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ESTIMATION OF PARAMETERS OF A LOGNORMAL DISTRIBUTION

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Abstract. To estimate the mean and the variance of a lognormal distribution, Finney (1941) derived the uniformly minimum variance unbiased estimators ($UMVUE$) in the form of infinite series. In this paper, we give an alternative derivation of the $UMVUE$s, and also obtain them in integral forms.

1. Introduction

Assume that $X_1, \ldots, X_n$ are independent random variables each having the same lognormal distribution with mean θ and variance η^2, both being unknown. To estimate θ and η^2, the usual approach is to use the transformation $Y_i = \ln X_i$, $i = 1, \ldots, n$, and the problem is reduced to that of estimation of parameters of a normal distribution. Suppose that Y_i is distributed as $N(\mu, \sigma^2)$ with mean μ and variance σ^2 so that

$$\begin{aligned} \theta &= \exp(\mu + \sigma^2/2), \\ \eta^2 &= \exp(2\mu + \sigma^2)\{\exp(\sigma^2) - 1\}. \end{aligned}$$

Obviously, $\bar{Y} = \sum_1^n Y_i/n$ and $S_Y^2 = \sum_1^n (Y_i - \bar{Y})^2$ are jointly sufficient and complete for μ and σ^2. We can get the maximum likelihood estimators (MLE) of θ and η^2 as

$$\begin{aligned} \tilde{\theta}_{MLE} &= \exp\left(\bar{Y} + \frac{1}{2n} S_Y^2\right), \\ \tilde{\eta}^2_{MLE} &= \exp\left(2\bar{Y} + \frac{1}{n} S_Y^2\right)\left\{\exp\left(\frac{1}{n} S_Y^2\right) - 1\right\}, \end{aligned}$$

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respectively. However, $\tilde{\theta}_{MLE}$ and $\tilde{\eta}^2_{MLE}$ are biased since

$$\begin{aligned} E(\tilde{\theta}_{MLE}) &= \theta \exp\Big\{-\frac{n-1}{n}\frac{\sigma^2}{2}\Big\}\Big(1-\frac{\sigma^2}{n}\Big)^{-(n-1)/2}, \\ E(\tilde{\eta}^2_{MLE}) &= \eta^2 \exp\Big\{\Big(\frac{2}{n}-1\Big)\sigma^2\Big\}\{\exp(\sigma^2)-1\}^{-1} \\ &\quad \cdot \Big\{\Big(1-\frac{4\sigma^2}{n}\Big)^{-(n-1)/2} - \Big(1-\frac{2\sigma^2}{n}\Big)^{-(n-1)/2}\Big\}. \end{aligned}$$

Finney (1941) defined the series

$$f(t) = 1 + t + \frac{n-1}{n+1}\frac{t^2}{2!} + \frac{(n-1)^2}{(n+1)(n+3)}\frac{t^3}{3!} + \cdots \tag{1.1}$$

and obtain the adjusted MLEs

$$\hat{\theta}_{MLE} = \exp(\bar{Y}) f\left(\frac{1}{2n}S_Y^2\right), \tag{1.2}$$

$$\hat{\eta}^2_{MLE} = \exp(2\bar{Y})\left\{f\left(\frac{2}{n}S_Y^2\right) - f\left(\frac{n-2}{n(n-1)}S_Y^2\right)\right\}, \tag{1.3}$$

which are unbiased for θ and η^2, and with asymptotic variances, respectively,

$$\begin{aligned} \mathrm{var}(\hat{\theta}_{MLE}) &\sim \frac{1}{n}\left(\sigma^2 + \frac{1}{2}\sigma^4\right)\exp(2\mu+\sigma^2), \\ \mathrm{var}(\hat{\eta}^2_{MLE}) &\sim \frac{2\sigma^2}{n}\exp(4\mu+2\sigma^2) \\ &\quad \cdot \{2[\exp(\sigma^2)-1]^2 + \sigma^2[2\exp(\sigma^2)-1]^2\}. \end{aligned}$$

By the Lehmann-Scheffé theorem, $\hat{\theta}_{MLE}$ and $\hat{\eta}^2_{MLE}$ are the uniformly minimum variance unbiased estimators ($UMVUE$) of θ and η^2, respectively.

Remark. The conditions $\sigma^2 < n$ and $\sigma^2 < n/4$ for computing $E(\hat{\theta}_{MLE})$ and $E(\hat{\eta}^2_{MLE})$ are missing in Finney (1941) and Kendall and Stuart (1979).

In this paper, we combine an orthogonal transformation and the Rao-Blackwell theorem to give an alternative derivation of the $UMVUE$s of θ and η^2, and also obtain $UMVUE$s in integral forms.

2. Preliminaries

In this section, we introduce the famous Helmert orthogonal transformation and some results which are needed in the sequel.

Assume, without loss of generality, $Y_1, \ldots, Y_n$, are distributed as standard normal. Let $\mathbf{y}' = (Y_1, \ldots, Y_n)$ and $\mathbf{z}' = (Z_1, \ldots, Z_n)$ be $n \times 1$ vectors. Define the orthogonal transformation by $\mathbf{z} = \Gamma \mathbf{y}$ where

$$\Gamma = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{1\cdot 2}} & \frac{-1}{\sqrt{1\cdot 2}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{-2}{\sqrt{2\cdot 3}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \cdots & \frac{-(n-1)}{\sqrt{(n-1)n}} \end{bmatrix}$$

is $n \times n$ Helmert orthogonal matrix and the absolute value of the Jacobian of the transformation is one. Since $\mathbf{z}'\mathbf{z} = \mathbf{y}'\mathbf{y}$, the joint pdf of $Z_1, \ldots, Z_n$ is $f(Z_1, \ldots, Z_n) = (2\pi)^{-n/2} \exp\{-\mathbf{z}'\mathbf{z}/2\}$. Therefore, $Z_1, \ldots, Z_n$ are *iid* $N(0,1)$.

Lemma 2.1. *Let X and Y be two independent random variables distributed as standard normal and chi-squared with k (≥ 1) d.f., respectively. Define $V = X/\sqrt{X^2+Y}$. Then the pdf of V is*

$$f(v) = \left[B\left(\frac{1}{2}, \frac{k}{2}\right)\right]^{-1} (1-v^2)^{(k-2)/2}, \qquad -1 < v < 1;$$

and $E(V^{2m+1}) = 0, m = 0, 1, 2, \ldots$.

Proof. The joint pdf of X and Y is

$$f(x,y) = c\exp\{-x^2/2\} y^{(k-2)/2} \exp\{-y/2\},$$

where $c = [\sqrt{\pi}(\sqrt{2})^{k+1}\Gamma(\frac{k}{2})]^{-1}$. Define $V = X/\sqrt{X^2+Y}$ and $W = Y$. Then the Jacobian of the transformation is $w^{1/2}(1-v^2)^{-3/2}$. Hence, the joint pdf of V and W is

$$f(v,w) = c(1-v^2)^{-3/2} \exp\left\{-\frac{1}{2}\frac{w}{1-v^2}\right\} w^{(k-1)/2}.$$

Therefore, the marginal pdf of V is

$$\begin{aligned} f(v) &= c(1-v^2)^{-3/2} \int_0^\infty \exp\left\{-\frac{1}{2}\frac{w}{1-v^2}\right\} w^{(k-1)/2} dw \\ &= c^*(1-v^2)^{(k-2)/2}, \qquad -1 < v < 1, \end{aligned}$$

where $c^* = \Gamma(\frac{k+1}{2})/\sqrt{\pi}\Gamma(\frac{k}{2})$. Furthermore,

$$E(V^{2m+1}) = c^* \int_{-1}^{1} v^{2m+1}(1-v^2)^{(k-2)/2} dv = 0$$

for $m = 0, 1, 2, \ldots$, since the integrand is an odd function. ■

3. UMVUE of θ

Since $\bar{Y}$ and S_Y^2 are jointly sufficient and complete for μ and σ^2, and $\exp(Y_n)$ is an unbiased estimator of θ, then by the Rao-Blackwell theorem, $E\{\exp(Y_n)|\bar{Y}, S_Y^2\}$ is the *UMVUE* of θ.

Note that $z_1^2 = n(\bar{Y})^2$, $S_Y^2 = \sum_2^n z_i^2$ and $Y_n - \bar{Y} = -\sqrt{\frac{n-1}{n}} z_n$. Then

$$\frac{Y_n - \bar{Y}}{S_Y} = -\sqrt{\frac{n-1}{n}} z_n / \left(\sum_{i=2}^{n} z_i^2\right)^{1/2} = -\sqrt{\frac{n-1}{n}} U,$$

where $U = z_n/(\sum_2^n z_i^2)^{1/2} = z_n/(z_n^2 + \sum_{i=2}^{n-1} z_i^2)^{1/2}$ and, by Lemma 2.1, the pdf of U is

$$f(u) = \left[B\left(\frac{1}{2}, \frac{n-2}{2}\right)\right]^{-1} (1-u^2)^{(n-4)/2}, \qquad -1 < u < 1. \tag{3.1}$$

Furthermore, by Basu's theorem (Lehmann, 1983), $(Y_n - \bar{Y})/S_Y$ is independent of $\bar{Y}$ and S_Y. Now consider

$$\begin{aligned} & E\{\exp(Y_n)|\bar{Y}, S_Y^2\} \\ &= E\left\{\exp\left(\bar{Y} + \frac{Y_n - \bar{Y}}{S_Y} S_Y\right) |\bar{Y}, S_Y^2\right\} \\ &= E\{\exp(\bar{Y})|\bar{Y}, S_Y^2\} E\left\{\exp\left(-\sqrt{\frac{n-1}{n}} S_Y U\right) |\bar{Y}, S_Y^2\right\} \\ &= \exp(\bar{Y}) E\left\{\exp\left(-\sqrt{\frac{n-1}{n}} S_Y U\right)\right\}. \end{aligned} \tag{3.2}$$

Since the expectation in (3.2) is conditional on S_Y, we have, by expanding $E\{\exp(-\sqrt{\frac{n-1}{n}} S_Y U)\}$ in infinite series and applying the second part of Lemma 2.1,

$$E\left\{\exp\left(-\sqrt{\frac{n-1}{n}}S_Y U\right)\right\} = \sum_{m=0}^{\infty}\frac{(-\sqrt{\frac{n-1}{n}}S_Y)^{2m}}{(2m)!}E(U^{2m})$$
$$= \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(\frac{n-1}{n}S_Y^2)^m}{(2m)!}\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{n-1}{2}+m)}$$

since U^2 is distributed as Beta$(\frac{1}{2}, \frac{n-2}{2})$. Note that

$$\frac{\Gamma(\frac{2m+1}{2})}{\sqrt{\pi}(2m)!} = (m!2^{2m})^{-1},$$
$$\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-1}{2}+m)} = \frac{2^m}{(n-1)(n+1)(n+3)\cdots(n-1+2m-2)},$$
$$\left(\frac{n-1}{n}S_Y^2\right)^m = \left(\frac{1}{2n}S_Y^2\right)^m 2^m(n-1)^m,$$

and thus

$$E\left\{\exp\left(-\sqrt{\frac{n-1}{n}}S_Y U\right)\right\}$$
$$= 1 + \sum_{m=1}^{\infty}\frac{\Gamma(\frac{n-1}{2})(\frac{n-1}{n}S_Y^2)^m\Gamma(\frac{1}{2}+m)}{\sqrt{\pi}(2m)!\Gamma(\frac{n-1}{2}+m)}$$
$$= 1 + \sum_{m=1}^{\infty}\frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{2m+1}{2})(\frac{1}{2n}S_Y^2)^m 2^m(n-1)^m}{\Gamma(\frac{n-1}{2}+m)\sqrt{\pi}(2m)!}$$
$$= 1 + \sum_{m=1}^{\infty}\left(\frac{1}{2n}S_Y^2\right)^m\frac{1}{m!}\frac{(n-1)^m}{(n-1)(n+1)\cdots(n+2m-3)}$$
$$= f\left(\frac{1}{2n}S_Y^2\right),$$

where f is as in (1.1). Hence, the $UMVUE$ of θ derived by using the Rao-Blackwell theorem is indeed identical to $\hat{\theta}_{MLE}$ in (1.2).

Furthermore, we may use (3.1) to compute the expectation on the right hand side of (3.2) and then obtain an $UMVUE$ of θ in an integral form as

$$\hat{\theta}^*_{UMVUE} = E\{\exp(Y_n)|\bar{Y}, S_Y^2\} = \exp(\bar{Y})E\left\{\exp\left(-\sqrt{\frac{n-1}{n}}S_Y U\right)\right\}$$
$$= \exp(\bar{Y})\left[B\left(\frac{1}{2}, \frac{n-2}{2}\right)\right]^{-1}\int_{-1}^{1}\exp\left(-\sqrt{\frac{n-1}{n}}S_Y u\right)(1-u^2)^{(n-4)/2}du.$$

4. UMVUE of η^2

Since $\exp(2Y_n) - \exp(Y_n + Y_{n-1})$ is an unbiased estimator of η^2, so by the Rao-Blackwell theorem, $E\{\exp(2Y_n) - \exp(Y_n + Y_{n-1})|\bar{Y}, S_Y^2\}$ is an $UMVUE$ of η^2. Now we use the same idea as in the previous section to obtain the $UMVUE$ of η^2. Clearly

$$
\begin{aligned}
&E\{\exp(2Y_n)|\bar{Y}, S_Y^2\} \\
&= E\left\{\exp\left(2\bar{Y} - 2\sqrt{\frac{n-1}{n}}S_Y U\right)|\bar{Y}, S_Y^2\right\} \\
&= \exp(2\bar{Y})E\left\{\exp\left(-2\sqrt{\frac{n-1}{n}}S_Y U\right)\right\} \qquad (4.1)\\
&= \exp(2\bar{Y})\frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(4\frac{n-1}{n}S_Y^2)^m}{(2m)!}\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{n-1}{2}+m)} \\
&= \exp(2\bar{Y})\left[B\left(\frac{1}{2}, \frac{n-2}{2}\right)\right]^{-1} \\
&\quad \cdot \int_{-1}^{1}\exp\left(-2\sqrt{\frac{n-1}{n}}S_Y u\right)(1-u^2)^{(n-4)/2}du. \qquad (4.2)
\end{aligned}
$$

For the second term, note that

$$
\begin{aligned}
Y_n + Y_{n-1} - 2\bar{Y} &= \frac{-(n-2)}{\sqrt{n(n-1)}}Z_n - \sqrt{\frac{n-2}{n-1}}Z_{n-1} \\
&= aZ_n + bZ_{n-1},
\end{aligned}
$$

where $a = \frac{-(n-2)}{\sqrt{n(n-1)}}$ and $b = -\sqrt{\frac{n-2}{n-1}}$, and let

$$V = (Y_n + Y_{n-1} - 2\bar{Y})/S_Y = (aZ_n + bZ_{n-1})/S_Y.$$

Then, by Basu's theorem, V is independent of $\bar{Y}$ and S_Y. Hence, we have

$$E\{\exp(Y_n + Y_{n-1})|\bar{Y}, S_Y^2\} = \exp(2\bar{Y})E\{\exp(S_Y V)|\bar{Y}, S_Y^2\}. \qquad (4.3)$$

Now, consider the orthogonal transformation such that $Z_2^* = (aZ_n + bZ_{n-1})/\sqrt{a^2+b^2}$ and $\sum_2^n Z_i^2 = \sum_2^n Z_i^{*2}$. Hence, $Z_2^*, \ldots, Z_n^*$ are iid $N(0,1)$. Then $V = \sqrt{a^2+b^2}Z_2^*/(\sum_2^n Z_i^{*2})^{1/2}$ and, similarly, we have

$$
\begin{aligned}
&E\{\exp(S_Y V)|\bar{Y}, S_Y^2\} = E\{\exp(S_Y V)\} \\
&= \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(2\frac{n-2}{n}S_Y^2)^m}{(2m)!}\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{n-1}{2}+m)} \qquad (4.4)
\end{aligned}
$$

$$(4.5) \quad = \left[B\left(\frac{1}{2}, \frac{n-2}{2}\right)\right]^{-1} \int_{-1}^{1} \exp\left(\sqrt{\frac{2(n-2)}{n}} S_Y v\right)(1-v^2)^{(n-4)/2} dv.$$

Finally, combining (4.1), (4.3), (4.4) and similar equalities used in deriving $\hat{\theta}_{UMVUE}$, we get the $UMVUE$ of η^2 in the form of an infinite series as

$$\begin{aligned}
&\hat{\eta}^{2*}_{UMVUE} = E\{\exp(2Y_n) - \exp(Y_n + Y_{n-1}) | \bar{Y}, S_Y^2\} \\
&= \exp(2\bar{Y}) \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(\frac{2}{n} S_Y^2)^m}{(2m)!} \frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{n-1}{2}+m)} [(2n-2)^m - (n-2)^m] \\
&= \exp(2\bar{Y}) \left\{ f\left(\tfrac{2}{n} S_Y^2\right) - f\left(\tfrac{n-2}{n(n-1)} S_Y^2\right) \right\},
\end{aligned}$$

where f is as in (1.1). Hence, the $UMVUE$ of η^2 derived by using the Rao-Blackwell theorem is indeed identical to $\hat{\eta}^2_{MLE}$ in (1.3). Or, combining (4.2), (4.3) and (4.5), we have the $UMVUE$ of η^2 in an integral form as

$$\begin{aligned}
\hat{\eta}^{2*}_{UMVUE} \quad = \quad & \exp(2\bar{Y}) \left[B\left(\frac{1}{2}, \frac{n-2}{2}\right)\right]^{-1} \\
& \cdot \int_{-1}^{1} \left[\exp\left(-2\sqrt{\frac{n-1}{n}} S_Y v\right)\right. \\
& \left. - \exp\left(\sqrt{\frac{2(n-2)}{n}} S_Y v\right)\right] (1-v^2)^{(n-4)/2} dv.
\end{aligned}$$

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