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A LARGE DEVIATION PRINCIPLE OF REFLECTING DIFFUSIONS

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Abstract. In this paper, we will prove that the solution of stochastic differential equation with a small diffusion coefficient in a nonsmooth domain normally reflected at boundary satisfies a large deviation principle and converges to a deterministic path in L^p.

1. Introduction

Let D be a domain in $\mathbb{R}^d$ and for each $x \in \partial D$, let

$$\mathcal{N}_{x,r} = \{n \in \mathbb{R}^d : |n| = 1,\ B(x - rn, r) \cap D = \emptyset\},$$
$$\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r},$$

where

$$B(z,r) = \{y \in \mathbb{R}^d : |y - z| < r\}, \quad z \in \mathbb{R}^d, r > 0.$$

Assume D satisfies the following two conditions:

(A) There exists $r_0 > 0$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$.

(B) There exist constants $\alpha > 0$ and $\beta > 0$ such that for any $x \in \partial D$, there is a unit vector ℓ_x such that

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$$(\ell_x, \mathbf{n}) \geq \beta, \quad \forall\, \mathbf{n} \in \bigcup_{y \in B(x,\alpha)} \mathcal{N}_y,$$

where $(\, , \,)$ is the usual inner product in $\mathbb{R}^d$.

For $T > 0$, let $C = C[0,T] = \{f : f$ is a continuous map from $[0,T]$ to $\mathbb{R}^d$ such that $f(0) \in \overline{D}\}, \overline{C} = \overline{C}[0,T] = \{f : f$ is a continuous map from $[0,T]$ to $\overline{D}\}$, let $\|f\|_T = \sup_{s \in [0,T]} |f(s)|$. Let $BV = \{f : f$ is a continuous map from $[0,T]$ to $\mathbb{R}^d$ with bounded variation, $f(0) = 0\}$. For $f \in BV$, let $|f|_t =$ total variation of f in the interval $[0,t] \subset [0,T]$. For $\phi \in C, \psi \in \overline{C}, \eta \in BV, (\phi, \psi, \eta)$ are *associated* if

(i) $\psi(t) = \phi(t) + \eta(t), \quad \forall t \in [0,T]$, and

(ii) $\eta(t) = \int_0^t \mathbf{n}_s d|\eta|_s, |\eta|_t = \int_0^t 1_{\partial D}(\psi(s)) d|\eta|_s,$

where $1_{\partial D}$ is the indicator function, $\mathbf{n}_s \in \mathcal{N}_{\psi(s)}$.

Given a $\phi \in C$, Tanaka [8] showed that there is a unique pair $(\psi, \eta), \psi \in \overline{C}$, $\eta \in BV$, such that (ϕ, ψ, η) are associated whenever D is convex (hence condition (A) is automatically satisfied for $r_0 = \infty$). Lions and Sznitman [5] extended Tanaka's results to the case when D is C^2. Saisho [6] relaxed their conditions to (A) and (B).

The Skorohod equation has a stochastic counterpart. Let $\sigma : \overline{D} \to \mathbb{R}^d \otimes \mathbb{R}^d, b : \overline{D} \to \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Suppose $(B_t)_{t \geq 0}$ is a d-dimensional $\mathcal{F}_t$-Brownian motion. Consider the problem of solving the following stochastic differential equation (called the *Skorohod SDE*): find a $\overline{D}$-valued continuous $\mathcal{F}_t$- semimartingale $(X_t)_{t \geq 0}$ and a continuous bounded variation process $(\eta_t)_{t \geq 0}$ such that

$$X_t = x_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds + \eta_t,$$
$$|\eta|_t = \int_0^t 1_{\partial D}(X_s) d|\eta|_s, \quad \eta_t = \int_0^t \mathbf{n}_s d|\eta|_s, \ \mathbf{n}_s \in \mathcal{N}_{X_s}.$$

Saisho [6] showed that the Skorohod SDE has a pathwise unique solution under the following condition:

(C) Both σ and b are bounded and uniformly Lipschitz; namely, there is an absolute constant $c > 0$ such that
$$|\sigma(x)| + |b(x)| \leq c,$$
$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq c|x - y|,$$
for every $x, y \in \overline{D}$.

In other words, $(x_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, X_t, \eta_t)$ are associated for almost all paths.

2. Large Deviations

Now, replace σ by $\sqrt{\varepsilon}\sigma$ and let $(X_t^\varepsilon)_{t\geq 0}$ be a solution of the corresponding Skorohod SDE. We will show X_t^ε satisfies a large deviation principle. First, for an associated triple (ϕ, ψ, η), let $\psi = F(\phi)$. It is known that F is continuous (see Lions and Sznitman [5] and Saisho [6]). In addition to the conditions (A), (B), (C), we will need the following condition:

(D) The matrix $a(x) = \sigma(x)\sigma'(x)$ (the '' means transpose) is uniformly elliptic; namely, there is a $\lambda > 0$ such that

$$a(x) \geq \lambda |x|^2, \quad \forall x \in \overline{D}.$$

Given σ, b satisfying conditions (C) and (D), let

$$S(\psi) = \frac{1}{2} \inf_{\phi \in F^{-1}(\psi)} \int_0^T \left(\dot{\phi}(s) - b(\psi(s))\right)' a^{-1}(\psi(s)) \left(\dot{\phi}(s) - b(\psi(s))\right) ds$$

for $\psi \in \overline{C}[0,T]$, with the understanding that $S(\psi) = \infty$ if $F^{-1}(\psi)$ is empty or $\phi(s)$ is not absolutely continuous (ϕ is the derivative of ϕ). The following lemma collects some simple facts about $S(\psi)$; see Stroock [6] or Varadhan [8].

Lemma 2.1. *We have*

1. $S(\psi)$ *is lower semi-continuous in* ψ,
2. $\{\psi \in \overline{C}[0,T] : S(\psi) \leq h\}$ *is compact for each* $h \geq 0$,
3. *If* $S(\psi) < \infty$, *then there is a* $\phi \in C[0,T]$ *such that* $F(\phi) = \psi$ *and*

$$S(\psi) = \frac{1}{2} \int_0^T \left(\dot{\phi}(s) - b(\psi(s))\right)' a^{-1}(\psi(s)) \left(\dot{\phi}(s) - b(\psi(s))\right) ds.$$

Theorem 2.2. $X^\varepsilon = (X_t^\varepsilon)$ *satisfies a large deviation principle with rate function* $S(\psi)$. *That is, for every Borel set* $A \subseteq \overline{C}[0,T]$, *we have*

$$\overline{\lim}_{\varepsilon\downarrow 0} \varepsilon \log P(X^\varepsilon \in A) \leq - \inf_{\psi \in \overline{A}} S(\psi)$$

and

$$\underline{\lim}_{\varepsilon\downarrow 0} \varepsilon \log P(X^\varepsilon \in A) \geq - \inf_{\psi \in A^\circ} S(\psi),$$

where $\overline{A}$ *is the closure of* A, A° *the interior of* A.

Proof. This follows directly from the so-called contraction principle and a large deviation principle for diffusions (see Stroock [7] or Varadhan [9], p. 5).

Similar results were obtained by Anderson and Orey [2], Doss and Priouret [3] when D is smooth, and by Dupuis [4] when D is convex.

Lemma 2.3. *If (ϕ, ψ, η) are associated and if ϕ is Hölder continuous: $|\phi(t) - \phi(s)| \le K|t-s|^\gamma, 0 < \gamma \le 1, K > 0$, then*

$$|\eta|_t \le K e^{k_1 \|\phi\|_t + k_2},$$

where k_1, k_2 are positive constants, k_1 depends only on $r_0, \alpha, \beta, \gamma; k_2$ depends only on $r_0, \alpha, \beta, \gamma$ and t; r_0 comes from condition (A); α and β come from condition (B).

Proof. By Theorem 4.2 in Saisho [5], we have

$$|\eta|_t \le k \sup_{0 \le s_1 < s_2 \le t} |\phi(s_1) - \phi(s_2)|,$$

where k is a positive constant depending only on $r_0, \alpha, \beta, t, \|\phi\|_t$, and the modulus of uniform continuity of ϕ on $[0, t]$. In fact, by looking carefully at Saisho's proof and using the Hölder continuity of ϕ, one sees k can be written as

$$k = K e^{k_1 \|\phi\|_t + k_2},$$

where $k_1 = k_1(r_0, \alpha, \beta, \gamma) > 0, k_2 = k_2(r_0, \alpha, \beta, \gamma, t) > 0$. Hence the result follows.

$\psi \in \overline{C}$ is called an *equilibrium point* of S if $S(\psi) = 0$.

Theorem 2.4. *Let $X^\varepsilon = (X_t^\varepsilon)_{t \ge 0}$ be a solution of Skorohod SDE and ψ an equilibrium point. Then for each $p, 0 < p < \infty$,* each $t > 0$, *we have*

$$E\|X^\varepsilon - \psi\|_t^p \to 0 \quad as \quad \varepsilon \to 0.$$

Proof. Let $X^\varepsilon = (X_t^\varepsilon), Y = (Y_t^\varepsilon)$, where

$$X_t^\varepsilon = Y_t^\varepsilon + \eta_t, \quad Y_t^\varepsilon = x_0 + \int_0^t \sqrt{\varepsilon} \sigma(X_s) dB_s + \int_0^t b(X_s) ds.$$

By Theorem 2.2, X^ε converges exponentially fast to ψ in probability as $\varepsilon \to 0$ and hence it is enough to show $\|X^\varepsilon\|_t^p$ is uniformly integrable in ε. We know for any positive number $k > 0$, by condition (C),

$$\sup_{0 \le \varepsilon \le 1} E\left(e^{k\|Y^\varepsilon\|_t}\right) < \infty.$$

By the well-known Borel inequality (see, e.g., Adler [1] p. 43), we know if

$$Z = \sup_{0 \le s_1 < s_2 \le t} \frac{|B_{s_2} - B_{s_1}|}{|s_2 - s_1|^{1/3}},$$

then

$$E(Z^p) < \infty \text{ for each } p, \quad 0 < p < \infty.$$

Since σ and b are bounded, for almost all ω we have

$$\left|Y^\varepsilon_{s_2}(\omega) - Y^\varepsilon_{s_1}(\omega)\right| \le K(\omega)|s_2 - s_1|^{1/3}, \quad 0 \le s_1 < s_2 \le t,$$

with

$$E(K^p) < \infty, \quad 0 < p < \infty.$$

Then by Lemma 2.3, we get for $0 < p < \infty$, $\sup\limits_{0 \le \varepsilon \le 1} E(|\eta|_t^p) < \infty$. Obviously, $\sup\limits_{0 \le \varepsilon \le 1} E\|Y^\varepsilon\|_t^p < \infty$. Hence, $\sup\limits_{0 \le \varepsilon \le 1} E\|X^\varepsilon\|_t^p < \infty$, $\forall p > 0$, which implies $\|X^\varepsilon\|_t^{p'}$ is uniformly integrable $\forall p'$, $0 < p' < p$. But this is enough since p is any positive number.

Remark 2.5. ψ is an equilibrium point if and only if

$$\begin{aligned} \psi(t) &= x_0 + \int_0^t b(\psi(s))ds + \eta(t), \\ \eta(t) &= \int_0^t \mathbf{n}_s d|\eta|_s, \qquad \mathbf{n}_s \in \mathcal{N}_{\psi(s)}, \\ |\eta|_t &= \int_0^t 1_{\partial D}(\psi(x))d|\eta|_s. \end{aligned}$$

Therefore, ψ is a solution of the following problem: a particle starts initially at $x_0 \in \overline{D}$. It moves according to the velocity field $b(x), x \in \overline{D}$. Whenever it reaches the boundary, it bounces back normally. Such ψ exists uniquely because it is a special care corresponding to $\sigma = 0$ in the Skorohod SDE.

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