TAIWANESE JOURNAL OF MATHEMATICS
Vol. 2, No. 3, pp. 353-360, September 1998

THE PATH-PARTITION PROBLEM IN BIPARTITE DISTANCE-HEREDITARY GRAPHS

Hong-Gwa Yeh* and Gerard J. Chang*

Abstract. A path partition of a graph is a collection of vertex-disjoint paths that cover all vertices of the graph. The path-partition problem is to find a path partition of minimum size. This paper gives a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs.

1. Introduction

A *path partition* of a graph is a collection of vertex-disjoint paths that cover all vertices of the graph. The *path-partition problem* is finding the *path-partition number* $p(G)$ that is the minimum size of a path partition of G. Note that G has a Hamiltonian path if and only if $p(G) = 1$. Since the Hamiltonian path problem is $\mathcal{NP}$-complete for planar graphs [7], bipartite graphs [8], chordal graphs [8], chordal bipartite graphs [12], and strongly chordal graphs [12], so is the path-partition problem. On the other hand, the path-partition problem is polynomially solvable for trees [11, 14], interval graphs [1, 3], cographs [4, 5], and block graphs [15, 16]. In this paper we present a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs. For technical reasons, we consider the following generalization of the path-partition problem. For a set S of vertices in a graph $G = (V, E)$, an *S-path partition* is a path partition $\mathcal{P}$ in which every vertex of S is an endpoint of a path in $\mathcal{P}$. The *S-path-partition problem* is to determine the *S-path-partition number* $p(G, S)$ that is the minimum size of an S-path partition of G. Note that the path-partition problem is a special case of the S-path-partition problem, since $p(G) = p(G, \emptyset)$.

Received December 27 1996; revised November 25, 1997.
Communicated by S.-Y. Shaw.
1991 *Mathematics Subject Classification*: 05C85, 05C38.
Key words and phrases: Path partition, distance-hereditary graph, Hamiltonian path.
*Supported in part by the National Science Council under grant NSC84-2121-M009-023.

We now review distance-hereditary graphs. Suppose A and B are two sets of vertices in a graph $G = (V, E)$. $G[A]$ denotes the *subgraph of* G *induced* by A. The *deletion* of A from G, denoted by $G - A$, is the graph $G[V - A]$. The *neighborhood* $N_A(B)$ of B in A is the set of vertices in A that are adjacent to some vertex in B. The *closed neighborhood* $N_A[B]$ of B in A is $N_A(B) \bigcup B$. For simplicity, $N_A(v)$, $N_A[v]$, $N(B)$, and $N[B]$ stand for $N_A(\{v\})$, $N_A[\{v\}]$, $N_V(B)$, and $N_V[B]$, respectively. The *degree* of a vertex v is $\deg(v) = |N(v)|$. A vertex x is called a *leaf* of G if $\deg(x) = 1$. The *distance* $d_G(x, y)$ between two vertices x and y in G is the minimum length of an x-y path in G. The *hanging* h_u of a connected graph $G = (V, E)$ at a vertex $u \in V$ is the collection of sets $L_0(u),\ L_1(u),\ \ldots,\ L_t(u)$ (or $L_0,\ L_1,\ \ldots, L_t$ if there is no ambiguity), where $t = \max_{v \in V}\ d_G(u, v)$ and $L_i(u) = \{v \in V : d_G(u, v) = i\}$ for $0 \le i \le t$. For any $1 \le i \le t$ and any $v \in L_i$, let $N'(v) = N(v) \bigcap L_{i-1}$. Note that the notion $N'(v)$ depends on the hanging h_u. A vertex $v \in L_i$ with $1 \le i \le t$ has a *minimal neighborhood* in L_{i-1} if $N'(v) \subseteq N'(w)$ or $N'(v) \cap N'(w) = \emptyset$ for any $w \in L_i$.

A graph is *distance-hereditary* if every two vertices in a connected induced subgraph have the same distance as in the original graph. Distance-hereditary graphs were introduced by Howorka [10]. Characterizations and recognition of distance-hereditary graphs were studied in [2, 6, 9]. The following theorem contains some useful properties used in this paper.

Theorem 1. ([2, 9]) *Suppose $h_u = (L_0, L_1, \ldots, L_t)$ is a hanging of a connected distance-hereditary graph G at u. For each $1 \le i \le t$ and any two vertices $x, y \in L_i$, $N'(x) \cap N'(y) = \emptyset$ or $N'(x) \subseteq N'(y)$ or $N'(y) \subseteq N'(x)$. Consequently, for each $1 \le i \le t$, L_i contains a vertex v having a minimal neighborhood in L_{i-1}. In addition, for such a vertex v, we have $N_{V-N'(v)}(x) = N_{V-N'(v)}(y)$ for every pair of vertices x and y in $N'(v)$.*

Note that for any bipartite distance-hereditary graph G with a hanging $h_u = (L_0, L_1, \ldots, L_t)$, each $G[L_i]$ contains no edges. Consequently, $N(x) = N'(x)$ for any $x \in L_t$. We shall frequently use this fact in Sections 2 and 3.

In this paper, we use the following notation. For a graph G and vertices w, x, y, we use $G - x$ for $G - \{x\}$, $G - x - y$ for $(G - \{x\}) - \{y\} \cong G - \{x, y\}$, and $G - w - x - y$ for $G - \{w, x, y\}$ ... etc. For a set A and elements x and y, we use $A - x$ for $A - \{x\}$, $A + x$ for $A \cup \{x\}$, $A - x - y$ for $(A - \{x\}) - \{y\} = A - \{x, y\}$, $A - x + y$ for $(A - \{x\}) \cup \{y\}$ ... etc.

2. Path Partition in Bipartite Distance-hereditary Graphs

To give a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs, we first establish three basic lemmas that are used later.

Lemma 2. *If x is a leaf of G, then $p(G, S) = p(G, S + x)$.*

In the following two lemmas, suppose $G = (V, E)$ is a connected bipartite distance-hereditary graph with a hanging $h_u = (L_0, \ldots, L_t)$ at u and $t \geq 1$. According to Lemma 2, for each vertex x in G, we may assume that either $x \in S$ or $x \notin S$ with $|N(x)| \geq 2$.

Lemma 3. *Suppose $x \in L_t$ has a minimal neighborhood in L_{t-1} and $N(x) \subseteq S$.*

(1) *If $x \in S$, then $p(G, S) = p(G - x - y, S - x - y) + 1$ for any $y \in N(x)$.*

(2) *If $x \notin S$ and $|N(x)| \geq 2$, then $p(G, S) = p(G - w - x - y, S - w - y) + 1$ for any two distinct vertices w, $y \in N(x)$.*

Proof. (1) Since an $(S-x-y)$-path partition of $G-x-y$, together with the path xy, forms an S-path partition of G, $p(G, S) \leq p(G-x-y, S-x-y)+1$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Since $N[x] \subseteq S$, for any optimal S-path partition $\mathcal{P}$ of G either $x \in \mathcal{P}$ or $xy' \in \mathcal{P}$ for some $y' \in N(x)$. For the case in which $x \in \mathcal{P}$, let y be an endpoint of some $P \in \mathcal{P}$. Then, $\mathcal{P}' = \mathcal{P}-x-P+xy+(P-y)$ is another optimal S-path partition of G. So, in any case, we may assume that $xy' \in \mathcal{P}$ for some $y' \in N(x)$. Since x has a minimal neighborhood in L_{t-1}, by Theorem 1, $N(y') = N(y)$ and thus we may interchange the roles of y' and y to assume that $xy \in \mathcal{P}$. Hence, $\mathcal{P} - xy$ is an $(S - x - y)$-path partition of $G - x - y$. Thus, $p(G, S) - 1 \geq p(G - x - y, S - x - y)$. Therefore, $p(G, S) = p(G - x - y, S - x - y) + 1$.

(2) Since an $(S - w - y)$-path partition of $G - w - x - y$, together with the path wxy, forms an S-path partition of G, $p(G, S) \leq p(G - w - x - y, S - w - y) + 1$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Let P be the path of $\mathcal{P}$ that contains x. By $N(x) \subseteq S$, $|N(x)| \geq 2$, and $x \notin S$, we have that P is x or xy' or $w'xy'$. By an argument similar to that for (1), we may assume that $wxy \in \mathcal{P}$. Hence, $\mathcal{P} - wxy$ is an $(S - w - y)$-path partition of $G - w - x - y$. Thus, $p(G, S) - 1 \geq p(G - w - x - y, S - w - y)$. Therefore, $p(G, S) = p(G - w - x - y, S - w - y) + 1$. ∎

Lemma 4. *Suppose $x \in L_t$ has a minimal neighborhood in L_{t-1} and $N(x) \not\subseteq S$.*

(1) *If $x \in S$, then $p(G, S) = p(G - x, S - x + y)$ for any $y \in N(x) - S$.*

(2) *If $x \notin S$ and $|N(x)| \geq 2$, then $p(G, S) = p(G - x - y, S)$ for any $y \in N(x) - S$.*

Proof. (1) Suppose $\mathcal{P}$ is an optimal $(S-x+y)$-path partition of $G-x$ such that y is an endpoint of some path $P \in \mathcal{P}$. Then, $\mathcal{P}-P+Px$ is an S-path partition of G and so, $p(G,S) \le p(G-x, S-x+y)$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Suppose the path P in $\mathcal{P}$ containing x is $xv_1v_2\dots v_r$, where $r \ge 0$. For the case of $r=0$, let P_1yP_2 be the path of $\mathcal{P}$ that contains y. Then $\mathcal{P}-x-P_1yP_2+P_1y+P_2$ is an $(S-x+y)$-path partition of $G-x$. For the case of $r \ge 1$, we have $y, v_1 \in N(x)$. Since x has a minimal neighborhood in L_{t-1}, by Theorem 1, $N(y)=N(v_1)$. Thus, we may interchange the roles of y and v_1 and assume that $xyv_2\dots v_r \in \mathcal{P}$. Then, $\mathcal{P}-P+yv_2\dots v_r$ is an $(S-x+y)$-path partition of $G-x$. In any case, $p(G,S) \ge p(G-x, S-x+y)$. Therefore, $p(G,S) = p(G-x,S-x+y)$.

(2) Suppose $\mathcal{P}$ is an optimal S-path partition of $G-x-y$. Since $|N_G(x)| \ge 2$, without loss of generality, we may assume that $\mathcal{P}$ has a path $P = v_0v_1\dots v_iv_{i+1}\dots v_k$ such that $v_i \in N_G(x)$. Since x has a minimal neighborhood in L_{t-1}, by Theorem 1, $N(v_i)=N(y)$. Thus, $P' = v_0v_1\dots v_ixyv_{i+1}\dots v_k$ is a path of G. Therefore, $\mathcal{P}-P+P'$ is an S-path partition of G and so $p(G,S) \le p(G-x-y,S)$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Consider first the case in which x and y lie on path

$$P = v_0v_1\dots v_ixv_{i+1}\dots v_jyv_{j+1}\dots v_k \in \mathcal{P}.$$

By Theorem 1, v_i is adjacent to v_j and v_{i+1} is adjacent to v_{j+1}. Hence,

$$P' = v_0v_1\dots v_{i-1}v_iv_jv_{j-1}v_{j-2}\dots v_{i+2}v_{i+1}v_{j+1}v_{j+2}\dots v_k$$

is a path in $G-x-y$ containing all vertices of P except x and y. Therefore, $\mathcal{P}-P+P'$ is an S-path partition of $G-x-y$. Next consider the case in which x and y lie on two distinct paths

$$P_1 = v_0v_1\dots v_ixv_{i+1}\dots v_{k-1}v_k \in \mathcal{P} \text{ and } P_2 = u_0u_1\dots u_jyu_{j+1}\dots u_{k'-1}u_{k'} \in \mathcal{P}.$$

By Theorem 1, v_i is adjacent to u_{j+1} and u_j is adjacent to v_{i+1}. Hence,

$$P_1' = v_0v_1\dots v_{i-1}v_iu_{j+1}u_{j+2}\dots u_{k'-1}u_{k'}$$

and

$$P_2' = u_0u_1\dots u_{j-1}u_jv_{i+1}v_{i+2}\dots v_{k-1}v_k$$

are paths in $G-x-y$ containing all vertices of P_1 and P_2 except x and y. Therefore, $\mathcal{P}-P_1-P_2+P_1'+P_2'$ is an S-path partition of $G-x-y$. In any case, we have that $p(G,S) \ge p(G-x-y,S)$. Therefore, $p(G,S)=p(G-x-y,S)$. ∎

Based on Lemmas 2 to 4, we have the following algorithm for the S-path partition problem in bipartite distance-hereditary graphs.

Algorithm PP-dh. Find the S-path partition number of a connected bipartite distance-hereditary graph.

Input: A connected bipartite distance-hereditary graph $G = (V, E)$ and $S \subseteq V$.
Output: The S-path partition number $p(G, S)$.
Method:

```
P(G,S) ⟵ 0;
determine the hanging h_u = (L_0, L_1, ..., L_t) of G at a vertex u;
for  i = t to 1 step −1 do
{       let L_i = {x_1, x_2, ..., x_j};
        sort L_i such that |N'(x_{i_1})| ≤ |N'(x_{i_2})| ≤ ... ≤ |N'(x_{i_j})|;
        for  k = 1 to  j do
        {       x ⟵ x_{i_k};
                if deg(x) = 1 then S ⟵ S + x;
                if N(x) ⊆ S
                then {P(G,S) ⟵ P(G,S) + 1;
                        if x ∈ S  then {pick y ∈ N(x);
                                          G ⟵ G − x − y;
                                          S ⟵ S − x − y;
                                          L_{i−1} ⟵ L_{i−1} − y;}
                                   else  {pick w, y ∈ N(x);
                                          G ⟵ G − w − x − y;
                                          S ⟵ S − w − y;
                                          L_{i−1} ⟵ L_{i−1} − w − y;}
                        }
                else  {pick y ∈ N(x) − S;
                        if x ∈ S  then {G ⟵ G − x;
                                          S ⟵ S − x + y;}
                                   else  {G ⟵ G − x − y;
                                          L_{i−1} ⟵ L_{i−1} − y;}
                        }
        }
}
```

Theorem 5. *Algorithm PP-dh finds the S-path-partition number of a bipartite distance-hereditary graph $G = (V, E)$ with $S \subseteq V$ in linear time.*

Proof. The correctness of the theorem follows from Lemmas 2 to 4. In order to make the running time linear, we can use a bucket-sort to sort L_i. ∎

3. Discussion

This paper gives a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs by using the concepts of hanging and a vertex with a minimal neighbor. The same idea also works for the Hamiltonian-cycle problem in bipartite distance-hereditary graphs.

Lemma 6. *Suppose $G = (V, E)$ is a connected bipartite distance-hereditary graph with a hanging $h_u = (L_0, L_1, \ldots, L_t)$ at u such that $t \geq 2$ and $|V| \geq 5$. If $x \in L_t$ has a minimal neighborhood in L_{t-1} and $\deg(x) \geq 2$, then for every $y \in N(x)$, G has a Hamiltonian cycle if and only if $G-x-y$ has a Hamiltonian cycle.*

Proof. Suppose G has a Hamiltonian cycle $C = v_1v_2v_3 \ldots v_nv_1$ with $x = v_1$. We first consider the case in which $y = v_i$ with $3 \leq i \leq n-1$. Since x has a minimal neighbor in L_{t-1}, by Theorem 1, $N(v_2) = N(v_i)$. Therefore, we may interchange the roles of v_2 and v_i and assume that $v_1v_iv_3v_4 \ldots v_{i-1}v_2v_{i+1} \ldots v_nv_1$ is a Hamiltonian cycle of G. So, without loss of generality, we may assume that $v_2 = y$ in C. Now consider the Hamiltonian cycle C of G. Since $v_n, v_2 \in N(x)$, by Theorem 1, $N(v_n) = N(v_2)$ and so v_n is adjacent to v_3. Therefore, $G-\{x, y\}$ has a Hamiltonian cycle $v_3v_4v_5 \ldots v_nv_3$.

Conversely, suppose $G - x - y$ has a Hamiltonian cycle $v_1v_2v_3 \ldots v_{n-2}v_1$. Since $\deg(x) \geq 2$, we may assume $v_1 \in N(x)$. Since $y, v_1 \in N(x)$, by Theorem 1, $N(y) = N(v_1)$ and so y is adjacent to v_2 in G. Therefore, G has a Hamiltonian cycle $v_1xyv_2v_3 \ldots v_{n-1}v_1$. ∎

Based on Lemma 6, we have the following algorithm for the Hamiltonian cycle problem in bipartite distance-hereditary graphs.

Algorithm HC-dh. Determine whether or not a connected bipartite distance-hereditary graph has a Hamiltonian cycle.

Input: A connected bipartite distance-hereditary graph $G = (V, E)$.
Output: "G has a Hamiltonian cycle" or "G has no Hamiltonian cycle."
Method:

determine the hanging $h_u = (L_0, L_1, \ldots, L_t)$ of G at a vertex u;

for $i = t$ **to** 1 **step** -1 **do**
{ let $L_i = \{x_1, x_2, \ldots, x_j\}$;
sort L_i such that $|N'(x_{i_1})| \leq |N'(x_{i_2})| \leq \ldots \leq |N'(x_{i_j})|$;
for $k = 1$ **to** j **do**
{ **if** $|V(G)| \leq 4$ **then if** $G \cong C_4$ **then goto** (y) **else goto** (n);
if $\deg(x_{i_k}) \leq 1$ **then goto** (n);
choose $y \in N(x_{i_k})$;
$G \longleftarrow G - x_{i_k} - y$;
$L_{i-1} \longleftarrow L_{i-1} - y$;
}
}
(y) print "G has a Hamiltonian cycle"; **stop**;
(n) print "G has no Hamiltonian cycle";

Theorem 7. *Algorithm HC-dh determines whether or not a connected bipartite distance-hereditary graph has a Hamiltonian cycle in linear time.*

Proof. The correctness of the algorithm follows from Lemma 6. In order to make the running time of the algorithm linear, we can use a bucket-sort to sort L_i. ∎

Acknowledgment

The authors thank the referee for useful suggestions on the revision of this paper.

References

1. S. R. Arikati and C. Pandu Rangan, Linear algorithm for optimal path cover problem on interval graphs, *Inform. Process. Lett.* **35** (1990), 149-153.
2. H. J. Bandelt and H. M. Mulder, Distance-hereditary graphs, *J. Combin. Theory Ser. B* **41** (1986), 182-208.
3. M. A. Bonuccelli and D. P. Bovet, Minimum node disjoint path covering for circular-arc graphs, *Inform. Process. Lett.* **8** (1979), 159-161.
4. G. J. Chang and D. Kuo, The $L(2,1)$-labeling problem on graphs, *SIAM J. Discrete Math.* **9** (1996), 309-316.

5. D. G. Corneil, H. Lerchs and L. Stewart, Complement reducible graphs, *Discrete Appl. Math.* **3** (1981), 163-174.
6. A. D'Atri and M. Moscarini, Distance-hereditary graphs, Steiner trees, and connected domination, *SIAM J. Comput.* **17** (1988), 521-538.
7. M. R. Garey, D. S. Johnson and R. E. Tarjan, The planar Hamiltonian circuit problem is NP-complete, *SIAM J. Comput.* **5** (1976), 704-714.
8. M. C. Golumbic, Algorithmic Graph Theorem and Perfect Graphs, Academic Press, New York, 1980.
9. P. H. Hammer and F. Maffray, Completely separable graphs, *Discrete Appl. Math.* **27** (1990), 85-99.
10. E. Howorka, A characterization of distance-hereditary graphs, *Quart. J. Math. Oxford Ser.* (2) **28** (1977), 417-420.
11. H. A. Jung, On a class of posets and the corresponding comparability graphs, *J. Combin. Theory Ser. B* **35** (1978), 125-133.
12. H. Müller, Hamiltonian circuits in chordal bipartite graphs, *Discrete Math.* **156** (1996), 291-298.
13. H. Müller and F. Nicolai, Polynomial time algorithms for Hamiltonian problems on bipartite distance-hereditary graphs, *Inform. Process. Lett.* **46** (1993), 225-230.
14. Z. Skupien, Path partitions of vertices and Hamiltonicity of graphs, in Proceedings of the Second Czechoslovakian Symposium on Graph Theory, Prague, 1974.
15. R. Strikant, Ravi Sundaram, Karan Sher Singh and C. Pandu Rangan, Optimal path cover problem on block graphs and bipartite permutation graphs, *Theoret. Comput. Sci.* **115** (1993), 351-357.
16. J.-H. Yan and G. J. Chang, The path-partition problem in block graphs, *Inform. Process. Lett.* **52** (1994), 317-322.
17. J.-H. Yan, G. J. Chang, S. M. Hedetniemi and S. T. Hedetniemi, k-Path partition in trees, *Discrete Appl. Math.* **78** (1997), 227-233.

Gerard J. Chang
Department of Applied Mathematics, National Chiao Tung University
Hsinchu 30050, Taiwan
E-mail: gjchang@math.nctu.edu.tw