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A CHARACTERIZATION OF HOLOMORPHIC GENERATORS ON THE CARTESIAN PRODUCT OF HILBERT BALLS

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Abstract. We present a necessary and sufficient condition for a holomorphic mapping to be a generator of a flow on any finite Cartesian product of Hilbert balls. A related null point theorem is also established.

Let X be a Banach space and let X^* be its dual. For a point $x \in X$ and a functional $x^* \in X$ we use the pairing $\langle x, x^* \rangle$ to denote $x^*(x)$. The duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) := \{x^* \in X^* : \mathrm{Re}\langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in X$.

In particular, if $X = H$ is a Hilbert space, then $\langle \cdot, \cdot \rangle$ is the inner product in H and $J : H \to H$ is the identity mapping. Let now D be the open unit ball in X, and let $C(\bar{D}, X)$ denote the class of continuous mappings from $\bar{D}$ into X. Suppose that f belongs to $C(\bar{D}, X)$ and satisfies the following boundary condition:

$$(*) \qquad \inf_{x^* \in J(x)} \mathrm{Re}\langle f(x), x^* \rangle \geq 0,$$

for each $x \in \partial D$.

Following [10] we call this condition a "*one-sided estimate*". We recall that such estimates have been systematically used in many areas of analysis, e.g., boundary value problems ([9], [5], [16]), nonlinear integral equations [6], and monotone operator theory [4]. For an extension of condition $(*)$ to topological vector spaces, with applications, we refer the reader to a paper by Fan [7].

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If $D = B$ is the open unit ball in a Hilbert space H, and $f : B \to H$ is a completely continuous vector field on $\bar{B}$ (i.e. $f \in C(\bar{B}, X)$ and $I - f$ is compact), then by Krasnoselskii's theorem [9] condition $(*)$ implies the existence of a null point of f in $\bar{B}$. As a matter of fact, this also follows from the Leray-Schauder Theorem because the mapping $I - f$ is compact.

A similar assertion was proved by Shinbrot [16] under the assumptions that f is weakly continuous and H is separable. This result was applied by him to a class of quasi-linear partial differential equations and to Navier-Stokes equations.

Suppose now that H is complex and that $f : B \to H$ is a holomorphic mapping in B. As we proved in [2], the compactness condition in this case can be replaced by the condition of uniform continuity of f on $\bar{B}$. However, examples show (see [2]) that such an assertion is no longer true for every Banach space. Nevertheless, we will show in the sequel that Theorem 2 in [2] can be generalized to the case when X is the Cartesian product of complex Hilbert spaces with the maximum norm. The key to the solution of such a problem is the following observation related to another issue, namely evolution equations and a characterization of infinitesimal holomorphic generators.

First we note that if D is a ball in a Banach space X and $f \in C(\bar{D}, X)$, then condition $(*)$ is equivalent to the following "flow invariance condition":

$$\text{(FIC)} \qquad \lim_{h \to 0^+} \frac{\operatorname{dist}(x - hf(x), \bar{D})}{h} = 0$$

(see [12]).

If we now suppose that f satisfies the condition:

For some $\delta > 0$ there exists a continuous family $F_t : [0, \delta) \to C(\bar{D}, X)$, $F_t(\bar{D}) \subset \bar{D}$, $t \in [0, \delta)$, such that for each $x \in D$,

$$f(x) = \lim_{t \to 0^+} \frac{x - F_t(x)}{t},$$

then it is clear that f satisfies (FIC) and hence the one-sided estimate $(*)$. This happens, in particular, if f is a strong generator of a one-parameter semigroup.

The converse assertion, generally speaking, is not clear. Usually, its validity can be ensured by additional conditions, such as accretivity (see, for example, [11]).

If X is complex and f is holomorphic in D and uniformly continuous on $\bar{D}$, then condition $(*)$ is equivalent to the assumption that f is an infinitesimal generator inside D (see [2]).

Thus the existence of an interior null point of f under the condition $(*)$ or (FIC) is equivalent in this case to the existence of a stationary point of the flow $\{F_t\}$, $t > 0$, defined by the Cauchy problem:

(CP)
$$\begin{cases} \frac{dF_t(x)}{dt} + f(F_t(x)) = 0, \quad t > 0, \\ \lim_{t\to 0^+} F_t(x) = x \end{cases}$$

(see [13]).

The following question now arises: Are there interior characterizations of f to be a generator on the open unit ball D such that if f has a continuous extension to $\bar{D}$ one can derive condition $(*)$ (or (FIC))? For the one-dimensional case an implicit characterization of f to be an infinitesimal generator of a one-parameter semigroup of holomorphic self-mappings in $D = \Delta$ (the unit disk in $\mathbb{C}$) was obtained by E. Berkson and H. Porta [3]. They proved the following assertion.

Let $f : \Delta \to \mathbb{C}$ be a holomorphic mapping in Δ. Then the Cauchy problem (CP) has a global solution on $\mathbb{R}^+ = (0, \infty)$ if and only if f admits the representation:

(BPC)
$$f(x) = (y - x)(\bar{y}x - 1)g(x)$$

for some $y \in \bar{\Delta}$ and for some holomorphic mapping $g : \Delta \to \mathbb{C}$ with Re $g(x) \geq 0$ for all $x \in \Delta$.

This characterization was used in [3] to study semi-groups of composition operators on Hardy spaces of the unit disk. Recently M. Abate [1] established a different characterization of holomorphic generators on the open unit ball B of $\mathbb{C}^n$ with the Euclidean norm (i.e., a finite-dimensional Hilbert space) by using the differentiability (in this case) of the Kobayashi metric. In our setting his characterization of $f : B \to \mathbb{C}^n$ to be a generator has the form

(AC)
$$\begin{aligned} &2[\| g(x) \|^2 - |\langle g(x), x\rangle|^2]\mathrm{Re}\langle g(x), x\rangle \\ &+(1 - \|x\|^2)^2\mathrm{Re}\langle f'(x)f(x), g(x)\rangle \geq 0, \end{aligned}$$

where $g(x) = (1 - \|x\|^2)f(x) + \langle f(x), x\rangle x$.

In particular, if $n = 1$, (AC) becomes

(AC)$'$)
$$\mathrm{Re} f(x)\bar{x} \geq -\frac{1}{2}\mathrm{Re} f'(x)(1 - |x|^2), \quad x \in \Delta.$$

It was also shown in [1] how to deduce (BPC) from (AC)$'$ and conversely. However, a deficiency of these conditions is that it is not clear how to derive the condition

$(*)'$)
$$\mathrm{Re}\langle f(x), x\rangle \geq 0, \quad x \in \partial D$$

when f has a continuous extension to $\bar{D}$. The difficulty is, of course, the presence of the derivative in (AC) (or (AC)$'$), which generally speaking may be unbounded (consider, for example, $f(x) = x - 1 + \sqrt{1-x}$). Observe that when $n = 1$, condition $(*)'$ can be written in the form

$$\operatorname{Re}\left(\frac{F(x) - F(0)}{x}\right) \leq 1 - \operatorname{Re}(\overline{F(0)}x), \quad x \in \partial\Delta,$$

where $F(x) := x - f(x),\ x \in \bar{\Delta}$. Since both the left and the right hand sides of the last inequality are harmonic functions, it continues to hold throughout $\bar{\Delta}$. Multiplying now by $|x|^2$ and returning to $f = I - F$, we obtain

$$(**) \qquad \operatorname{Re}(f(x)\bar{x}) \geq \operatorname{Re}(f(0)\bar{x})(1 - |x|^2), \quad x \in \bar{\Delta}.$$

As a matter of fact, as we will see below, this condition (with $x \in \Delta$) characterizes holomorphic generators on Δ even when f is not assumed to have a continuous extension to $\bar{\Delta}$.

In another direction, a careful study of the notion of monotonicity in the hyperbolic sense has led us [14] to conclude that a bounded holomorphic mapping f on the open unit ball B of a complex Hilbert space H is a generator if and only if

$$\frac{\operatorname{Re}\langle x, f(x)\rangle}{1 - \|x\|^2} + \frac{\operatorname{Re}\langle y, f(y)\rangle}{1 - \|y\|^2} \geq \operatorname{Re}\frac{\langle f(x), y\rangle + \langle x, f(y)\rangle}{1 - \langle x, y\rangle}$$

for all x and y in B.

Setting $y = 0$ we obtain the condition

$$(**)') \qquad \operatorname{Re}\langle f(x), x\rangle \geq \operatorname{Re}\langle f(0), x\rangle(1 - \|x\|^2), \quad x \in B,$$

which reduces to $(**)$ in the one-dimensional case. Actually, it turns out that $(**)'$ is also sufficient for f to be a generator. However, once again a crucial point of the arguments in [14] is the smoothness of the hyperbolic metric on B.

In the present paper we present an entirely different, but simple enough, approach to derive an analogous condition to $(**)'$ as a necessary and sufficient condition for f to be a generator on any finite Cartesian product of Hilbert balls.

Let $X = H^n$ be the Cartesian product of n copies of a complex Hilbert space H, and let D be the open unit ball in X with the maximum norm, i.e., $D = B^n$, where B is the open unit ball in H. By $\operatorname{Hol}(D, \tilde{D})$ we denote the family of holomorphic mappings from D into a subset $\tilde{D}$ of X.

We will say that $f \in \mathrm{Hol}(D, X)$ is a generator of a flow on D if for some $\delta > 0$ there is a continuous one-parameter semigroup $F_t : [0, \delta) \to \mathrm{Hol}(D, D)$ such that the strong limit

$$\lim_{t \to 0^+} \frac{x - F_t(x)}{t} = f(x) \tag{1}$$

exists for all $x \in D$.

Theorem 1. *Let $f \in \mathrm{Hol}(D, X)$, where $D = B^n$ and $X = H^n$.*

1. *If f is the generator of a flow on D, then it satisfies the following condition for all $x \in D$ and $x^* \in J(x)$:*

$$\mathrm{Re}\langle f(x), x^* \rangle \geq \mathrm{Re}\langle f(0), x^* \rangle (1 - \|x\|^2). \tag{2}$$

2. *Conversely, if f is bounded on each subset strictly inside D, and for each $x \in D$ there is $x^* \in J(x)$ such that*

$$\mathrm{Re}\langle f(x), x^* \rangle \geq \mathrm{Re}\langle f(0), x^* \rangle (1 - \|x\|^2), \tag{2)'}$$

then f is a generator of a flow on D.

Proof. Recall that for each $b \in B$ we can define the Möbius transformation $M_b : B \to B$ by

$$M_b(z) = (\sqrt{1 - \| b \|^2} Q_b + P_b) m_b(z),$$

where

$$m_b(z) = \frac{z + b}{1 + \langle z, b \rangle}, \; P_b(z) = \frac{\langle z, b \rangle b}{\|b\|^2}, \text{ and } Q_b = I - P_b.$$

(See, for example, [15] and [8].)

Let f be the generator of a flow $F_t = (F_t^1, F_t^2, \ldots, F_t^n)$. For each $t \geq 0$ we now consider the holomorphic mapping $G_t = (G_t^1, G_t^2, \ldots, G_t^n) : D \to D$, $D = B^n$, defined by

$$G_t^k(x) := M_{-F_t^k(0)}(F_t^k(x)), \; x \in D, \; 1 \leq k \leq n.$$

Note that since $G_t(0) = 0$, we have

$$\|G_t(x)\| \; \leq \; \|x\| \; , \quad x \in D, \tag{3}$$

by the Schwarz lemma.

To differentiate G_t at the origin we calculate

$$
\begin{aligned}
&\lim_{t\to 0^+} \frac{1}{t}\left(x^k - G_t^k(x)\right)\\
(4)\qquad &= \lim_{t\to 0^+} \frac{x^k - \langle F_t^k(x), F_t^k(0)\rangle x^k + F_t^k(0) - \sqrt{1-\|F_t^k(0)\|^2}F_t^k(x)}{t(1-\langle F_t^k(x), F_t^k(0)\rangle)}\\
&- \lim_{t\to 0^+} \frac{(1-\sqrt{1-\|F_t^k(0)\|^2})\langle F_t^k(x), F_t^k(0)\rangle F_t^k(0)}{t(1-\langle F_t^k(x), F_t^k(0)\rangle)\|F_t^k(0)\|^2}.
\end{aligned}
$$

Since $F_t^k(0) \to 0$ and $\frac{\left(1-\sqrt{1-\|F_t^k(0)\|^2}\right)}{\|F_t^k(0)\|^2} \to \frac{1}{2}$ as $t \to 0^+$, the second limit in (4) is zero, and

$$
\frac{x^k - \sqrt{1-\|F_t^k(0)\|^2}\; F_t^k(x)}{t} \to f^k(x),
$$

as $t \to 0+$.

Hence

$$
(5)\qquad g^k(x) := \lim_{t\to 0^+} \frac{1}{t}\left(x^k - G_t^k(x)\right) = f^k(x) + \langle x^k, f^k(0)\rangle x^k - f^k(0).
$$

By (3) we have, for any $z \in J(x)$,

$$
(6)\qquad \operatorname{Re}\langle g(x), z\rangle \geq 0, \quad x \in D,
$$

where $g = (g^1, g^2, \dots, g^n)$.

We observe now that for each $x = (x^1, x^2, \dots, x^n) \in H^n$, $z = (z^1, z^2, \dots, z^n) \in J(x)$, *and* $1 \leq k \leq n$,

$$
z^k = \alpha_k x^k, \text{ where } 0 \leq \alpha_k \leq 1 \text{ and } \sum_{k=1}^n \alpha_k = 1.
$$

Moreover, if $\|x^k\| < \|x\| = \max\{\|x^j\| : 1 \leq j \leq n\}$, then $\alpha_k = 0$. Therefore for each $x \in D$ and $z \in J(x)$,

$$
\begin{aligned}
0 \leq \operatorname{Re}\langle g(x), z\rangle &= \operatorname{Re}\left(\sum_{k=1}^n \langle g^k(x), z^k\rangle\right)\\
&= \operatorname{Re}\langle f(x), z\rangle + \operatorname{Re}\left(\sum_{k=1}^n \langle x^k, f^k(0)\rangle\langle x^k, z^k\rangle\right) - \operatorname{Re}\left(\sum_{k=1}^n \langle f^k(0), z^k\rangle\right)\\
&= \operatorname{Re}\langle f(x), z\rangle + \sum_{k=1}^n \alpha_k \operatorname{Re}\langle f^k(0), x^k\rangle(\|x^k\|^2 - 1)\\
&= \operatorname{Re}\langle f(x), z\rangle + (\|x\|^2 - 1)\operatorname{Re}\langle f(0), z\rangle.
\end{aligned}
$$

This yields (2) and the first assertion of the theorem is proven.

Conversely, by Theorem 1.2 in [14] it is sufficient to prove that under the assumptions of assertion 2, $f \in \mathrm{Hol}(D, X)$ satisfies the following range condition:
For each $r > 0$ and for each $y \in D$, the equation

$$x + rf(x) = y \tag{7}$$

has a unique solution $x \in D$.

Indeed, fix $r > 0$ and $y \in D$, and consider the mapping $G \in \mathrm{Hol}(D, X)$ defined by the formula

$$G(x) = y - rf(x). \tag{8}$$

For each $t \in (0, 1)$, $\|y\| < s < 1$, and $x \in D$ with $\|x\| = s$, there exists by $(2)'$ a functional $x^* \in J(x)$ such that

$$\begin{aligned} \mathrm{Re}\langle x - tG(x), x^*\rangle &= \|x\|^2 - t\mathrm{Re}\langle y, x^*\rangle + tr\mathrm{Re}\langle f(x), x^*\rangle \\ &\geq s^2 - ts\|y\| - trs\|f(0)\|(1 - s^2) \\ &= s^2 \left[1 - t\left(\frac{\|y\|}{s} + \frac{r\|f(0)\|(1 - s^2)}{s}\right)\right]. \end{aligned}$$

If we choose now s close enough to 1, we obtain

$$\|x - tG(x)\| \; \|x\| \geq \mathrm{Re}\langle x - tG(x), x^*\rangle \geq \|x\|^2(1 - tK),$$

$$\text{with} \quad K = \frac{\| y \|}{s} + \frac{r \| f(0) \| (1 - s^2)}{s} < 1.$$

Hence it follows by Lemma 1 in [2] that $G : D \to X$ has a unique fixed point $x \in D$.

This fixed point is the solution of the equation (7). This concludes the proof of the theorem. ■

Combining this theorem with our results in [13] and [14] we deduce the following results.

Corollary 1. *Let $D = B^n$, and let $f \in \mathrm{Hol}(D, X)$ be bounded on each ball strictly inside D. Then the following are equivalent:*

(i) *For each $x \in D$ there exist $x^* \in J(x)$ and $m \in R$ such that*

$$\mathrm{Re}\langle f(x), x^*\rangle \geq m(1 - \|x\|^2);$$

(ii) *For each* $x \in D$ *and all* $x^* \in J(x)$,

$$\operatorname{Re}\langle f(x), x^*\rangle \geq \operatorname{Re}\langle f(0), x^*\rangle (1 - \|x\|^2);$$

(iii) *For some* $\delta > 0$ *there exists a continuous family* $F_t : [0, \delta) \to \operatorname{Hol}(D, D)$ *such that*

$$\lim_{t \to 0^+} \frac{1}{t}(x - F_t(x)) = f(x)$$

for each $x \in D$;

(iv) *The Cauchy problem* (CP) *has a unique solution on* $\mathbb{R}^+$ *for each* $x \in D$;

(v) *For each* $r > 0$ *the mapping* $J_r = (I + rf)^{-1}$ *is well-defined on* D *and belongs to* $\operatorname{Hol}(D, D)$.

Corollary 2. *Let* D *and* f *be as above and assume that* f *has a uniformly continuous extension to* $\bar{D}$. *Then the following assertions are equivalent:*

(i) *For each* $x \in \partial D$ *there exists* $x^* \in J(x)$ *such that*

$$\operatorname{Re}\langle f(x), x^*\rangle \geq 0;$$

(ii) *For each* $x \in \partial D$

$$\inf_{x^* \in J(x)} \operatorname{Re}\langle f(x), x^*\rangle \geq 0;$$

(iii) *For each* $x \in \partial D$, f *satisfies the flow invariance condition* (FIC):

$$\lim_{h \to 0^+} \frac{1}{h} \operatorname{dist}(x - hf(x),\ \bar{D}) = 0;$$

(iv) *The mapping* f *generates a flow* (*one-parameter semigroup*) $\{F_t\}_{t>0} \subset \operatorname{Hol}(D, D)$.

Corollary 3. *If* $D = B^n$ *and* $f \in \operatorname{Hol}(D, X)$ *is a generator of a flow on* D, *then the linear operator* $A = f'(0)$ *is accretive.*

Proof. Let us represent f in the form

$$f(x) = f(0) + Ax + h(x),$$

where $\displaystyle\lim_{\|x\| \to 0} \frac{1}{\| x \|} h(x) = 0$. Then it follows by Theorem 1 that for $x \in D$,

$$\begin{aligned} \operatorname{Re}\langle f(x), x^*\rangle &= \operatorname{Re}\langle f(0), x^*\rangle + \operatorname{Re}\langle Ax, x^*\rangle + \operatorname{Re}\langle h(x), x^*\rangle \\ &\geq \operatorname{Re}\langle f(0), x^*\rangle (1 - \|x\|^2) \end{aligned}$$

for all $x^* \in J(x)$. This yields the inequality

$$\mathrm{Re}\langle Ax, x^* \rangle \geq -\mathrm{Re}\left(\langle h(x), x^* \rangle + \|x\|^2 \langle f(0), x^* \rangle\right).$$

Let now $y \in \partial D$ be arbitrary and set $x = ty,\ 0 < t < 1$. The last inequality implies

$$t^2 \mathrm{Re}\langle Ay, y^* \rangle \geq -\mathrm{Re}\left(\langle h(ty), ty^* \rangle + \ t^3 \langle f(0), y^* \rangle\right).$$

Hence for $0 < t < 1$,

$$\mathrm{Re}\langle Ay, y^* \rangle \geq -\mathrm{Re}\left(\langle \frac{1}{t} h(ty), y^* \rangle + t\langle f(0), y^* \rangle\right).$$

But the right hand side of this inequality converges to zero as $t \to 0^+$.

It follows that for each y with $\| y \| = 1$ and each $y^* \in J(y)$,

$$\mathrm{Re}\langle Ay, y^* \rangle \geq 0.$$

In other words, A is an accretive linear operator. ■

Returning now to the existence of null points, we consider for simplicity only the case $n = 2$.

Theorem 2. *Let B be the open unit ball in a complex Hilbert space H, and let $D = B^2$. Suppose that a bounded $f \in \mathrm{Hol}(D, H^2)$ has a uniformly continuous extension to $\bar{D}$. If for each $x \in \partial D$ there exists $x^* \in J(x)$ such that*

$$\mathrm{Re}\langle f(x), x^* \rangle \geq 0, \tag{9}$$

then f has a null point in $\bar{D}$.

For the proof we need the following lemmata.

Lemma 1. *Let B be the open unit ball in a complex Hilbert space H, and let Ω be a domain in a complex reflexive Banach space X. Suppose that $g : B \times \Omega \to H$ is a bounded holomorphic mapping such that for each $\lambda \in \Omega$ the mapping $g(\cdot, \lambda)$ has a uniformly continuous extension to $\bar{B}$ and satisfies the condition*

$$\mathrm{Re}\langle g(x, \lambda), x \rangle \geq 0, \quad x \in \partial B. \tag{10}$$

Then

1) *The equation*

$$g(x, \lambda) = 0 \tag{11}$$

has a holomorphic solution $x : \Omega \to \bar{B}$*;*

2) *If for some* $\lambda_0 \in \Omega$ *the equation*

$$g(x, \lambda_0) = 0$$

has no solution on ∂B*, then for each* $\lambda \in \Omega$*, equation* (11) *has a unique solution* $x = x(\lambda)$ *in* B*.*

This lemma can be obtained by combining Theorem 8.1 of [13] with Theorem 2 of [2]. For information on the hyperbolic metric, see, for example, [8, p. 98].

Lemma 2. *Let* $\rho(\cdot, \cdot)$ *be the hyperbolic metric on the open unit ball* B *of a complex Hilbert space* H*. Let* $\{z_n\}$ *and* $\{w_n\}$ *be two sequences in* B *such that* $\{z_n\}$ *converges to* $e \in \partial B$ *as* $n \to \infty$*, and for some sequence* $t_n \in (0, 1)$*,* $t_n \to 1-$*, the following condition holds for all* $n \in N$ *:*

$$\rho\left(\frac{1}{t_n} z_n, w_n\right) \leq \rho(z_n, w_n). \tag{12}$$

Then $\{w_n\}$ *converges to* e *as* $n \to \infty$*.*

Proof. It is not difficult to see that if there exists a subsequence of $\{\langle z_n, w_n\rangle\}$ which does not converge to 1, then condition (12) leads to a contradiction. Therefore $\{\langle z_n, w_n\rangle\} \to 1$, and $\{z_n - w_n\} \to 0$ as $n \to \infty$. ■

Proof of Theorem 2. Let $f = (f_1, f_2)$, where each $f_i : B^2 \to H,\ i = 1, 2$, is a bounded holomorphic mapping on B^2 which is uniformly continuous on $\overline{B^2}$. It follows from condition (9) that for each fixed $x_2 \in B$ and for each fixed $x_1 \in B$, the mappings $f_1(\cdot, x_2)$ and $f_2(x_1, \cdot)$ satisfy the boundary conditions

$$\operatorname{Re}\langle f_1(x_1, x_2), x_1\rangle \geq 0, \quad x_1 \in \partial B,\ x_2 \in B, \tag{13}$$

and

$$\operatorname{Re}\langle f_2(x_1, x_2), x_2\rangle \geq 0, \quad x_2 \in \partial B,\ x_1 \in B. \tag{14}$$

Lemma 1 and condition (13) imply that for each $x_2 \in B$, the mapping $f_1(\cdot, x_2)$ has a null point $x_1 = \varphi(x_2)$ in $\bar{B}$. If for some $x_2 \in B$, $f_1(\cdot, x_2)$ has no null point in B, then it has no null point in B for all $x_2 \in B$, and therefore the function $x_1 = \varphi(x_2)$ is a constant $e_1 \in \partial B$ by the maximum principle. In other words, $f_1(e_1, x_2) \equiv 0$ for all $x_2 \in B$. But by (14) and continuity, the mapping $f_2(e_1, \cdot) : \bar{B} \to H$ has a null point $e_2 \in \bar{B}$ and therefore $e = (e_1, e_2)$ is a null point of $f = (f_1, f_2)$.

Thus we can suppose that for at least one $x_2 \in B$ and hence for all $x_2 \in B$, the mapping $f_1(\cdot, x_2)$ has a null point $x_1 = \varphi(x_2)$ in B. In addition we can assume that $f_2(x_1, \cdot)$ has a null point $x_2 = \psi(x_1) \in B$, since otherwise the same considerations as above yield the result. Thus we arrive at the following system:

$$\begin{cases} f_1(\varphi(x_2), x_2) = 0, & x_2 \in B, \\ f_2(x_1, \psi(x_1)) = 0, & x_1 \in B, \end{cases} \tag{15}$$

where $\varphi(\cdot)$ and $\psi(\cdot)$ are holomorphic self-mappings of B.

We now claim that the equations

$$\begin{aligned} x_1 - J_1(x_1, x_2) &= f_1(J_1(x_1, x_2), x_2), \\ x_2 - J_2(x_1, x_2) &= f_2(x_1, J_2(x_1, x_2)) \end{aligned} \tag{16}$$

have unique holomorphic solutions $J_i(\cdot, \cdot) : B^2 \to B$, $i = 1, 2$. To see this, consider the mappings $g_i : B \times B^2 \to H$ defined by the formulas

$$\begin{aligned} g_1(y, x_1, x_2) &:= y + f_1(y, x_2) - x_1, \\ g_2(y, x_1, x_2) &:= y + f_2(x_1, y) - x_2, \end{aligned}$$

where $y \in B$. Setting in Lemma 1, $B^2 = \Omega$ and $\lambda = (x_1, x_2) \in B^2$, we see that the mappings $g_i(\cdot, \lambda)$, $i = 1, 2$, have uniformly continuous extensions to $\bar{B}$, and therefore we have by (13) and (14),

$$\mathrm{Re}\langle g_i(y, \lambda), y\rangle \geq 1 - \|x_i\|\ , \ y \in \partial B, \quad i = 1, 2.$$

Thus assertion (2) of Lemma 1 implies the existence and uniqueness of holomorphic solutions $y = J_i(x_1, x_2)$, $i = 1, 2$, to the equations $g_i(y, \lambda) = 0$, $i = 1, 2$, which are equivalent to (16). In addition, the uniqueness of $J_i : B^2 \to B$, $i = 1, 2$, and (15) imply that the mappings $\varphi(\cdot)$ and $\psi(\cdot)$ satisfy the following equations:

$$\begin{aligned} \varphi(x_2) &= J_1(\varphi(x_2), x_2), \\ \psi(x_1) &= J_2(x_1, \psi(x_1)). \end{aligned} \tag{17}$$

Now we consider the holomorphic mappings $F_i : B \to B$, $i = 1, 2$, defined as follows:

$$\begin{aligned} F_1 &:= J_1(\cdot, \psi(\cdot)), \\ F_2 &:= J_2(\varphi(\cdot), \cdot). \end{aligned}$$

Suppose that one of them, say F_1, has a fixed point $z \in B$. That is, $z = J_1(z, \psi(z))$ and $\psi(z) = J_2(z, \psi(z))$ by (17). Hence it follows from (16) that the point $(z, \psi(z)) \in B_2$ is a null point of $f = (f_1, f_2)$.

Finally, assume that neither F_1 nor F_2 has a fixed point in B. In this case, it is known (see [8]) that the approximating curves

$$z(t) = tF_1(z(t))$$

and

$$w(t) = tF_2(w(t))$$

converge as $t \to 1^-$ to points a and b, respectively, on ∂B. If ρ is the hyperbolic metric on B, then we have

$$\tag{18} \begin{aligned} \rho\left(\frac{1}{t}z(t), \varphi(w(t)\right) &= \rho(F_1(z(t)), \varphi(w(t))) \\ &= \rho(J_1(z(t), \psi(z(t))), \varphi(w(t))) \\ &= \rho(J_1(z(t), \psi(z(t))), J_1(\varphi(w(t)), w(t))) \\ &\le \max\{\rho(z(t), \varphi(w(t))); \rho(\psi(z(t)), w(t))\} = m(t). \end{aligned}$$

In a similar way we also get

$$\tag{19} \rho\left(\frac{1}{t}w(t), \psi(z(t))\right) \le m(t).$$

Suppose that there is a sequence $t_n \to 1^-$ such that $m(t_n) = \rho(z(t_n), \varphi(w(t_n)))$. By Lemma 2, we have $\varphi(w(t_n)) \to a$ strongly and hence $f_1(a, b) = \lim_{n\to\infty} f_1(\varphi(w(t_n)), w(t_n)) = 0$ by (15). To show that $f_2(a, b) = 0$ we use (16) and the following simple calculations:

$$\begin{aligned} f_2(a, b) &= \lim_{n\to\infty} f_2(\varphi(w(t_n)), \frac{1}{t_n}w(t_n)) \\ &= \lim_{n\to\infty} f_2(\varphi(w(t_n)), J_2(\varphi(w(t_n)), w(t_n))) \\ &= \lim_{n\to\infty} [w(t_n) - J_2(\varphi(w(t_n)), w(t_n))] \\ &= \lim_{n\to\infty} (t_n - 1)J_2(\varphi(w(t_n)), w(t_n)) = 0. \end{aligned}$$

If, on the other hand, there is a sequence $t_n \to 1^-$ such that $m(t_n) = \rho(\psi(z(t_n)), w(t_n))$, then we can use (18), and once again the same arguments as above show that $f(a, b) = (f_1(a, b), f_2(a, b)) = (0, 0) \in H^2$. This concludes the proof of Theorem 2. ∎

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