_j}$ be the finite element function extension of $u_h - c$ from $\tilde{\Omega}_j \setminus \Omega_{j+1}$ to $\tilde{\Omega}_j$, as in Lemma 2, such that $\tilde{u}_{h_j}$ is in $V^{h_j}(\tilde{\Omega}_j)$. It is possible. We actually don't impose any boundary value constraint on the extended finite element function $\tilde{u}_{h_j}$ because $\partial\tilde{\Omega}_j \cap \partial\Omega_{j+1} = \emptyset$. Now $H_{j+1}^j(u_h - c)$ and $\tilde{u}_{h_j}$ have the same boundary values on $\partial\Omega_{j+1}$. However $H_{j+1}^j(u_h - c)$ is the minimal energy extension of $u_h - c$

from $\tilde{\Omega}_j \backslash \Omega_{j+1}$ to $\tilde{\Omega}_j$. Therefore by Lemmas 2 and 3,

$$a_{\Omega_{j+1}}(H^j_{j+1}u_h, H^j_{j+1}u_h) = a_{\Omega_{j+1}}(H^j_{j+1}(u_h - c), H^j_{j+1}(u_h - c)) \le a_{\Omega_{j+1}}(\tilde{u}_{h_j}, \tilde{u}_{h_j})$$

$$\le \|\tilde{u}_{h_j}\|^2_{\Omega_{j+1}} \le C_1^2(\tilde{\Omega}_j \backslash \Omega_{j+1}) \|u_h - c\|^2_{H^1(\tilde{\Omega}_j \backslash \Omega_{j+1})}$$

$$\le C_1^2(\tilde{\Omega}_j \backslash \Omega_{j+1})(1 + C_2^2(\tilde{\Omega}_j \backslash \Omega_{j+1}) H^2_{\tilde{\Omega}_j \backslash \Omega_{j+1}}) a_{\tilde{\Omega}_j \backslash \Omega_{j+1}}(u_h, u_h)$$

$$= C_1^2(\tilde{\Omega}_j \backslash \Omega_{j+1})(1 + C_2^2(\tilde{\Omega}_j \backslash \Omega_{j+1}) H^2_{\tilde{\Omega}_j \backslash \Omega_{j+1}}) a_{\Omega_j \backslash \Omega_{j+1}}(u_h, u_h).$$

Here C_1 and C_2 are the constants which appear in Lemmas 2 and 3. Note that we have regarded u_h as defined on $\Omega \cup \tilde{\Omega}_j$ by zero extension.

Finally we can use a simple dilation argument to remove the dependence of the constant upon $H_{\tilde{\Omega}_j \backslash \Omega_{j+1}}$. ∎

Note that no additional assumptions on the mesh sizes h_i are needed in the above lemma.

We now state a result about a uniform lower bound for AFAC2 operator which was described in [9].

Theorem 1. *Under Assumption* 1, *the eigenvalues of* $P_a^{(2)}$ *are bounded from below by a constant which is independent of k and the mesh sizes h_i,* $1 \le i \le k$.

Actually, the above theorem holds under a more general assumption than Assumption 1; we can allow $\Omega_j = \Omega_{j+1}$ in Assumption 1. This is so, because in this case we can form a new AFAC2 operator $P_{a,n}^{(2)}$, which satisfies Assumption 1 and can be shown to be equal to the original operator $P_a^{(2)}$, by using the basic property of projections to cancel some projection operators.

We will next prove the following theorem by using Lemma 1. Therefore the only tool is the standard extension theorem for finite element functions.

Theorem 2. *Under Assumption* 1, *the operator* $P_a^{(2)}$ *has a condition number which is bounded by* $const \cdot \sqrt{k}$, *but is independent of the mesh sizes* h_i, $1 \le i \le k$.

Proof. Let $u_i = (P_i^i - P_i^{i-1})u_h$ for $i = 1, 2, \cdots, k$. Then $P^{(2)}u_h = u_1 + u_2 + \cdots + u_k$. Also note that if $j > i$, then $a(u_j, u_i) = a(u_j, u_i + v)$ for all $v \in V^{h_{j-1}} \cap H_0^1(\Omega_j)$. Therefore we can choose v such that

$$a(u_j, u_i + v) = a(u_j, H^{j-1}_j u_i) = a(u_j, u_i).$$

Hence if we use this equation, Lemma 1 and Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
a(P^{(2)}u_h, P^{(2)}u_h) &= \sum_{i=1}^{k} a(u_i, u_i) + 2\sum_{j>i} a(u_i, u_j) = \sum_{i=1}^{k} a(u_i, u_i) + 2\sum_{j>i} a(u_j, H_j^{j-1}u_i) \\
&\leq \sum_{i=1}^{k} a(u_i, u_i) + 2\sum_{i=1}^{k}\sum_{j=i+1}^{k} a(u_j, u_j)^{1/2} \cdot a_{\Omega_j}(H_j^{j-1}u_i, H_j^{j-1}u_i)^{1/2} \\
&\leq \sum_{i=1}^{k} a(u_i, u_i) + C\sum_{i=1}^{k}\sum_{j=i+1}^{k} a(u_j, u_j)^{1/2} \cdot a_{\Omega_{j-1}\setminus\Omega_j}(u_i, u_i)^{1/2} \\
&\leq \sum_{i=1}^{k} a(u_i, u_i) + C\sum_{i=1}^{k}\left[\sum_{j=i+1}^{k} a(u_j, u_j)\right]^{1/2} \cdot \left[\sum_{j=i+1}^{k} a_{\Omega_{j-1}\setminus\Omega_j}(u_i, u_i)\right]^{1/2} \\
&= \sum_{i=1}^{k} a(u_i, u_i) + C\sum_{i=1}^{k}\left[\sum_{j=i+1}^{k} a(u_j, u_j)\right]^{1/2} \cdot a(u_i, u_i)^{1/2} \\
&\leq \sum_{i=1}^{k} a(u_i, u_i) + C\left[\sum_{i=1}^{k}\sum_{j=i+1}^{k} a(u_j, u_j)\right]^{1/2} \cdot \left[\sum_{i=1}^{k} a(u_i, u_i)\right]^{1/2} \\
&\leq \sum_{i=1}^{k} a(u_i, u_i)(1 + C\sqrt{k}) \leq C'\sqrt{k}a(P^{(2)}u_h, u_h).
\end{aligned}
$$

From this result and Theorem 1, Theorem 2 easily follows.   ∎

We remark that the above theorem holds under more general assumption by modifying the corresponding result of Lemma 1. For example, we can make the following assumption.

Assumption 2. *For each j, there exists a bounded Lipschitz polyhedral region $\tilde{\Omega}_{N(j)}$, with $1 \leq N(j) \leq j$ and $j - N(j)$ uniformly bounded with respect to j, such that $\Omega_{N(j)} \subset \tilde{\Omega}_{N(j)}$, $(\tilde{\Omega}_{N(j)} \setminus \Omega_{N(j)}) \cap \Omega = \emptyset$, $\partial\tilde{\Omega}_{N(j)} \cap \partial\Omega_{j+1} = \emptyset$ and the Lipschitz constants of $\tilde{\Omega}_{N(j)} \setminus \Omega_{j+1}$ are uniformly bounded.*

Under Assumption 2, we can prove the counterpart of Lemma 1 by applying Lemma 2 to the region $\tilde{\Omega}_{N(j)} \setminus \Omega_{j+1}$. Next we will prove that $P_a^{(2)}$ has a uniformly bounded condition number under the following additional assumption. Later we will apply some trick to remove this assumption.

Assumption 3. *The mesh sizes h_i are bounded from above and below by $const \cdot q^i$ uniformly for all i. Here q is some constant less than 1.*

Before giving the proof, we need to describe some theoretical results which we will apply later on. One is an interpolation theorem of Hilbert scales.

Another one is related to the elliptic regularity assumption. We first review an interpolation theorem of Hilbert scales and modify this theorem for finite element spaces.

Let X and Y be two separable Hilbert spaces with $X \subset Y$ and let X be dense in Y with a continuous injection. For example, we may take $X = H^1(\Omega)$ and $Y = L^2(\Omega)$. A functional $K(t, u)$ is defined by

$$K(t, u) = \inf_{u = u_0 + u_1} (\|u_1\|_X^2 + t^2 \|u_0\|_Y^2)^{1/2}.$$

Here we write general functions u in Y as $u = u_0 + u_1$ where $u_1 \in X$ and $u_0 \in Y$. Then we define the Hilbert space $[X, Y]_s$ to be the space of all functions u in Y for which the norm

$$(\|u\|_Y^2 + \int_0^\infty t^{-(3-2s)}(K(t, u))^2 dt)^{1/2}, \quad 0 \le s \le 1,$$

is finite. When $s = 0$, the space is Y. When $s = 1$, the space is X. The above procedure is called the interpolation of Hilbert scales. The following interpolation theorem can be found in Lions and Magenes [14], or Triebel [26].

Theorem 3. *Let A be a continuous linear functional over $\mathbb{R}$ on X and Y. Then there exists a constant $C(s)$, depending only on s, $0 \le s \le 1$, such that*

$$\|A\|_{[X,Y]_s \to \mathbb{R}} \le C(s) \cdot \|A\|_{X \to \mathbb{R}}^s \cdot \|A\|_{Y \to \mathbb{R}}^{1-s}.$$

Let $H^s(\Omega)$ denote the fractional Sobolev space of order s for $0 \le s \le 1$. It can be shown that $[H^1(\Omega), L^2(\Omega)]_s = H^s(\Omega)$ and that the norms in these spaces are equivalent with the equivalence constants only depending on Ω and s. The proof can be found in Lions and Magenes [14]. We will use the K-method to establish an interpolation theorem for finite-dimensional subspaces of finite element functions.

Let Ω be a bounded Lipschitz polyhedral domain in $\mathbb{R}^n$. We introduce a triangulation on Ω such that the triangulation has a quasi-uniform mesh size h. For the triangulation, the finite element space V^h is defined as the Lagrange finite element space of type p, p being fixed. Then we have the following lemma which is a variant of the above theorem.

Lemma 4. *Let A be a continuous linear functional over $\mathbb{R}$ on V^h both in $H^1(\Omega)$ and $L^2(\Omega)$. Then there exists a constant $C(\Omega, s)$, $0 \le s \le 1$, which is independent of h, such that*

$$\|A\|_{V^h \subset H^s(\Omega) \to \mathbb{R}} \le C(\Omega, s) \cdot \|A\|_{V^h \subset H^1(\Omega) \to \mathbb{R}}^s \cdot \|A\|_{V^h \subset L^2(\Omega) \to \mathbb{R}}^{1-s}.$$

Proof. Let us do the interpolation procedure between the Hilbert spaces $V^h \subset H^1(\Omega)$ and $V^h \subset L^2(\Omega)$ and apply the above interpolation theorem. To prove this lemma, it is sufficient to prove that the norm induced by the interpolation procedure between the space $V^h \subset H^1(\Omega)$ and the space $V^h \subset L^2(\Omega)$ is equivalent to the norm induced by the interpolation procedure between the space $H^1(\Omega)$ and the space $L^2(\Omega)$ for each s. Let us define

$$K_h(t, u_h) = \inf_{u_h = u_{0h} + u_{1h}} (\|u_{1h}\|^2_{H^1(\Omega)} + t^2 \|u_{0h}\|^2_{L^2(\Omega)})^{1/2},$$

where u_{1h}, u_{0h}, and u_h are in V^h. In fact, it suffices to prove that there exist constants C_1 and C_2, which are independent of h, such that

$$C_1 K(t, u_h) \le K_h(t, u_h) \le C_2 K(t, u_h), \quad \forall u_h \in V^h.$$

Here $K(t, u)$ is the K-functional corresponding to $X = H^1(\Omega)$ and $Y = L^2(\Omega)$. It is obvious from the definitions that

$$K(t, u_h) \le K_h(t, u_h), \quad \forall u_h \in V^h.$$

Let Q_h denote the standard L^2-projection or quasi-interpolant; cf. Bramble and Xu [5] or Cheng [7]. Then there exists a constant C, which is independent of h, such that

$$\|Q_h u\|_{H^1(\Omega)} \le C \|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Now given any u_h in V^h, we write u_h as $u_0 + u_1$ where $u_0 \in L^2(\Omega)$ and $u_1 \in H^1(\Omega)$. Then

$$u_h = Q_h u_h = Q_h u_0 + Q_h u_1.$$

Therefore

$$
\begin{aligned}
K_h(t, u_h) &\le \inf_{u_h = u_0 + u_1} (\|Q_h u_1\|^2_{H^1(\Omega)} + t^2 \|Q_h u_0\|^2_{L^2(\Omega)})^{1/2} \\
&\le C \cdot \inf_{u_h = u_0 + u_1} (\|u_1\|^2_{H^1(\Omega)} + t^2 \|u_0\|^2_{L^2(\Omega)})^{1/2} = C \cdot K(t, u_h).
\end{aligned}
$$

Then the lemma easily follows.   ∎

Next we use the elliptic regularity assumption to set up some technical lemmas. We consider the Dirichlet boundary value problem

$$-\sum_i \sum_j \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial U}{\partial x_j} = F \quad in \ \Omega, \ \ U = 0 \ \ on \ \partial\Omega,$$

in a weak sense, i.e., $U \in H^1_0(\Omega)$ is the solution of

$$a(U, \chi) = (F, \chi)_{L^2(\Omega)}, \quad \forall \chi \in H^1_0(\Omega).$$

We make the following elliptic regularity assumption: There exists a constant $s \in (0,1]$ such that

(7) $$\|U\|_{H^{1+s}(\Omega)} \leq C\|F\|_{H^{s-1}(\Omega)}$$

for the solution U, where C is a constant depending only on the domain Ω. In Grisvard [12], it is established that the above assumption holds for some $s \in (1/2, 1]$ if Ω is a Lipschitz polyhedral bounded domain in two- or three-dimensional space and the coefficients $\{a_{ij}(x)\}$ are in $W^{1,\infty}(\Omega)$.

For such a Lipschitz polyhedral domain $\Omega_0 \subset \mathbb{R}^n$, we introduce a sequence of triangulations $\mathcal{T}_l$ of Ω_0. The triangulation $\mathcal{T}_l$ has a quasi-uniform mesh size h_l. For these triangulations, the finite element spaces $\tilde{V}^{h_l}$ are defined as the Lagrange finite element space of type p. We assume that

$$\tilde{V}^{h_1} \subset \tilde{V}^{h_2} \subset \tilde{V}^{h_3} \subset \cdots \subset \tilde{V}^{h_l} \subset \cdots,$$

and define the Galerkin projection $P_l : H_0^1(\Omega_0) \to \tilde{V}^{h_l} \cap H_0^1(\Omega_0)$ by

$$a(P_l u, v) = a(u, v), \quad \forall v \in \tilde{V}^{h_l} \cap H_0^1(\Omega_0).$$

The following well-known lemma can be proved by using the regularity assumption (7) and the standard finite element approximation theory; cf. Xu [29].

Lemma 5. *There exists an $s \in (1/2, 1]$ and a constant C such that*

$$\|(I - P_l)u\|_{H^{1-s}(\Omega_0)} \leq C h_l^s \|(I - P_l)u\|_{H^1(\Omega_0)}, \quad \forall u \in H_0^1(\Omega_0).$$

The following lemma is proved for the special case of linear element and the differential operator being Laplacian in Yserentant [30] or Xu [29]. However, it is possible to prove the general case of Lagrange elements with higher degree and general self-adjoint differential operator. To simplify the presentation, we omit the proof.

Lemma 6. *Assume that the coefficient matrix $\{a_{ij}(x)\}$ is in $W^{1,\infty}(\Omega_0)$ and that Assumption 3 holds. Then*

$$a(u, v) \leq C' q^{(j-i)/2} h_j^{-1} |u|_{H^1(\Omega_0)} \|v\|_{L^2(\Omega_0)}, \quad \forall u \in \tilde{V}^{h_i}, v \in \tilde{V}^{h_j}$$

if $i \leq j$. Here C' depends only upon the shape regularity and the bound of $\{a_{ij}(x)\}$ in $W^{1,\infty}(\Omega_0)$.

We can also replace the L^2-norm in the above lemma by the H^s-norm by using Lemma 4.

Lemma 7. *Assume that the coefficient matrix $\{a_{ij}(x)\}$ is in $W^{1,\infty}(\Omega_0)$. Then*

$$a(u,v) \leq C'(\Omega_0, s)(q^{(j-i)/2}h_j^{-1})^s |u|_{H^1(\Omega_0)}\|v\|_{H^{1-s}(\Omega_0)}, \quad \forall u \in \tilde{V}^{h_i}, \quad \forall v \in \tilde{V}^{h_j}.$$

Here $C'(\Omega_0, s)$ is a generic constant which only depends upon the domain Ω_0, s, the shape regularity and the bound of $\{a_{ij}(x)\}$ in $W^{1,\infty}(\Omega_0)$.

Proof. For each fixed $u \in \tilde{V}^{h_i}$, there is a bounded linear functional $A_u : \tilde{V}^{h_j} \to \mathbb{R}$ such that

$$A_u(v) = a(u,v).$$

We observe that

$$|A_u(v)| \leq C|u|_{H^1(\Omega_0)}\|v\|_{H^1(\Omega_0)}.$$

Therefore

$$\|A_u\|_{\tilde{V}^{h_j}\subset H^1(\Omega_0)\to R} \leq C|u|_{H^1(\Omega_0)}.$$

By Lemma 6, we also have

$$\|A_u\|_{\tilde{V}^{h_j}\subset L^2(\Omega_0)\to R} \leq C'q^{(j-i)/2}h_j^{-1}|u|_{H^1(\Omega_0)}.$$

We now can apply Lemma 4 to get a good bound for $\|A_u\|_{\tilde{V}^{h_j}\subset H^{1-s}(\Omega_0)\to\mathbb{R}}$. We have

$$\begin{aligned}
\|A_u\|_{\tilde{V}^{h_j}\subset H^{1-s}(\Omega_0)\to\mathbb{R}} &\leq & C(\Omega_0, s)\|A_u\|_{\tilde{V}^{h_j}\subset H^1(\Omega_0)\to\mathbb{R}}^{1-s} \cdot \|A_u\|_{\tilde{V}^{h_j}\subset L_2(\Omega_0)\to\mathbb{R}}^{s} \\
&\leq & C'(\Omega_0, s)(q^{(j-i)/2}h_j^{-1})^s \cdot |u|_{H^1(\Omega_0)}.
\end{aligned}$$

From this result, the lemma easily follows. ∎

We can now return to our original problem on composite meshes. Using the following lemma, we can prove that the additive Schwarz operator $P_a^{(2)}$ has an optimal condition number under Assumptions 1 and 3.

Lemma 8. *Assume that the coefficient matrix $\{a_{ij}(x)\}$ is in $W^{1,\infty}(\Omega)$ and that Assumption 3 holds. Then*

$$a(u,v) \leq C(\Omega_j, s)q^{(j-i-2)s/2}a(u,u)^{1/2} \cdot a(v,v)^{1/2},$$
$$\forall u \in \text{Range}\,(P_i^i), \quad \forall v \in \text{Range}\,(P_j^j - P_j^{j-1}).$$

Here $C(\Omega_j, s)$ is a constant which only depends upon the subdomain Ω_j, s, the shape regularity and the bound of $\{a_{ij}(x)\}$ in $W^{1,\infty}(\Omega)$.

Proof. If we apply the results of Lemmas 5 and 7 for each subdomain Ω_j which will need the elliptic regularity assumption (7) for each Ω_j and observe that $P_j^j - P_j^{j-1} = (I - P_j^{j-1})P_j^j$, we obtain

$$
\begin{aligned}
a(u,v) &= a_{\Omega_j}(u,v) \leq C'(\Omega_j,s)(q^{(j-i)/2}h_j^{-1})^s \cdot |u|_{H^1(\Omega_j)} \cdot \|v\|_{H^{1-s}(\Omega_j)} \\
&\leq C'(\Omega_j,s)(q^{(j-i)/2}h_j^{-1})^s \cdot |u|_{H^1(\Omega_j)} \cdot C \cdot h_{j-1}^s \cdot \|v\|_{H^1(\Omega_j)} \\
&\leq C(\Omega_j,s)(q^{(j-i)/2}h_j^{-1})^s |u|_{H^1(\Omega_j)} \cdot C \cdot h_{j-1}^s |v|_{H^1(\Omega_j)} \\
&\leq C(\Omega_j,s) \cdot q^{(j-i-2)s/2} \cdot |u|_{H^1(\Omega_j)} \cdot |v|_{H^1(\Omega_j)} \\
&\leq C(\Omega_j,s) \cdot q^{(j-i-2)s/2} \cdot a(u,u)^{1/2} \cdot a(v,v)^{1/2}. \qquad \blacksquare
\end{aligned}
$$

We can now prove the main theorem in this section.

Theorem 4. *Under Assumptions* 1 *and* 3*, the operator* $P_a^{(2)}$ *has a condition number which is independent of k and the number of degrees of freedom.*

To prove that $P_a^{(2)}$ are uniformly bounded from above by a constant, we can use Lemma 8 and apply the trick in Widlund [28]. By combining this result with Theorem 1, we now complete the proof of Theorem 4.

However, we can replace Assumption 3 by a much weaker assumption such that Theorem 4 still holds.

Assumption 4. *The mesh sizes h_i are uniformly bounded from above and below by $const \cdot q^{N(i)}$. Here $N(i)$ is a strictly increasing sequence of positive integers and q is some constant less than* 1.

Theorem 5. *Under Assumptions* 1 *and* 4*, the operator* $P_a^{(2)}$ *has a condition number which is independent of k and the number of degrees of freedom.*

Proof. It is sufficient to prove that $P_a^{(2)}$ has a uniform upper bound. It is obvious from the special form of $P_a^{(2)}$ that we can introduce intermediate subdomains, mesh sizes and the corresponding projections such that the resulting new AFAC2 operator $P_{a,n}^{(2)}$ is the same as the original operator $P_a^{(2)}$ and satisfies Assumption 3. It follows that $P_{a,n}^{(2)}$ has a uniformly bounded upper bound. Therefore $P_a^{(2)}$ has a uniformly bounded upper bound. $\qquad \blacksquare$

4. A Convergence Rate Estimate for FAC with Inexact Solvers

We consider T_i^i which are spectrally equivalent to P_i^i for $i = 1, 2, \ldots, k$, in the sense that there exist constants ω_1 and ω_2 with $0 < \omega_1 < \omega_2 < 2$ such that $\omega_1 P_i^i \le T_i^i \le \omega_2 P_i^i$, for all $i = 1, 2, \ldots, k$. In this section, we will establish the optimality for FAC based on such T_i^i. We make the same assumptions about the domains $\Omega_1, \cdots, \Omega_k$ as in the last section. We remark that our idea comes from the multigrid theory in Xu [29] and Zhang [31]. However, their proof cannot be applied to our case. It is the reason why we need to establish our proof which is related to the operator $P_a^{(2)}$, but not $T_1^1 + \cdots + T_k^k$ which is the additive Schwarz operator corresponding to FAC algorithm with inexact solvers. The main difference of ingredients is that it is necessary to consider other operators $P_i^i - P_i^{i-1}$ which is not directly related to the original operators T_i^i in our specific problem. We may also refer to [4] for the analysis of multiplicative Schwarz algorithms in general cases.

Theorem 6. *Under Assumptions 1 and 3, the energy norm of the basic FAC operator based on the $T_i^i, 1 \le i \le k$, is bounded by a constant less than one with the constant independent of k and the number of degrees of freedom, but depending upon ω_1, ω_2 and the shape regularity. Therefore the corresponding symmetrized FAC algorithm has a condition number which depends only upon ω_1, ω_2 and the shape regularity.*

Proof. Let $E_i = (I - T_i^i) \cdots (I - T_1^1)$. Then $E_{i-1} - E_i = T_i^i E_{i-1}$, for $i = 1, 2, \cdots, k$, and we obtain

$$\|E_{i-1}u\|_a^2 - \|E_i u\|_a^2 = \|T_i^i E_{i-1}u\|_a^2 + 2a(T_i^i E_{i-1}u, E_i u)$$
$$= a(T_i^i(2I - T_i^i)E_{i-1}u, E_{i-1}u) \ge (2 - \omega_2)a(T_i^i E_{i-1}u, E_{i-1}u).$$

Summing these terms for $i = 1, 2, \cdots, k$, we get

$$\|u\|_a^2 - \|E_k u\|_a^2 \ge (2 - \omega_2) \sum_{i=1}^{k} a(T_i^i E_{i-1}u, E_{i-1}u).$$

We can write u as

$$u = T_1^1 u + (I - T_1^1)u = T_1^1 u + E_1 u = T_1^1 E_0 u + T_2^2 E_1 u + (I - T_2^2)E_1 u$$

$$= T_1^1 E_0 u + T_2^2 E_1 u + E_2 u = \cdots = \sum_{j=1}^{i-1} T_j^j E_{j-1}u + E_{i-1}u.$$

If we define θ_{ij} by

$$\theta_{ij} = \begin{cases} 1/\sqrt{\omega_1} & \text{if } i = j \\ \max_{v,w \in V^h} \dfrac{a((P_i^i - P_i^{i-1})v, T_j^j w)}{a((P_i^i - P_i^{i-1})v, v)^{\frac{1}{2}} \cdot a(T_j^j w, w)^{\frac{1}{2}}} & \text{if } i > j \\ \theta_{ji} & \text{if } i < j, \end{cases}$$

we obtain

$$
\begin{aligned}
a((P_i^i - P_i^{i-1})u, u) &= a((P_i^i - P_i^{i-1})u, E_{i-1}u) + \sum_{j=1}^{i-1} a((P_i^i - P_i^{i-1})u, T_j^j E_{j-1}u) \\
&\leq a((P_i^i - P_i^{i-1})u, u)^{\frac{1}{2}} \cdot a((P_i^i - P_i^{i-1})E_{i-1}u, E_{i-1}u)^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^{i-1} \theta_{ij} a((P_i^i - P_i^{i-1})u, u)^{\frac{1}{2}} \cdot a(T_j^j E_{j-1}u, E_{j-1}u)^{\frac{1}{2}} \\
&\leq a((P_i^i - P_i^{i-1})u, u)^{\frac{1}{2}} \cdot a(P_i^i E_{i-1}u, E_{i-1}u)^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^{i-1} \theta_{ij} a((P_i^i - P_i^{i-1})u, u)^{\frac{1}{2}} \cdot a(T_j^j E_{j-1}u, E_{j-1}u)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{\omega_1}} a((P_i^i - P_i^{i-1})u, u)^{\frac{1}{2}} \cdot a(T_i^i E_{i-1}u, E_{i-1}u)^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^{i-1} \theta_{ij} a((P_i^i - P_i^{i-1})u, u)^{\frac{1}{2}} \cdot a(T_j^j E_{j-1}u, E_{j-1}u)^{\frac{1}{2}} \\
&= a((P_i^i - P_i^{i-1})u, u)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{i} \theta_{ij} a(T_j^j E_{j-1}u, E_{j-1}u)^{\frac{1}{2}} \right).
\end{aligned}
$$

Thus, for $1 \leq i \leq k$, we get

$$
a((P_i^i - P_i^{i-1})u, u) \leq \left(\sum_{j=1}^{k} \theta_{ij} a(T_j^j E_{j-1}u, E_{j-1}u)^{\frac{1}{2}} \right)^2.
$$

Finally we conclude that

$$
a(P_a^{(2)}u, u) \leq \|\theta\|_2^2 \sum_{i=1}^{k} a(T_i^i E_{i-1}u, E_{i-1}u),
$$

where $P_a^{(2)}$ is the AFAC2 operator and $\|\theta\|_2$ is the l^2-norm of the matrix θ.

We now need to estimate the θ_{ij}. Observe that $\mathrm{Null}\,(T_j^j) = \mathrm{Null}\,(P_j^j)$. We also have $\mathrm{Range}\,(T_j^j) = \mathrm{Null}\,(T_j^j)^{\perp}$ and $\mathrm{Range}\,(P_j^j) = \mathrm{Null}\,(P_j^j)^{\perp}$ because T_j^j and P_j^j are energy-symmetric. We can then conclude that $\mathrm{Range}\,(T_j^j) =$

Range (P_j^j). It is also easy to see that Range $((T_j^j)^{\frac{1}{2}}) = $ Range (T_j^j). For $i > j$,

$$
\begin{aligned}
\theta_{ij} &= \max_{v,w \in V^h} \frac{a((P_i^i - P_i^{i-1})v, T_j^j w)}{a((P_i^i - P_i^{i-1})v, v)^{\frac{1}{2}} \cdot a(P_j^j (T_j^j)^{\frac{1}{2}} w, (T_j^j)^{\frac{1}{2}} w)^{\frac{1}{2}}} \\[2mm]
&\leq \sqrt{\omega_2} \max_{v,w \in V^h} \frac{a((P_i^i - P_i^{i-1})v, T_j^j w)}{a((P_i^i - P_i^{i-1})v, v)^{\frac{1}{2}} \cdot a(T_j^j (T_j^j)^{\frac{1}{2}} w, (T_j^j)^{\frac{1}{2}} w)^{\frac{1}{2}}} \\[2mm]
&= \sqrt{\omega_2} \max_{v,w \in V^h} \frac{a((P_i^i - P_i^{i-1})v, T_j^j w)}{a((P_i^i - P_i^{i-1})v, (P_i^i - P_i^{i-1})v)^{\frac{1}{2}} \cdot a(T_j^j w, T_j^j w)^{\frac{1}{2}}} \\[2mm]
&\leq \sqrt{\omega_2} \max_{u_1 \in \mathrm{Range}(P_i^i - P_i^{i-1}), u_2 \in \mathrm{Range}(P_j^j)} \frac{a(u_1, u_2)}{a(u_1, u_1)^{\frac{1}{2}} \cdot a(u_2, u_2)^{\frac{1}{2}}} \\[2mm]
&\leq \sqrt{\omega_2} C q^{(i-j-2)s/2}.
\end{aligned}
$$

The last step follows from Lemma 8 and thus we have $\|\theta\|_2^2 \leq \|\theta\|_1^2 \leq C'$. Therefore

$$
\begin{aligned}
a(P_a^{(2)} u, u) &\leq C' \sum_{i=1}^k a(T_i^i E_{i-1} u, E_{i-1} u) \leq \frac{C'}{2 - \omega_2}(\|u\|_a^2 - \|E_k u\|_a^2) \\[2mm]
&\equiv C_0(\|u\|_a^2 - \|E_k u\|_a^2),
\end{aligned}
$$

where C_0 is a constant which only depends on ω_1, ω_2, q, the subdomains and the shape regularity. By Theorem 1, we have $P_a^{(2)} \geq cI$, where $c > 0$ is a constant which is independent of k and the number of degrees of freedom, so we can conclude that $\|E_k u\|_a^2 \leq (C_0 - c)/C_0 \cdot \|u\|_a^2$. ∎

When $T_i^i = P_i^i$ for all i, we can prove this theorem under Assumptions 1 and 4 by using the same trick as in Theorem 5 and tracing the proof of above theorem. Consequently, we have obtained another proof about the optimality of basic FAC algorithm given in Widlund [28], which is stated below.

Theorem 7. *Under Assumptions 1 and 4, the energy norm of the basic FAC operator based on the P_i^i, $1 \leq i \leq k$, is bounded by a constant less than one with the constant independent of k and the number of degrees of freedom.*

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