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THE CRITICAL MASS OF COMPRESSIBLE VISCOUS GAS-STARS

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Abstract. Let γ be the adiabatic index of self-gravitating, spherically symmetric motion of compressible viscous gas-star. When $\gamma \in (1, 2]$, we prove the existence of nonisentropic equilibrium. Furthermore, at the adiabatic index $\gamma = \frac{4}{3}$, we found a family of particular solutions which corresponds to an expansive (contractive) gaseous star. Moreover, we find that there is a critical total mass M_0. If the total mass M of star is less than M_0, then the star expands infinitely. However, if $M \geq M_0$, then there is an "escape velocity" $v_e r$ associated with M and the initial configuration of the star. If $v(0, r) \geq v_e r$, then the star will expand infinitely. If $v(0, r) < v_e r$, then it will collapse after a finite time.

1. Introduction

In studying the evolution of a gaseous star, which consists of spherically symmetric movements of self-gravitating viscous gas, we have the following equations

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r} + \rho\frac{\partial v}{\partial r} + \frac{2}{r}\rho v = 0, \tag{1.1}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) + \frac{\partial p}{\partial r} + \frac{4\pi\rho}{r^2}\int_0^r \rho(t,\tau)\tau^2 d\tau = \nu\left\{\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} - \frac{2}{r^2}v\right\}, \tag{1.2}$$

$$\frac{\partial S}{\partial t} + v\frac{\partial S}{\partial r} = 0, \tag{1.3}$$

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$$p = e^S \rho^\gamma, \tag{1.4}$$

where $t \geq 0, r \geq 0$ (see, e.g., [10, 11, 12, 15]). Here, the unknown variable ρ is the density of the gas, v is the outward velocity, and S is entropy, p is the pressure, $\gamma \in (1,2]$ is the adiabatic exponent, and ν is the viscosity coefficient.

The problem originates in Newtonian (non-relativistic) astrophysical theory. One of the main problems in studying (1.1) $\sim$ (1.4) is the existence of temporarily global solution for a given set of initial data at $t = 0$. However, when $\gamma = \frac{4}{3}, \nu = 0$, Makino [9] found there is a family of particular solutions that tend toward the delta function after a finite time, i.e., a model for gravitational collapse of a gaseous star even in Newtonian theory. In [3], Fu and Lin studied the total mass M of these solutions and found that there is a critical total mass M_0. If $M < M_0$, then the star expands infinitely. However, if $M \geq M_0$, then there is an "escape velocity", $v_e r$, associated with M and the initial configuration of the star. If the star expands at an initial velocity of $v(0,r) \geq v_e r$, then it will expand as in the case in which $M < M_0$. If the initial velocity $v(0,r) < v_e r$, then it will collapse in finite time. In [3, 9], $\nu = 0$ and they considered S to be constant, i.e., the gas flow is isentropic. When $\nu > 0$, we can consider nonisentropic flow. In this paper, we extend the results in [3, 9] under some assumptions about S including the special case $S \equiv$ constant.

The paper is organized as follows: in Section 2, we study the existence of a Ball-type stationary solution of (1.1) $\sim$ (1.4) for $\gamma \in$(1,2]. The definition of a Ball-type solution is given below. In Section 3 we study a family of special solutions of (1.1)$\sim$(1.4) for $\gamma = \frac{4}{3}$, after which we compare the total mass of these solutions with the Ball-type solution, which yields a very interesting result.

2. Star in Equilibrium - Stationary Solution

We seek a bounded stationary solution of the following form

$$\rho(t,r) = \left(\frac{q+1}{4\pi}\right)^{\frac{q}{q-1}} y^q(r), \tag{2.1}$$

$$v(t,r) = 0, \tag{2.2}$$

$$s(t,r) = (q+1)S(r), \tag{2.3}$$

where $q = \frac{1}{\gamma - 1}$ and $S(r)$ is a given function that satisfies the following assumptions:

(S-1) $S \in C^1(0,\infty)$ and is bounded;

(S-2) S is nondecreasing.

According to (2.1) $\sim$ (2.3), y satisfies

$$e^{(q+1)S}(y'+yS') + \frac{1}{r^2}\int_0^r y^q \tau^2 d\tau = 0, \tag{2.4}$$

$$y'(0) + y(0)S'(0) = 0, \quad y(0) > 0. \tag{2.5}$$

It is easy to see that (2.4) $\sim$ (2.5) is equivalent to

$$\begin{aligned} y(r) = e^{-(q+1)^S} \Bigg(& y(0)e^{(q+1)S(0)} + \int_0^r q e^{(q+1)S} S' y d\tau \\ & - \int_0^r \tau \left(1 - \frac{\tau}{r}\right) y^q d\tau \Bigg). \end{aligned} \tag{2.6}$$

Using standard methods we obtain the following:

Proposition 2.1. *If S satisfies* ***(S-1)****, then for all $y(0) > 0$, there is an r_0 dependent on S and $y(0)$ such that (2.4) $\sim$ (2.5) has a unique solution in $C([0, r_0])$, which is C^2 in $(0, r_0)$.*

Proof. Let $y_1 = e^{(q+1)S}y$. Then y_1 satisfies

$$y_1(r) = y_1(0) + \int_0^r qS'y_1 d\tau + \int_0^r \tau\left(1 - \frac{\tau}{r}\right) e^{-q(q+1)S} y_1^q d\tau. \tag{2.7}$$

Let us denote by $Ty_1(r)$ the right-hand side of (2.7). Choosing $M > y_1(0)$, we consider the set of functions $F = \{\eta \in C[0, r_0] : \sup\limits_{o \le r \le r_0} |\eta(r)| \le M\}$. Then there exists an r_0 dependent on $y_1(0)$, M, $S_m = \inf\limits_{0 \le r \le \infty} S(r)$, $S_M = \sup\limits_{0 \le r < \infty} S(r)$, such that T is a contraction mapping with respect to the metric $d(\eta_1, \eta_2) = \|\eta_1 - \eta_2\|_\infty$. (2.6) then admits a unique solution in F, which is the fixed point of T. Since it is easy to deduce the estimate of r_0, we omit the computation here. The proof is complete. ∎

If, in addition, S satisfies **(S-2)**, from (2.4) we have $y' < 0$ for $r > 0$. Let us continue $y = y(r, y(0))$ to the right as long as possible.

Let $\overline{R} = \sup\{\tilde{r} | y > 0 \text{ in } [0, \tilde{r})\}$. We need the following definition.

Definition 2.2. (i) If $\overline{R} < \infty$, then we say y is a Ball-type solution.
(ii) *If $\overline{R} = \infty$, then we say y is a ground-state solution.*

A Ball-type solution means that we have a gaseous star of finite radius. In order to know when we have Ball-type or ground-state solution, we deduce the generalized Pohozaev identity. Let

$$x(r) = ye^{S}.$$

$x(r)$ then satisfies

$$(r^2 e^{qS} x')' + r^2 e^{-qS} x^q = 0, \tag{2.8}$$

$$x(0) = y(0)e^{S(0)}, \quad x'(0) = 0. \tag{2.9}$$

Let

$$g(r) = r^2 e^{qS}, \tag{2.10}$$

and

$$h(r) = g \int_r^\infty g^{-1}(\tau) d\tau. \tag{2.11}$$

We then have the Pohozaev identity for (2.8) $\sim$ (2.9).

Lemma 2.3. *Let x satisfy* (2.8) $\sim$ (2.9) *and g, h be given as in* (2.10) *and* (2.11)*. Then,*

$$\begin{aligned} &\frac{d}{dr}\left\{(gx')(hx'+x) + \frac{2gh}{q+1} e^{-2qS} x^{q+1}\right\} \\ &\quad = \frac{2}{1+q} g e^{-2qS} \left(\frac{4h}{r} - \frac{3+q}{2}\right) x^{q+1}. \end{aligned} \tag{2.12}$$

Proof. By (2.8), (2.10) and (2.11), it is easy to see that x satisfies

$$(hx' + x)' + he^{-2qS} x^q = 0. \tag{2.13}$$

By (2.8), (2.10), (2.11) and (2.13), we have (2.12). ∎

Remark 2.4. Let $H(r) = \frac{4h}{r} - \frac{3+q}{2}$. Using a partial integration, we have

$$H(r) = \frac{5-q}{2} - 4re^{qS} \int_r^\infty q\tau^{-1} S' e^{-qS} d\tau. \tag{2.14}$$

If $S \equiv$ constant, then $H(r) = \frac{5-q}{2}$, and (2.12) reduces to the usual Pohozaev identity.

From (2.4), (2.5), **(S-1)** and **(S-2)**, it is easy to see that y is decreasing to zero as $r \to \infty$ when y is a ground-state solution. We next give some asymptotic behavior of x, when $y = e^{-S}x$ is a ground-state solution.

Lemma 2.5. *Assume S satisfies **(S-1)** and **(S-2)**. If x is the ground-state solution of* (2.8) *and* (2.9)*, then we have*

(i) $x(r) \geq c_1 r^{-1}, x' \leq c_2 r^{-2}$ *for all sufficiently large r, where $c_1 > 0, c_2 < 0$ are constants,*

(ii) $x(r) \leq c_3(q) r^{\frac{-2}{q-1}}$ *for all sufficiently large r if $q > 1$, and $rx(r) \to \infty$ as $r \to \infty$ if $q = 3$.*

The constants c_1, c_2 and $c_3(q)$ are independent of r.

Proof. Since S is bounded and nondecreasing, according to the argument used in [14], Theorems 2.1, and 2.2, we have asymptotic behavior for x if x is the ground-state solution of (2.7) and (2.8).

By comparing the asymptotic behavior of $x(r) \geq c_1 r^{-1}$, if $1 < q < 3, rx(r) \to \infty$ when $q = 3$, with $x(r) \leq c_3(q) r^{\frac{-2}{q-1}}$, we have an immediate contradiction. Thus, $1 < q \leq 3$ and all solutions of (2.8)∼(2.9) are Ball-type. The proof is complete. ■

Before we state the result for the full range of q, we make the following assumption:

(S-3) $H(r) \geq 0$ for any $r \geq 0$.

Indeed, as $1 < q < 5$, if $S' \geq 0$ and $S(0) - S(\infty) \geq ln(\frac{q+3}{8})^{\frac{1}{q}}$ then $H(r) \geq 0$ for $r \geq 0$. We can now state the following:

Proposition 2.6. *If S satisfies **(S-1)** and **(S-2)**, and if x is the solution of* $(2.8) \sim (2.9)$*, then:*

(i) *if $1 < q \leq 3$, then $\overline{R} < \infty$;*

(ii) *if $3 < q < 5$ and in addition, S satisfies **(S-3)**, then $\overline{R} < \infty$;*

(iii) *if $q \geq 5$, then $\overline{R} = \infty$.*

Proof. For $3 < q < 5$, since we have a Pohozaev identity and S satisfies **(S-3)**, we can use the argument for Theorem 3.1 [14] and draw the appropriate conclusion.

For $q \geq 5$, if $\overline{R} < \infty$, then integrating (2.12) from 0 to $\overline{R}$, since $S' \geq 0$, therefore $H(r) < 0$ and we have a contradiction. The proof is complete. ■

We can now state the following:

Theorem 2.7. *Assume $S(r)$ satisfies **(S-1)** and **(S-2)**, and let $(\rho, v, S)_{(t,r)}$ be the solution given for* $(2.1) \sim (2.3)$.

(i) *If $\frac{4}{3} \leq \gamma < 2, (\rho, v, S)$ is a Ball-type solution.*

(ii) *If* $\frac{6}{5} < \gamma < \frac{4}{3}$, *and* $S(r)$ *satisfies* ***(S-3)***, *then* (ρ, v, S) *is a Ball-type solution.*

(iii) *If* $1 < \gamma < \frac{6}{5}$, *then* (ρ, v, S) *is a ground-state solution.*

Remark 2.8. It is interesting to know the mass-radius diagram ($M - R$ diagram) from [1]. The total mass M of a Ball-type solution is given by

$$M = 4\pi C_q \int_0^{\overline{R}} y^q r^2 dr < \infty,$$

where $C_q = (\frac{q+1}{4\pi})^{\frac{q}{q-1}}$.

To understand the $M - R$ diagram, it is useful to study the following two problems.

Problem 1. Given $y(0) = \alpha > 0, M > 0$, how many solutions of (2.4) and (2.5) are there?

Problem 2. Given $y(0) = \alpha > 0, \overline{R} > 0$, how many solutions of (2.4) and (2.5) are there?

In [7], $S \equiv$ constant, we know the $M - R$ diagram for $1 < q \leq 3$ looks like the following Fig. 1

FIG. 1.

But in (2.2) and (2.3), when $S \not\equiv$constant, the computation of $\frac{dM}{d\alpha}$ in Problem 1 or $\frac{d\overline{R}}{d\alpha}$ in Problem 2 is more difficult than the case in which $S \equiv$ constant.

3. The Relation of Mass and Expanding of Star

In this section we shall study a particular solution for nonisentropic gas. Following [9], we adopt the following transformation to seek a particular class of solutions. Let

$$r = a(t)z, \tag{3.1}$$

$$\rho(t,r) = Aa^{-3}(t)y^3(z), \tag{3.2}$$

$$v(t,r) = \dot{a}(t)z \quad and \tag{3.3}$$

$$s(t,r) = 4S(z). \tag{3.4}$$

The positive r and $\rho \geq 0$ require $z > 0, y(z) \geq 0$ and $a(t) > 0$. It is easy to verify that (1.1) $\sim$ (1.3) are satisfied by (3.1) $\sim$ (3.4) and (1.2) becomes

$$a^2\ddot{a}z^3 + A^{\gamma-1}z^2e^{4S}a^{-3\gamma+4}y^{3\gamma-4}(3\gamma y' + 4S'y) + 4\pi A\int_0^z y^3\xi^2 d\xi = 0. \tag{3.5}$$

Furthermore, if $\gamma = \frac{4}{3}$ and we let $A = \pi^{-3/2}$, then (3.5) becomes

$$\frac{1}{4\pi A}a^2\ddot{a}z^3 + z^2e^{4S}(y' + S'y) + \int_0^z y^3\xi^2 d\xi = 0. \tag{3.6}$$

Now, (3.6) can be solved by the method of separation of variables. Indeed, let

$$a^2\ddot{a}(t) = \frac{4}{3}\pi A\lambda. \tag{3.7}$$

Then (3.6) becomes

$$z^2e^{4S}(y' + yS') + \int_0^z \xi^2(y^3 + \lambda)d\xi = 0. \tag{3.8}$$

We consider the initial condition

$$y'(0) + y(0)S'(0) = 0, \quad y(0) > 0. \tag{3.9}$$

Remark 3.1. We denote the solution of (3.8) and (3.9) by $y_\lambda(z) = y(z, \lambda, y(0))$. As $\lambda = 0$, the equation for $y_0(z)$ is the same as (2.1) and (2.2) for $q = 3$. Henceforth, we will omit the subscript λ, which causes no confusion.

(3.8) and (3.9) are equivalent to

$$y(z) = e^{-4S}\left\{y(0)e^{4S(0)} + \int_0^r 3e^{4S}S'yd\xi - \int_0^z \xi\left(1 - \frac{\xi}{z}\right)\left(y^3 + \lambda\right)d\xi\right\}. \tag{3.10}$$

Using standard methods as in Proposition 2.1, we have a local solution $y(z)$ for (3.8) and (3.9) near $z = 0$ if S satisfies **(S-1)**. We continue $y(z)$ to the right as long as possible. Furthermore, if S satisfies **(S-2)**, then $y' < 0$ for $r \geq 0$ as $\lambda \geq 0$. On the other hand, y' may change signs as $\lambda < 0$. In order to get more information about the solution $y(z)$, let

$$x = e^S y. \tag{3.11}$$

Differentiating (3.8) once, we obtain

$$\left(z^2 e^{3S} x'\right)' + z^2 e^{-3S} x^3 + \lambda z^2 = 0, \tag{3.12}$$

$$x(0) = y(0)e^{S(0)}, \quad x'(0) = 0. \tag{3,13}$$

Lemma 3.2 *Let* $x(z)$ *be the solution of* (3.12) $\sim$ (3.13). *Let* $Z = Z(\lambda) = \sup\{z | x(z) > 0$ *in* $(0, z)\}$, $\varphi(z) = \frac{\partial x}{\partial \lambda}$. *Assume* S *satisfies* **(S-1)** *and* **(S-2)**. *If* $x'(z) \leq 0$ *in* $(0, Z)$, *then* $\varphi(z) < 0$ *in* $(0, Z]$.

Proof. By (3.11), (3.12) and (3.13), it is easy to see that $\varphi(z)$ satisfies

$$(z^2 e^{3S} \varphi')' + 3z^2 e^{-3S} x^2 \varphi + z^2 = 0 \tag{3.14}$$

and

$$\varphi(0) = 0 = \varphi'(0). \tag{3.15}$$

Define

$$\varphi\lambda(x) = \frac{1}{2} e^{3S} \varphi'^2 + \int_0^z (3e^{-3S} x^2 \varphi + 1)\varphi' d\xi. \tag{3.16}$$

By partial integration, we have

$$\begin{aligned} \varphi_\lambda(z) = {} & \frac{1}{2} e^{3S} \varphi'^2 + \varphi + \frac{3}{2} e^{-3S} x^2 \varphi^2 \\ & + \int_0^z \frac{3}{2} \varphi^2 (3S' e^{-3S} x^2 - 2e^{-3S} x x') d\xi. \end{aligned} \tag{3.17}$$

Then by (3.14), we have

$$\frac{d\varphi_\lambda}{dz} = -\left(\frac{2}{z} + \frac{3}{2} S'\right) e^{3S} \varphi'^2 \leq 0 \quad for\ z > 0. \tag{3.18}$$

Since $\varphi_\lambda(0) = 0$,

$$\varphi_\lambda(z) < 0 \ \ in\ (0, Z]. \tag{3.19}$$

By (3.17), (3.19), **(S-2)** and our assumption that $x'(z) \le 0$, we have $\varphi(z) < 0$ in $(0, Z]$. ∎

Observe that $x(z, \lambda)$ is C^1 in λ, and if $x'(Z(\lambda)) \neq 0$, then by the implicit function theorem $Z(\lambda)$ is C^1 in λ. If $Z < \infty$, we have

$$\frac{\partial x}{\partial z}(Z(\lambda), \lambda)\frac{dZ}{d\lambda} + \varphi(Z(\lambda), \lambda) = 0. \tag{3.20}$$

Hence, if $x'(z) \le 0$ in $(0, Z)$, then, by Lemma 3.2, $\frac{dZ}{d\lambda} < 0$. Thus it is important to know when $x'(z)$ is nonpositive in $(0, Z]$.

Lemma 3.3. *Assume S satisfies* **(S-1)** *and* **(S-2)**. *If $x(z)$ is the solution of* (3.12) $\sim$ (3.13)*, then there is $\overline{\lambda} < 0$ such that for all $\lambda > \overline{\lambda}$, there exists $Z = Z(\lambda) < \infty$ such that $x(z) > 0$ in $(0, Z), x(Z) = 0$ and $x'(z) < 0$ in $(0, Z]$.*

Proof. If $\lambda \ge 0$, by (3.12) $\sim$ (3.13)

$$z^2 e^{3S} x'(z) = -\int_0^z (\xi^2 e^{-3S} x^3 + \lambda \xi^2) d\xi. \tag{3.21}$$

Hence,

$$x'(z) < 0 \quad for \quad z > 0, \quad \lambda \ge 0. \tag{3.22}$$

As $\lambda = 0$, the equation of (3.12) is the same as (2.8). Thus, by Proposition 2.6(i), $Z(0) < \infty$. Furthermore, as $\lambda > 0$, by (3.22) and Lemma 3.2, we have $\varphi(Z) < 0$. Hence, $\frac{dZ}{d\lambda} < 0$ for $\lambda > 0$.

Since for $\lambda = 0, x'(z) < 0$ in $(0, Z(0)]$, therefore as $\lambda < 0$ and $|\lambda|$ is sufficiently small, we can use the argument of continued dependence on parameter λ to obtain $Z(\lambda) < \infty$ such that $x(z) > 0$ in $(0, Z(\lambda)), x'(z) < 0$ in $(0, Z(\lambda))$. We may choose $\overline{\lambda}$ to be the smallest number that still allows this argument to hold. The proof is complete. ∎

Indeed, if $\lambda < 0$ and is small enough, then $x(z) > 0$ for all $z > 0$. We state the result below.

Lemma 3.4. *Assume S satisfies **(S-1)** and **(S-2)**. If $\lambda + \frac{1}{4}y^3(0) < 0$, then every solution $x(z)$ of* (3.11) $\sim$ (3.12) *is positive for any $z > 0$.*

Proof. We define the energy function

$$E(z) = \frac{1}{2}e^{3S}x'^2 + \int_0^z (e^{-3S}x^3 + \lambda)x' d\xi + \lambda x(0) + \frac{1}{4}x^4(0)e^{-3S(0)}. \tag{3.23}$$

By partial integration, we have

$$E(z) = \frac{1}{2}e^{3S}x'^2 + \lambda x + \frac{1}{4}e^{-3S}x^4 + \int_0^z \frac{3}{4}e^{-3S}S'x^4 d\xi. \tag{3.24}$$

Differentiating (3.23) once, we obtain

$$\frac{dE}{dz} = -\left(\frac{2}{z} + \frac{3}{2}S'\right)e^{3S}x'^2 \leq 0. \tag{3.25}$$

If there is any $Z(\lambda) < \infty$ such that $x(Z(\lambda)) = 0$, then by **(S-2)**,(3.23) and (3.24), $0 < E(Z(\lambda)) \leq E(0)$. Hence, if $\lambda + \frac{1}{4}y^3(0) = \frac{E(0)}{x(0)} < 0$, then we have a contradiction. The proof is complete. ■

The total mass M of solution (ρ, v, S) of (1.1)∼(1.4) is given by

$$M = 4\pi \int_0^{R(t)} \rho(t,r)r^2 dr, \tag{3.26}$$

where $R(t) \leq \infty$ is the first zero of $\rho(t,r)$ at time t. For solutions of the form (3.1) ∼ (3.4), $M(\lambda)$ is dependent only on λ and $y(0)$, with

$$M(\lambda) = 4\pi A \int_0^{Z(\lambda)} y^3 \xi^2 d\xi. \tag{3.27}$$

By (3.11),

$$M(\lambda) = 4\pi A \int_0^{Z(\lambda)} e^{-3S}x^3\xi^2 d\xi. \tag{3.28}$$

Thus,

$$\frac{dM}{d\lambda} = 4\pi A \int_0^{Z(\lambda)} 3e^{-3S}x^2 \cdot \frac{\partial x}{\partial \lambda}\xi^2 d\xi \tag{3.29}$$

if $Z(\lambda) < \infty$ and $x(Z(\lambda)) = 0$. By Lemmas 3.2, and 3.3, we have:

Lemma 3.5. *Assume S satisfies **(S-l)** and **(S-2)**, and let $x(0)$ be fixed. Then there is a $\overline{\lambda}$ dependent on $x(0)$ such that for all $\lambda > \overline{\lambda}$,*

$$\frac{dM}{d\lambda} < 0. \tag{3.30}$$

Proof. The result follows easily from Lemmas 3.2 and 3.3. The proof is complete. ■

By Lemma 3.5, we define

$$\overline{M} = \lim_{\lambda \to \overline{\lambda}} M(\lambda). \tag{3.31}$$

On the other hand, for the equation (3.7), we consider the initial-value problem

$$a(0) = a_0 > 0,\ \dot{a}(0) = a_1. \tag{3.32}$$

The solutions of (3.7) and (3.32) have been studied in [3, 9]. We merely review the results here.

Proposition 3.6. *Let $a(t)$ be the solution of* (3.7) *and* (3.32). *We then have the following results.*

(I) *If $\lambda > 0$, then $a(t) > 0$ for any $t > 0$, and $a(t) \to \infty$ as $t \to \infty$.*

(II) *If $\lambda = 0$, then $a(t) = a_1 t + a_0$.*

(III) *If $\lambda < 0$, let*

$$a_1^*(\lambda) = \left(\frac{S\pi A}{3}|\lambda|a_0^{-1}\right)^{1/2}. \tag{3.33}$$

If $a^1 \geq a_1^*(\lambda)$, then $a(t) > 0$ for any $t > 0$, and $a(t) \to \infty$ as $t \to \infty$.

If $a_1 < a_1^*(\lambda)$, then there is $T < \infty$ such that $a(t) > 0$ in $(0, T)$, and $a(t) \to 0$ as $t \to T^-$.

Denote by $M_0 = M(0)$. For any $\lambda \in (\overline{\lambda}, \infty), a_0 > 0, a_1 \in \mathbb{R}$, the solutions of (3.1) $\sim$ (3.4) are given by

$$\rho(t, r) = Aa^{-3}(t)y^3\left(\frac{r}{a(t)}\right), \tag{3.34}$$

$$v(t, r) = a^{-1}(t)\dot{a}(t)r \text{ and} \tag{3.35}$$

$$S(t, r) = S\left(\frac{r}{a(t)}\right), \tag{3.36}$$

with an initial velocity of

$$v(0, r; a_0, a_1) = a_0^{-1}a_1 r, \tag{3.37}$$

where $a(t) \equiv a(t; a_0, a_1)$ and S is any given function satisfying **(S-1)** and **(S-2)**. Denote the escape velocity, $v_e r$, as

$$v_e \equiv v_e(\lambda, a_0) = \left(\frac{8\pi A}{3}|\lambda|a_0^{-1}\right)^{1/2}. \tag{3.38}$$

Combining the results of Proposition 3.6 and Lemma 3.5, we obtain the following main result.

Theorem 3.7. *Assume S satisfies* **(S-1)**, **(S-2)**, *and the total mass $M \in (0, \overline{M})$. Then a gaseous star of the form* (3.1) $\sim$ (3.4) *is given by* (3.33) $\sim$ (3.35). *Furthermore, we have*:

(I) *If $M < M_0$, then the star will expand and the density eventually tends toward zero.*

(II) *If $M > M_0$, and the initial velocity $v(0, r) \geq v_e r$ for $r \in (0, R_0)$, where $\rho(0, R_0) = 0$, then the star behaves as in* (I). *On the other hand, if $v(0, r) < v_e r$ for $r \in (0, R_0)$, then the star will collapse toward its center in a finite time.*

(III) *If $M = M_0$, we have three cases*:
(i) *when $a_1 > 0$, the star behaves as in* (I);
(ii) *when $a_1 < 0$, the star collapses toward its center*;
(iii) *when $a_1 = 0$, the star is in equilibrium.*

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