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PSEUDOCONVEXITY OF RIEMANN DOMAINS OVER A PRODUCT OF COMPLEX PLANES

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Abstract. This paper is concerned with the pseudoconvex Riemann domain over the product space $\mathbb{C}^{\mathbb{N}}$. We investigate that the pseudoconvex Riemann domain is the product of $\mathbb{C}^{\mathbb{N}} - \{1, 2, \cdots, n\}$ and a pseudoconvex Riemann domain over $\mathbb{C}^n$.

1. Introduction

Recenty, H. Kazama [8] proved that any Stein neighborhood of $\mathbb{C}^k \times \mathbb{R}^l (1 \leq k, l < \infty)$ in $\mathbb{C}^k \times \mathbb{C}^l$ is a product of $\mathbb{C}^k$ and a domain of holomorphy in $\mathbb{C}^l$ and showed that $\mathbb{C}^k \times \mathbb{R}^l$ has no Stein neighborhood bases in $\mathbb{C}^k \times \mathbb{C}^l$. More recently, S. Ohgai and K. H. Shon [17] proved that a domain of holomorphy in the product space of the complex Glassmann manifold $M_{n,k}$ and $\mathbb{C}^l$ containing $M_{n,k} \times G$ $(G \subset \mathbb{R}^l)$ is a product of $M_{n,k}$ and a Stein connected open neighborhood of G in $\mathbb{C}^l$. J. Kajiwara, S. Ohgai and K. H. Shon [7] extended this result to complex Lie groups. M. Nishihara, K. H. Shon and N. Sugawara [14] solved this problem in a locally convex Hausdorff space over $\mathbb{C}$. H. Hamada and J. Kajiwara [2] solved this problem in the related pseudoconvex domain. Most recently, K. H. Shon [19] proved that a pseudoconvex Riemann domain containing the product space at a real domain and a complex vector space E is the product space of a pseudoconvex Riemann domain and E. The present paper is aimed at a Schlicht Riemann domain over the Cartesian product $\mathbb{C}^{\mathbb{N}}$ of a countable number of complex planes.

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2. Preliminaries and Notations

Let (Ω, φ) be a Riemann domain over Ω, i.e., Ω is a Hausdorff topological space and an analytic map $\varphi : \Omega \to E$ is locally an homeomorphism. Let A be a subset of E. A map $\mathcal{S} : A \to \Omega$ is called a *section* of (Ω, φ) over A if $\mathcal{S} : A \to \Omega$ is a continuous mapping such that $\mathcal{S} \circ \varphi = id_A$. Let E be a locally convex Haudorff space over $\mathbb{C}, (\Omega, \varphi)$ be a (connected) Riemann domain over E and cs(E) be the set of all continuous seminorms on E. We define the (boundary) distance functions $d^\alpha_\Omega : \Omega \to [0, +\infty]$ for $\alpha \in$ cs(E), and $\delta_\Omega : \Omega \times E \to (0, +\infty]$, as follows:

$$d^\alpha_\Omega(x) = \sup(\{r > 0 : \text{ there is a section } \sigma : B^\alpha_E(\varphi(x), r) \to \Omega \\ \text{with } \sigma \circ \varphi(x) = x\} \cup \{0\},$$

$$\delta_\Omega(x, a) = \sup\{r > 0 : \text{ there is a section } \sigma : D_E(\varphi(x), a, r) \to \Omega \\ \text{with } \ \sigma \circ \varphi(x) = x\},$$

where for $\xi, a \in E$ and $r > 0$ we write

$$B^\alpha_E(\xi, r) = \{\xi + b : b \in E, \alpha(b) < r\},$$
$$D_E(\xi, a, r) = \{\xi + \lambda a : \lambda \in \mathbb{C}, |\lambda| < r\}.$$

We let $D(\zeta, r)$ be the disc with center ζ and radius $r, \mathcal{H}(\Omega)$ the complex holomophic mappings in Ω. The domain Ω is said to be *pseudoconvex if the* function $-\log \delta_\Omega$ is plurisubharmonic on $\Omega \times E$ (see J. Mujica [11, 12]). The following is a well-known result (cf. P. Noverraz [15] or M. Schottenloher [18]).

Proposition 2.1. *Let (Ω, φ) be a Riemann domain over a locally convex space E. Then the following conditions are equivalent:*

(1) *Ω is a pseudoconvex domain.*

(2) *For any $\alpha \in cs(E), -\log d^\alpha_\Omega$ is plurisubharmonic on Ω.*

3. Pluripolar Sets on Pseudoconvex Domains Over Locally Convex Spaces

Let E_1 and E_2 be locally convex spaces over $\mathbb{C}$. We introduce the following definition. A set P in $\mathbb{C}$ is *polar* if, on some (or every) open set $U \supset P$, there is a subharmonic u on U (which may be chosen ≤ 0 if U is bounded) such that $P \subset u^{-1}(-\infty)$. Polar subsets of an open set Ω in a locally convex space E have several possible definitions (cf. M. Hervé [3, 4], P. Noverraz [16]).

Definition 3.1. A subset W of an open set X in a locally convex space is *pluripolar* in X if there is a plurisubharmonic function p on X (and < 0) such that $W \subset p^{-1}(-\infty)$.

We have proved the following theorem.

Theorem 3.2([19]). *Let (Ω, φ) be a Riemann domain over $E = E_1 \times E_2$. Suppose that there exist an open set G of E_2 and a section $\mathcal{S} : E_1 \times G \to \Omega$. Then Ω is a pseudoconvex domain if and only if there exist a pseudoconvex Riemann domain V over E_2 and a biholomorphic mapping $\sigma : E_1 \times V \to \Omega$ such that $V = \varphi^{-1}(\{0\} \times E_2)$ and $(id_{E_1}, \varphi|_V) = \varphi \circ \sigma$. Furthermore, if φ is injective, then $\Omega = E_1 \times V$.*

Let (Ω, φ) be a Riemann domain over $\mathbb{C}^{\mathbb{N}}$ such that $\mathcal{H}(\Omega)$ separates the points of Ω. M. Matos [10] proved that (Ω, φ) is a domain of holomorphy if and only if there exists a positive integer n such that (Ω, φ) is of order n in Ω and (Ω_n, φ_n) is a manifold of holomorphy spread over $\mathbb{C}^n$, where $\varphi_n = \varphi|_{\Omega_n}$ and $\Omega_n = \varphi^{-1}[\pi_n \circ \varphi(\Omega)]$, and π_n denotes the projection mapping from $\mathbb{C}^{\mathbb{N}}$ onto the space of the first n variables. A. Hirschowitz[5] proved it for the case in which (Ω, φ) is an open subset of $\mathbb{C}^{\mathbb{N}}$.

Theorm 3.3. *Let (Ω, φ) be a Schlicht Riemann domain over $\mathbb{C}^{\mathbb{N}}$. If Ω is a pseudoconvex domain, then there exists a number $n \in \mathbb{N}$ and a psedoconvex Riemann domain $(V, \varphi|_V)$ over $\mathbb{C}^n$ such that $\Omega = \mathbb{C}^{\mathbb{N}-\{1,2,\cdots,n\}} \times V$.*

Proof. For $x \in \Omega$, there is $\alpha \in cs\ (\mathbb{C}^{\mathbb{N}})$ with $d^{\alpha}_{\Omega}(x) \geq 1$. Thus for $z = (z_i)$, there exist $n \in \mathbb{N}$ and $c > 0$ such that

$$c\left(\sup_{1 \leq i \leq n} |z_i|\right) \geq \alpha(z).$$

Hence there exists a section

$$\mathcal{S} : B^{\alpha}_{\mathbb{C}^{\mathbb{N}}}(\varphi(x), 1) \to \Omega$$

satisfying $\mathcal{S} \circ \varphi(x) = x$. In fact, we have

$$\begin{aligned} B^{\alpha}_{\mathbb{C}^{\mathbb{N}}}(\varphi(x), 1) &= \{\varphi(x) + \zeta \in \mathbb{C}^{\mathbb{N}} : \alpha(\zeta) < 1\} \\ &\supset \left\{\varphi(x) + \zeta \in \mathbb{C}^{\mathbb{N}} : c\left(\sup_{1 \leq i \leq n} |\zeta_i|\right) < 1\right\} \end{aligned}$$

$$
\begin{aligned}
&= \Big\{\varphi(x) + (\zeta_1, \zeta_2, \cdots, \zeta_n, \zeta_{n+1}, \cdots) \in \mathbb{C}^{\mathbb{N}} : |\zeta_i| < \frac{1}{c}, \\
&\quad i = 1, 2, \cdots, n, \zeta_j \in \mathbb{C},\ j = n+1, n+2, \cdots \Big\} \\
&= \varphi(x) + \Big\{(\zeta_1, \zeta_2, \cdots, \zeta_n, 0, 0, \cdots) \in \mathbb{C}^{\mathbb{N}} : |\zeta_i| < \frac{1}{c}, i = 1, 2, \cdots, n\Big\} \\
&\quad + \{(0, 0, \cdots, 0, \zeta_{n+1}, \zeta_{n+2}, \cdots) \in \mathbb{C}^{\mathbb{N}} : \zeta_j \in \mathbb{C},\ j = n+1, n+2, \cdots\} \\
&= (p_1 \circ \varphi(x), p_2 \circ \varphi(x), \cdots, p_n \circ \varphi(x), 0, 0, \cdots, 0, \cdots) \\
&\quad + (0, 0, \cdots, 0, p_{n+1} \circ \varphi(x), p_{n+2} \circ \varphi(x), \cdots) \\
&\quad + \Big\{(\zeta_1, \zeta_2, \cdots, \zeta_n, 0, 0, \cdots, 0, \cdots) \in \mathbb{C}^{\mathbb{N}} : |\zeta_i| < \frac{1}{c}, i = 1, 2, \cdots, n\Big\} \\
&\quad + \{(0, 0, \cdots, 0, \zeta_{n+1}, \zeta_{n+2}, \cdots) \in \mathbb{C}^{\mathbb{N}} : \zeta_j \in \mathbb{C}, j = n+1, n+2, \cdots\} \\
&= \mathbb{C}^{\mathbb{N} - \{1, 2, \cdots, n\}} \times \prod_{j=1}^{n} D\Big(p_j \circ \varphi(x), \frac{1}{c}\Big).
\end{aligned}
$$

That is,

$$
\mathcal{S}|_{\mathbb{C}^{\mathbb{N}-\{1,2,\cdots,n\}} \times \prod_{j=1}^{n} D(p_j \circ \varphi(x), \frac{1}{c})} : B^{\alpha}_{\mathbb{C}^{\mathbb{N}}}(\varphi(x), 1) \to \Omega
$$

is a section satisfying $\mathcal{S}|_{\mathbb{C}^{\mathbb{N}-\{1,2,\cdots,n\}} \times \prod_{j=1}^{n} D(p_j \circ \varphi(x), \frac{1}{c})} \circ \varphi(x) = x$. From Theorem 3.2, we have the result. ■

Definition 3.4. Let Ω be an open set in a locally convex space E. A set $A \subset \Omega$ is *unipolar* in Ω if:

(1) for every complex line L in E and every connected component U of $L \cap \Omega$, $A \cap U$ is either polar or $A \cap U = U$,

(2) $\mathring{A} = \emptyset$ or, equivalently, A does not include any connected component of Ω.

Lemma 3.5. $\mathbb{R}$ *is not a polar set in* $\mathbb{C}$.

Proof. Assume that $\mathbb{R}$ is a polar set in $\mathbb{C}$. Then $\mathbb{C} - \mathbb{R}$ is connected. This is a contradiction. ■

Let $E(\mathbb{R})$ be any locally convex Hausdorff space over $\mathbb{R}$. For any $(a, b), (c, d) \in E(\mathbb{R}) \times E(\mathbb{R})$ and any complex number $\lambda = \alpha + \sqrt{-1}\beta$, we define

$$
(a, b) + (c, d) = (a + c, b + d),
$$

$$
\lambda \cdot (a, b) = (\alpha a - \beta b, \beta a + \alpha b).
$$

Then $E(\mathbb{R}) \times E(\mathbb{R})$ can be turned into a vector space over $\mathbb{C}$. If $\mathcal{B}$ is a fundamental system of 0-neighborhoods of $E(\mathbb{R})$ consisting of convex balanced sets, then the family

$$\mathcal{B}(\mathbb{C}) = \{\lambda(x, y); x \in V_1, y \in V_2, |\lambda| \leq 1\}_{V_1, V_2 \in \mathcal{B}}$$

forms a fundamental system of 0-neighborhoods of $E(\mathbb{R}) \times E(\mathbb{R})$ consisting of convex balanced sets (see, J. E. Colombeau [1]). We denote by $E(\mathbb{C})$ the vector space $E(\mathbb{R}) \times E(\mathbb{R})$ equipped with the topology defined by $\mathcal{B}(\mathbb{C})$. Then $E(\mathbb{C})$ is also a locally convex Hausdorff space over $\mathbb{C}$, and $E(\mathbb{C})$ is called the complexification of $E(\mathbb{R})$.

Proposition 3.6. *Let E be a locally convex space over $\mathbb{R}$, G be on open subset of E and $E(\mathbb{C})$ be the complexification of E. If Ω is an open subset of $E(\mathbb{C})$ with $G \subset \Omega$, then G is not a unipolar set in Ω.*

Proof. Assume that G is unipolar in Ω. Then there exists a complex line L in E with $L \cap G \neq \emptyset$. Since $G \cap L \subset E$, we have

$$L \cap G = L \cap E \cap G \subset L \cap E \subset L \cap E(\mathbb{C}).$$

Since $L \cap G \neq \emptyset$ and $L \cap G \not\subseteq L \cap \Omega, L \cap G$ is a polar set in $L \cap \Omega$. This contradicts Lemma 3.5. ■

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