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A STABILITY ANALYSIS FOR VIBRATING VISCOELASTIC SPHERICAL SHELLS
(The Effect of Damping on the LBB Constants of Vibration Problems)

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Abstract. This paper is a continuation of [3-6] and it is devoted to a stability analysis of multilayered vibrating viscoelastic spheres, both in vacuo and in an acoustical fluid. The analysis is done by investigating the effect of viscoelastic damping on the (continuous) Ladyzenskaya-Babuška-Brezzi (LBB) constants for the related boundary-value problems. The sphere is modeled using both 3-D viscoelasticity and the Kirchhoff-Love shell theory.

1. Introduction

The present study has been motivated by the numerical modeling of fluid-structure acoustic interaction problems presented in [7]. Based on the numerical experiments, it was found experimentally and analyzed theoretically in [5, 6] that, for a typical data (a thin steel shell in water) the radiation damping might be insufficient to guarantee stability (and, therefore, convergence) of the numerical simulations.

The natural choice in such a situation is to add to the model some damping phenomenon. Modeling of damping for real-life complex structures is a very complicated and unresolved issue (see, e.g. [12]) and in the present study we resort to the simplest, linear viscoelastic laws. The stability effect of the viscoelastic damping will be measured through the evaluation of the LBB constants for the appropriate boundary-value problems.

The investigations are done in context of spherical shells only. The spherical geometry allows for a modal decoupling and makes the whole analysis

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possible. In particular, we model the shells using both 3-D viscoelasticity and Kirchhoff-Love shell theories. As a by-product of the stability consideration we construct the exact 3-D solutions to the viscoelastic vibration and scattering problems. The solutions are indispensable in validating the numerical simulations mentioned earlier. In this aspect, the work is a continuation of [3, 4].

The plan of the paper is as follows. In Section 2, we review first the exact solutions for vibrating hollow viscoelastic spheres using the 3-D theory, and use it to construct the solution of the scattering problem. The stability considerations are presented in Section 3. After a short theoretical introduction, we perform the stability analysis for a viscoelastic spherical shell using first the shell theory and next the full 3-D theory. The paragraph is concluded with a generalization for multilayered shells. Finally, Section 4 is devoted to various numerical experiments investigating the effect of viscous damping on stability.

2. Steady-State Vibrations of a Viscoelastic Sphere

In this section, we consider several problems concerning steady-state vibrations of a 3-D viscoelastic hollow sphere. By means of the Helmholtz potentials, solutions of the steady-state form of the Navier equations for the sphere can be reduced to the solution of equivalent, three decoupled reduced wave equations for three Helmholtz potentials Φ, Ψ, and χ:

$$- c_l^{*2}\nabla^2\Phi - \omega^2\Phi = 0, \tag{2.1}$$

$$- c_s^{*2}\nabla^2\Psi - \omega^2\Psi = 0, \tag{2.2}$$

$$- c_s^{*2}\nabla^2\chi - \omega^2\chi = 0. \tag{2.3}$$

Here ω is the frequency of the vibrations and c_l^* and c_s^* are the viscous counterparts of the longitudinal wave velocity and the shear wave velocity, respectively,

$$c_l^* = \sqrt{\frac{\lambda^* + 2\mu^*}{\rho_s}},\ c_s^* = \sqrt{\frac{\mu^*}{\rho_s}} \tag{2.4}$$

with λ^*, μ^* denoting the complex viscoelastic moduli depending upon a particular model and frequency ω, and ρ_s denoting the density of the solid. For instance, for the simplest Kelvin model we have

$$\lambda^* = \lambda_e - i\omega\lambda_v, \tag{2.5}$$

$$\mu^* = \mu_e - i\omega\mu_v. \tag{2.6}$$

Here λ_e and μ_e are the elastic Lame constants and λ_v and μ_v are their viscous counterparts.

We shall restrict ourselves to axisymmetric vibrations only. Referring to [3, 4] for details, we list here only the final formulas

$$\begin{aligned}\sigma_{rr} = \sum_{n=0}^{\infty} \frac{2\mu^*}{r^2}\Big[A_n T_{11}^{(1)}(\alpha^* r) + B_n T_{13}^{(1)}(\beta^* r) + C_n T_{11}^{(2)}(\alpha^* r) \\ + D_n T_{13}^{(2)}(\beta^* r)\Big] P_n(\cos\theta).\end{aligned} \tag{2.7}$$

Following [8], we shall use a simplified notation of the form

$$\sigma_{rr} = \frac{2\mu^*}{r^2}\left[T_{11}^{(i)}(\alpha^* r) + T_{13}^{(i)}(\beta^* r)\right] P_n(\cos\theta) \tag{2.8}$$

with $T_{11}^{(i)}(\alpha^* r)$ replacing $A_n T_{11}^{(1)}(\alpha^* r) + C_n T_{11}^{(2)}(\alpha^* r)$, $T_{13}^{(i)}(\alpha^* r)$ replacing $B_n T_{13}^{(1)}(\beta^* r) + D_n T_{13}^{(2)}(\beta^* r)$, and the summation convention being used. Continuing formulas for the stresses, we have

$$\begin{aligned}\sigma_{\theta\theta} = \frac{2\mu^*}{r^2}\Big\{ & T_{21}^{(i)}(\alpha^* r) P_n(\cos\theta) + \hat{T}_{21}^{(i)}(\alpha^* r)\frac{1}{\sin^2\theta} \\ & \cdot\left[-n\cos^2\theta P_n(\cos\theta) + n\cos\theta P_{n-1}(\cos\theta)\right] \\ & + T_{23}^{(i)}(\beta^* r) P_n(\cos\theta) + \hat{T}_{23}^{(i)}(\beta^* r)\frac{1}{\sin^2\theta} \\ & \cdot\left[(-n\cos^2\theta) P_n(\cos\theta) + n\cos\theta P_{n-1}(\cos\theta)\right]\Big\},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\sigma_{\phi\phi} = \frac{2\mu^*}{r^2}\Big\{ & T_{31}^{(i)}(\alpha^* r) P_n(\cos\theta) + \hat{T}_{31}^{(i)}(\alpha^* r)\frac{1}{\sin^2\theta} \\ & \cdot\left[n\cos^2\theta P_n(\cos\theta) - n\cos\theta P_{n-1}(\cos\theta)\right] \\ & + T_{33}^{(i)}(\beta^* r) P_n(\cos\theta) + \hat{T}_{33}^{(i)}(\beta^* r)\frac{1}{\sin^2\theta} \\ & \cdot\left[(n\cos^2\theta) P_n(\cos\theta) - n\cos\theta P_{n-1}(\cos\theta)\right]\Big\},\end{aligned} \tag{2.10}$$

$$\begin{aligned}\sigma_{r\theta} = \frac{2\mu^*}{r^2}\Bigg\{ & T_{41}^{(i)}(\alpha^* r)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta} P_{n-1}(\cos\theta)\right] \\ & + T_{43}^{(i)}(\beta^* r)\cdot\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta} P_{n-1}(\cos\theta)\right]\Bigg\},\end{aligned} \tag{2.11}$$

$$\sigma_{r\phi} = \sigma_{\theta\phi} = 0, \tag{2.12}$$

where

$$T_{11}^{(i)}(\alpha^* r) = \left(n^2 - n - \frac{1}{2}\beta^{*2} r^2\right) Z_n^{(i)}(\alpha^* r) + 2\alpha^* r Z_{n+1}^{(i)}(\alpha^* r), \tag{2.13}$$

$$T_{13}^{(i)}(\beta^* r) = n(n+1)\left[(n-1) Z_n^{(i)}(\beta^* r) - \beta^* r Z_{n+1}^{(i)}(\beta^* r)\right], \tag{2.14}$$

$$T_{21}^{(i)}(\alpha^* r) = \left(-n^2 - \frac{1}{2}\beta^{*2} r^2 + \alpha^{*2} r^2\right) Z_n^{(i)}(\alpha^* r) - \alpha^* r Z_{n+1}^{(i)}(\alpha^* r), \tag{2.15}$$

$$\hat{T}_{21}^{(i)}(\alpha^* r) = Z_n^{(i)}(\alpha^* r), \tag{2.16}$$

$$T_{23}^{(i)}(\beta^* r) = -(n^2 + n)\left[n Z_n^{(i)}(\beta^* r) - \beta^* r Z_{n+1}^{(i)}(\beta^* r)\right], \tag{2.17}$$

$$\hat{T}_{23}^{(i)}(\beta^* r) = (n+1) Z_n^{(i)}(\beta^* r) - \beta^* r Z_{n+1}^{(i)}(\beta^* r), \tag{2.18}$$

$$T_{31}^{(i)}(\beta^* r) = \left(n - \frac{1}{2}\beta^{*2} r^2 + \alpha^{*2} r^2\right) Z_n^{(i)}(\alpha^* r) - \alpha^* r Z_{n+1}^{(i)}(\alpha^* r), \tag{2.19}$$

$$\hat{T}_{31}^{(i)}(\alpha^* r) = Z_n^{(i)}(\alpha^* r), \tag{2.20}$$

$$T_{33}^{(i)}(\beta^* r) = n(n+1) Z_n^{(i)}(\beta^* r), \tag{2.21}$$

$$\hat{T}_{33}^{(i)}(\beta^* r) = (n+1) Z_n^{(i)}(\beta^* r) - \beta^* r Z_{n+1}^{(i)}(\beta^* r), \tag{2.22}$$

$$T_{41}^{(i)}(\alpha^* r) = (n-1) Z_n^{(i)}(\alpha^* r) - \alpha^* r Z_{n+1}^{(i)}(\alpha^* r), \tag{2.23}$$

$$T_{43}^{(i)}(\beta^* r) = \left(n^2 - 1 - \frac{1}{2}\beta^{*2} r^2\right) Z_n^{(i)}(\beta^* r) + \beta^* r Z_{n+1}^{(i)}(\beta^* r), \tag{2.24}$$

$$\alpha^* = \frac{\omega}{c_l^*},\ \beta^* = \frac{\omega}{c_s^*},\ c_l^* = \sqrt{\frac{\lambda^* + 2\mu^*}{\rho}},\ c_s^* = \sqrt{\frac{\mu^*}{\rho}}, \tag{2.25}$$

$$Z_n^{(1)}(z) \equiv j_n(z) \equiv (\pi/2z)^{1/2} J_{n+1/2}(z), \tag{2.26}$$

$$Z_n^{(2)}(z) \equiv y_n(z) \equiv (\pi/2z)^{1/2} Y_{n+1/2}(z). \tag{2.27}$$

Here r, θ, ϕ are the spherical coordinates, $P_n(\eta)$ are the Legendre polynomials, $j_n(z)$, $J_{n+\frac{1}{2}}(z)$ and $y_n(z)$, $Y_{n+\frac{1}{2}}(z)$ are spherical and regular Bessel functions of first and second kind of complex argument z.

The corresponding displacement field takes the form

$$u_r = \frac{1}{r}\left[U_1^{(i)}(\alpha^* r) + U_3^{(i)}(\beta^* r)\right] P_n(\cos\theta); \tag{2.28}$$

$$u_\theta = \frac{1}{r}\left\{V_1^{(i)}(\alpha^* r) + V_3^{(i)}(\beta^* r)\right\}\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]; \tag{2.29}$$

$$u_\phi = 0, \tag{2.30}$$

where

$$U_1^{(i)}(\alpha^* r) = nZ_n^{(i)}(\alpha^* r) - \alpha^* r Z_{n+1}^{(i)}(\alpha^* r), \tag{2.31}$$

$$U_3^{(i)}(\beta^* r) = n(n+1)Z_n^{(i)}(\beta^* r), \tag{2.32}$$

$$V_1^{(i)}(\alpha^* r) = Z_n^{(i)}(\alpha^* r), \quad V_2^{(i)}(\beta^* r) = Z_n^{(i)}(\beta^* r), \tag{2.33}$$

$$V_3^{(i)}(\beta^* r) = (n+1)Z_n^{(i)}(\beta^* r) - \beta^* r Z_{n+1}^{(i)}(\beta^* r). \tag{2.34}$$

2-1. Free Vibrations in Vacuum

Imposing the traction-free boundary conditions, we obtain the characteristic modal equations for natural frequencies ω:

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & T_{11}^{(2)}(\alpha^* r_o) \\ T_{11}^{(1)}(\alpha^* r_i) & T_{11}^{(2)}(\alpha^* r_i) \end{vmatrix} = 0 \quad \text{for } n = 0, \tag{2.35}$$

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & T_{13}^{(1)}(\beta^* r_o) & T_{11}^{(2)}(\alpha^* r_o) & T_{13}^{(2)}(\beta^* r_o) \\ T_{11}^{(1)}(\alpha^* r_i) & T_{13}^{(1)}(\beta^* r_i) & T_{11}^{(2)}(\alpha^* r_i) & T_{13}^{(2)}(\beta^* r_i) \\ T_{41}^{(1)}(\alpha^* r_o) & T_{43}^{(1)}(\beta^* r_o) & T_{41}^{(2)}(\alpha^* r_o) & T_{43}^{(2)}(\beta^* r_o) \\ T_{41}^{(1)}(\alpha^* r_i) & T_{43}^{(1)}(\beta^* r_i) & T_{41}^{(2)}(\alpha^* r_i) & T_{43}^{(2)}(\beta^* r_i) \end{vmatrix} = 0 \quad \text{for } n > 0. \tag{2.36}$$

Here, r_i and r_o denote inner and outer radii, respectively. In order to obtain the corresponding eigenmodes, the corresponding systems of 2×2 ($n = 0$) or 4×4 ($n > 0$) equations for constants A_n, B_n, C_n, D_n must be solved.

2-2. Forced Vibrations in Vacuum

Expanding the excitation pressure in terms of the Legendre polynomials we reduce again the whole problem, as for the free vibrations, to separate modal systems of equations. The only different equation compared with the free vibrations is as follows:

$$\begin{aligned} \sigma_{rr}\,|_{r=r_0} = {} & \frac{2\mu^*}{r_o^2}\Big[A_n T_{11}^{(1)}(\alpha^* r_o) + B_n T_{13}^{(1)}(\beta r_o) + C_n T_{11}^{(2)}(\alpha^* r_o) \\ & + D_n T_{13}^{(2)}(\beta r_o)\Big] P_n(\cos\theta) \\ = {} & f_n P_n(\cos\theta), \end{aligned} \tag{2.37}$$

where f_n is the nth spectral component of the forcing term.
For $n = 0$, we obtain

$$\begin{aligned} A_0 T_{11}^{(1)}(\alpha^* r_o) + C_0 T_{11}^{(2)}(\alpha^* r_o) &= \frac{r_o^2}{2\mu^*} f_0, \\ A_0 T_{11}^{(1)}(\alpha^* r_i) + C_0 T_{11}^{(2)}(\alpha^* r_i) &= 0. \end{aligned} \tag{2.38}$$

Solving for A_0 and C_0 in (2.38), we obtain

$$A_0 = \frac{\begin{vmatrix} \frac{r_o^2}{2\mu^*} f_0 & T_{11}^{(2)}(\alpha^* r_o) \\ 0 & T_{11}^{(2)}(\alpha^* r_i) \end{vmatrix}}{\begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & T_{11}^{(2)}(\alpha^* r_o) \\ T_{11}^{(1)}(\alpha^* r_i) & T_{11}^{(2)}(\alpha^* r_i) \end{vmatrix}}, \quad C_0 = \frac{\begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & \frac{r_o^2}{2\mu^*} f_0 \\ T_{11}^{(1)}(\alpha^* r_i) & 0 \end{vmatrix}}{\begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & T_{11}^{(2)}(\alpha^* r_o) \\ T_{11}^{(1)}(\alpha^* r_i) & T_{11}^{(2)}(\alpha^* r_i) \end{vmatrix}}. \tag{2.39}$$

For $n > 0$, we have

$$\begin{aligned} A_n T_{11}^{(1)}(\alpha^* r_o) + B_n T_{13}^{(1)}(\beta^* r_o) + C_n T_{11}^{(2)}(\alpha^* r_o) + D_n T_{13}^{(2)}(\beta^* r_o) &= \frac{r_o^2}{2\mu^*} f_n, \\ A_n T_{11}^{(1)}(\alpha^* r_i) + B_n T_{13}^{(1)}(\beta^* r_i) + C_n T_{11}^{(2)}(\alpha^* r_i) + D_n T_{13}^{(2)}(\beta^* r_i) &= 0, \\ A_n T_{41}^{(1)}(\alpha^* r_o) + B_n T_{43}^{(1)}(\beta^* r_o) + C_n T_{41}^{(2)}(\alpha^* r_o) + D_n T_{43}^{(2)}(\beta^* r_o) &= 0, \\ A_n T_{41}^{(1)}(\alpha^* r_i) + B_n T_{43}^{(1)}(\beta^* r_i) + C_n T_{41}^{(2)}(\alpha^* r_i) + D_n T_{43}^{(2)}(\beta^* r_i) &= 0. \end{aligned} \tag{2.40}$$

Solving for coefficients $A_n,\ B_n,\ C_n$ and D_n, we obtain

$$A_n = \frac{\triangle_{n1}}{\triangle_n}, \quad B_n = \frac{\triangle_{n2}}{\triangle_n}, \quad C_n = \frac{\triangle_{n3}}{\triangle_n}, \quad D_n = \frac{\triangle_{n4}}{\triangle_n}, \tag{2.41}$$

where

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & T_{13}^{(1)}(\beta^* r_o) & T_{11}^{(2)}(\alpha^* r_o) & T_{13}^{(2)}(\beta^* r_o) \\ T_{11}^{(1)}(\alpha^* r_i) & T_{13}^{(1)}(\beta^* r_i) & T_{11}^{(2)}(\alpha^* r_i) & T_{13}^{(2)}(\beta^* r_i) \\ T_{41}^{(1)}(\alpha^* r_o) & T_{43}^{(1)}(\beta^* r_o) & T_{41}^{(2)}(\alpha^* r_o) & T_{43}^{(2)}(\beta^* r_o) \\ T_{41}^{(1)}(\alpha^* r_i) & T_{43}^{(1)}(\beta^* r_i) & T_{41}^{(2)}(\alpha^* r_i) & T_{43}^{(2)}(\beta^* r_i) \end{vmatrix}, \tag{2.42}$$

$$\triangle_{n1} = \begin{vmatrix} \frac{r_o^2}{2\mu^*} f_n & T_{13}^{(1)}(\beta^* r_o) & T_{11}^{(2)}(\alpha^* r_o) & T_{13}^{(2)}(\beta^* r_o) \\ 0 & T_{13}^{(1)}(\beta^* r_i) & T_{11}^{(2)}(\alpha^* r_i) & T_{13}^{(2)}(\beta^* r_i) \\ 0 & T_{43}^{(1)}(\beta^* r_o) & T_{41}^{(2)}(\alpha^* r_o) & T_{43}^{(2)}(\beta^* r_o) \\ 0 & T_{43}^{(1)}(\beta^* r_i) & T_{41}^{(2)}(\alpha^* r_i) & T_{43}^{(2)}(\beta^* r_i) \end{vmatrix}, \tag{2.43}$$

$$\triangle_{n2} = \begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & \frac{r_o^2}{2\mu^*} f_n & T_{11}^{(2)}(\alpha^* r_o) & T_{13}^{(2)}(\beta^* r_o) \\ T_{11}^{(1)}(\alpha^* r_i) & 0 & T_{11}^{(2)}(\alpha^* r_i) & T_{13}^{(2)}(\beta^* r_i) \\ T_{41}^{(1)}(\alpha^* r_o) & 0 & T_{41}^{(2)}(\alpha^* r_o) & T_{43}^{(2)}(\beta^* r_o) \\ T_{41}^{(1)}(\alpha^* r_i) & 0 & T_{41}^{(2)}(\alpha^* r_i) & T_{43}^{(2)}(\beta^* r_i) \end{vmatrix}, \tag{2.44}$$

$$\triangle_{n3} = \begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & T_{13}^{(1)}(\beta^* r_o) & \frac{r_o^2}{2\mu^*} f_n & T_{13}^{(2)}(\beta^* r_o) \\ T_{11}^{(1)}(\alpha^* r_i) & T_{13}^{(1)}(\beta^* r_i) & 0 & T_{13}^{(2)}(\beta^* r_i) \\ T_{41}^{(1)}(\alpha^* r_o) & T_{43}^{(1)}(\beta^* r_o) & 0 & T_{43}^{(2)}(\beta^* r_o) \\ T_{41}^{(1)}(\alpha^* r_i) & T_{43}^{(1)}(\beta^* r_i) & 0 & T_{43}^{(2)}(\beta^* r_i) \end{vmatrix}, \tag{2.45}$$

$$\triangle_{n4} = \begin{vmatrix} T_{11}^{(1)}(\alpha^* r_o) & T_{13}^{(1)}(\beta^* r_o) & T_{11}^{(2)}(\alpha^* r_o) & \frac{r_o^2}{2\mu^*} f_n \\ T_{11}^{(1)}(\alpha^* r_i) & T_{13}^{(1)}(\beta^* r_i) & T_{11}^{(2)}(\alpha^* r_i) & 0 \\ T_{41}^{(1)}(\alpha^* r_o) & T_{43}^{(1)}(\beta^* r_o) & T_{41}^{(2)}(\alpha^* r_o) & 0 \\ T_{41}^{(1)}(\alpha^* r_i) & T_{43}^{(1)}(\beta^* r_i) & T_{41}^{(2)}(\alpha^* r_i) & 0 \end{vmatrix}. \tag{2.46}$$

It is useful to define a modal mechanical impedance in vacuum for the 3-D theory as follows

$$Z_n = \frac{f_n}{-i\omega u_{rn}} \tag{2.47}$$

with

$$u_{rn} = \frac{1}{r_o}\left\{A_n U_1^{(1)}(\alpha^* r) + C_n U_1^{(2)}(\alpha^* r) + B_n U_3^{(1)}(\beta^* r) + D_n U_3^{(2)}(\beta^* r)\right\}, \tag{2.48}$$

and the coefficients A_n, B_n, C_n, and D_n given by (2.39) and (2.41). Note that Z_n is independent of f_n.

2-3. Radiation Problem

The radiation pressure distribution for the outer surface $r = r_o$ vibrating with acceleration $\ddot{u}_r$ can be written as [10]

$$p(r,\theta) = -\frac{\rho_w}{k}\sum_{n=0}^{\infty} \ddot{u}_{rn} P_n(\cos\theta)\frac{h_n(kr)}{h_n'(kr_o)}. \tag{2.49}$$

By setting $r = r_o$, we obtain the surface pressure in terms of the modal specific acoustic impedance

$$p(r_o,\theta) = \sum_{n=0}^{\infty} z_n \dot{u}_{rn} P_n(\cos\theta), \tag{2.50}$$

where z_n is the *modal specific acoustic impedance*

$$z_n = r_n - i\omega m_n = i\rho_w c_w \frac{h_n(kr_o)}{h_n'(kr_o)} \tag{2.51}$$

with *modal resistance* r_n and *modal accession to inertia* m_n given by the formulas

$$r_n = \Re\left[i\rho_w c_w \frac{h_n(kr_o)}{h'_n(kr_o)}\right], \tag{2.52}$$

$$m_n = \frac{1}{\omega}\Im\left[-i\rho_w c_w \frac{h_n(kr_o)}{h'_n(kr_o)}\right]. \tag{2.53}$$

As usual, the bar symbol denotes the complex conjugate, ρ_w is the density of water, h_n and h'_n are Hankel's functions and derivatives of Hankel's functions, respectively.

2-4. Free Vibrations in Fluid

We begin by recalling the formula for the normal displacement component

$$\begin{aligned} u_r = &\frac{1}{r}\left[A_n U_1^{(1)}(\alpha^* r) + C_n U_1^{(2)}(\alpha^* r) + B_n U_3^{(1)}(\beta^* r) + D_n U_3^{(2)}(\beta^* r)\right] \\ &\cdot P_n(\cos\theta), \end{aligned} \tag{2.54}$$

where

$$U_1^{(i)}(\alpha^* r) = n Z_n^{(i)}(\alpha^* r) - \alpha^* r Z_{n+1}^{(i)}(\alpha^* r), \tag{2.55}$$

$$U_3^{(i)}(\beta^* r) = n(n+1) Z_n^{(i)}(\beta^* r). \tag{2.56}$$

Substituting (2.54) into (2.51), we get the normal stress

$$\begin{aligned} \sigma_{rr}(r_o,\theta) = &-\sum_{n=0}^{\infty} z_n \dot{u}_{rn} P_n(\cos\theta) \\ = &\sum_{n=0}^{\infty}(i\omega r_n + \omega^2 m_n)\frac{1}{r_o}[A_n U_1^{(1)}(\alpha^* r_o) + C_n U_1^{(2)}(\alpha^* r_o) \\ &+ B_n U_3^{(1)}(\beta^* r_o) + D_n U_3^{(2)}(\beta^* r_o)] P_n(\cos\theta). \end{aligned} \tag{2.57}$$

Finally, we impose the traction boundary condition

$$\begin{aligned} \sigma_{rr}|_{r=r_0} = &\frac{2\mu^*}{r_o^2}[A_n T_{11}^{(1)}(\alpha^* r_o) + B_n T_{13}^{(1)}(\beta^* r_o) \\ &+ C_n T_{11}^{(2)}(\alpha^* r_o) + D_n T_{13}^{(2)}(\beta^* r_o)] P_n(\cos\theta). \end{aligned} \tag{2.58}$$

The resulting systems of modal equations look as follows.
For $n = 0$, we get

$$
\begin{aligned}
& A_n T_{11}^{(1)}(\alpha^* r_o) + C_n T_{11}^{(2)}(\alpha^* r_o) = \frac{r_0}{2\mu^*}(i\omega r_n + \omega^2 m_n) \\
& \qquad \cdot \left[A_n U_1^{(1)}(\alpha^* r) + C_n U_1^{(2)}(\alpha^* r)\right], \\
& A_n T_{11}^{(1)}(\alpha^* r_i) + C_n T_{11}^{(2)}(\alpha^* r_i) = 0.
\end{aligned}
\tag{2.59}
$$

For $n > 0$, we obtain

$$
\begin{aligned}
& A_n T_{11}^{(1)}(\alpha^* r_o) + B_n T_{13}^{(1)}(\beta^* r_o) + C_n T_{11}^{(2)}(\alpha^* r_o) + D_n T_{13}^{(2)}(\beta^* r_o) \\
& = \frac{r_0}{2\mu^*}(i\omega r_n + \omega^2 m_n)\Big[A_n U_1^{(1)}(\alpha^* r_o) + C_n U_1^{(2)}(\alpha^* r_o) \\
& \quad + B_n U_3^{(1)}(\beta^* r_o) + D_n U_3^{(2)}(\beta^* r_o)\Big], \\
& A_n T_{11}^{(1)}(\alpha^* r_i) + B_n T_{13}^{(1)}(\beta^* r_i) + C_n T_{11}^{(2)}(\alpha^* r_i) + D_n T_{13}^{(2)}(\beta^* r_i) = 0, \\
& A_n T_{41}^{(1)}(\alpha^* r_o) + B_n T_{43}^{(1)}(\beta^* r_o) + C_n T_{41}^{(2)}(\alpha^* r_o) + D_n T_{43}^{(2)}(\beta^* r_o) = 0, \\
& A_n T_{41}^{(1)}(\alpha^* r_i) + B_n T_{43}^{(1)}(\beta^* r_i) + C_n T_{41}^{(2)}(\alpha^* r_i) + D_n T_{43}^{(2)}(\beta^* r_i) = 0.
\end{aligned}
\tag{2.60}
$$

Equating the corresponding determinants to zero, we obtain the corresponding modal characteristic equations from which resonant (scattering) frequencies [11] are determined. Contrary to the vibrations in vacuo, the characteristic equations are transcendental (not algebraic) as both modal resistance r_n and modal accession to inertia m_n depend upon frequency ω.

2-5. Forced Vibrations in Fluid

The problem is equivalent to forced vibrations in vacuo with the loading term equal to the sum of the actual loading and the radiation loading

$$Z_n \dot{u}_{rn} = f_n - z_n \dot{u}_{rn}. \tag{2.61}$$

Solving for $\dot{u}_{rn}$ we get

$$\dot{u}_{rn} = \frac{f_n}{Z_n + z_n}, \tag{2.62}$$

where the mechanical impedance corresponding to the 3-D theory is defined by (2.47).

2-6. Rigid Scattering of a Plane Wave

Consider a spherical scatterer with midsurface radius a, and a distant source generating a train of sound waves which impinge on the outer boundary of the sphere. Let the rigid scattered pressure be denoted by $p^{s,\infty}$ and the total pressure by p:

$$p = p^{inc} + p^{s,\infty}. \tag{2.63}$$

Since the boundary is rigid, the resultant particle acceleration must have a zero component in the radial direction on the scatterer.

$$\ddot{u}_r^{s,\infty} + \ddot{u}_r^{inc} = 0 \ \text{ at } \ r = r_o, \tag{2.64}$$

where $\ddot{u}_r^{inc}$ and $\ddot{u}_r^{s,\infty}$ are the normal acceleration components corresponding to incident and scattered waves, and r_o denotes the outer radius, respectively. Consequently

$$\ddot{u}_r^{s,\infty} = -\ddot{u}_r^{inc} = \frac{1}{\rho_w}\frac{\partial p^{inc}(r,\theta)}{\partial r} \ \text{ at } \ r = r_o. \tag{2.65}$$

Thus the rigid scattering problem reduces simply to the radiation problem with the vibrating boundary acceleration $\ddot{u}_{rn}$ specified by (2.65). We restrict now ourselves to an incident *plane wave* only, representing it in terms of spherical harmonics

$$\begin{aligned} p^{inc}(r,\theta) &= P_{inc}e^{ikr\cos\theta} \\ &= P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta) j_n(kr), \end{aligned} \tag{2.66}$$

where P_{inc} is a prescribed coefficient. Substituting (2.66) into (2.65) and comparing with equation (2.49), we obtain finally

$$p^{s,\infty}(r,\theta) = -P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\frac{j_n'(kr_o)}{h_n'(kr_o)}h_n(kr). \tag{2.67}$$

2-7. Scattering of a Plane Wave from a Viscoelastic Sphere

We begin again by expanding the incident plane wave in terms of the spherical harmonics

$$\begin{aligned} p^{inc}(r,\theta) &= P_{inc}e^{ikr\cos\theta} \\ &= P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta) j_n(kr), \end{aligned} \tag{2.68}$$

where P_{inc} is a prescribed coefficient.

The total viscoelastic-body scattered pressure $p^s(r,\theta)$ is equal to the sum of rigid-body scattered pressure $p^{s,\infty}(r,\theta)$

$$p^{s,\infty}(r,\theta) = -P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\frac{j_n'(kr_o)}{h_n'(kr_o)}h_n(kr) \tag{2.69}$$

and pressure $p^{s,r}(r,\theta)$ radiated by the viscoelastic shell in fluid loaded with forcing term, equal to $p^{s,\infty}(r,\theta) + p^{inc}(r,\theta)$.

$$\begin{aligned} p^{s,r}(r,\theta) &= i\rho_w c_w \sum_{n=0}^{\infty} \dot{u}_{rn} P_n(\cos\theta)\frac{h_n(kr)}{h_n'(kr_o)} \\ &= i^n \rho_w c_w \sum_{n=0}^{\infty} P_n(\cos\theta)\frac{(2n+1)P_{inc}h_n(kr)}{(kr_o)^2(Z_n+z_n)(h_n'(kr_o))^2}\,, \end{aligned} \tag{2.70}$$

where $\dot{u}_{rn}$ denotes the velocity component for the nth mode

$$\dot{u}_{rn} = -\frac{p_n}{Z_n+z_n} = -\frac{i^{n+1}(2n+1)P_{inc}}{(kr_o)^2 h_n'(kr_o)(Z_n+z_n)} \tag{2.71}$$

with the forcing term equal to the sum of p^{inc} and $p^{s,\infty}$

$$p_n = \frac{i^{n+1}(2n+1)P_{inc}}{(kr_o)^2 h_n'(kr_o)} \tag{2.72}$$

and modal mechanical impedance Z_n defined by (2.47) and modal specific acoustic impedance z_n defined by (2.51).

Using the spectral decomposition, we obtain the final pressure field

$$p = p^{inc}(r,\theta) + p^{s,\infty}(r,\theta) + p^{s,r}(r,\theta) \tag{2.73}$$

with

$$\begin{aligned} p^{inc}(r,\theta) &= P_{inc}e^{ikr\cos\theta} \\ &= P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)j_n(kr), \end{aligned} \tag{2.74}$$

$$p^{s,\infty}(r,\theta) = -P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\frac{j_n'(kr_o)}{h_n'(kr_o)}h_n(kr), \tag{2.75}$$

$$p^{s,r}(r,\theta) = i\rho_w c_w \sum_{n=0}^{\infty} \dot{u}_{rn} P_n(\cos\theta)\frac{h_n(kr)}{h_n'(kr_o)}, \tag{2.76}$$

$$\dot{u}_{rn} = -\frac{p_n}{Z_n+z_n} = -\frac{i^{n+1}(2n+1)P_{inc}}{(kr_o)^2 h_n'(kr_o)(Z_n+z_n)}. \tag{2.77}$$

3. Stability Analysis. The Effect of Damping on the LBB Constant

3-1. Theoretical Foundations

Given a Hilbert space V, we consider an abstract variational problem of the form

$$\left\{ \begin{array}{ll} \text{Find } \boldsymbol{u} \in V & \text{such that} \\ b(\boldsymbol{u}, \boldsymbol{v}) = \ell(\boldsymbol{v}) & \forall \boldsymbol{v} \in V, \end{array} \right. \tag{3.78}$$

where $b(\boldsymbol{u}, \boldsymbol{v})$ is a continuous sesquilinear form on $V \times V$ and $\ell(\mathbf{v})$ is a continuous antilinear form on V.

The sesquilinear form b defines a linear operator B prescribed on V with values in the topological dual V',

$$B : V \rightarrow V'; \;\; < B\boldsymbol{u}, \boldsymbol{v} >= b(\boldsymbol{u}, \boldsymbol{v}) \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in V \tag{3.79}$$

which allows us to rewrite (3.78) in the operator form

$$B\boldsymbol{u} = \ell, \;\; \boldsymbol{u} \in V, \tag{3.80}$$

consistently with the classical theory of linear operators in Banach spaces (see, e.g. [13]); operator B is postulated to be *bounded below*

$$\| B\boldsymbol{u} \|_{V'} \geq \gamma \| \boldsymbol{u} \|_V, \tag{3.81}$$

where $\| \cdot \|_V$ and $\| \cdot \|_{V'}$ are the norms in the original Hilbert space and its dual, respectively. The optimal (smallest) constant γ is known as the LBB (Ladyzenskaya-Babuška-Brezzi) constant

$$\gamma = \inf_{u \neq 0} \frac{\| \boldsymbol{Bu} \|_{V'}}{\| \boldsymbol{u} \|_V}. \tag{3.82}$$

Introducing the Riesz operator

$$R : V \rightarrow V'; \;\; < R\boldsymbol{u}, \boldsymbol{v} > = \; (\boldsymbol{u}, \boldsymbol{v}) \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.83}$$

where $(\cdot, \cdot)$ denotes the inner product in space V, we can eliminate inner product $(\cdot, \cdot)$ from the dual norm

$$\begin{aligned} \gamma^2 &= \inf_{u \neq 0} \frac{\| \boldsymbol{Bu} \|_{V'}^2}{\| \boldsymbol{u} \|_V^2} = \inf_{u \neq 0} \frac{\| \boldsymbol{R}^{-1}\boldsymbol{Bu} \|_V^2}{\| \boldsymbol{u} \|_V^2} \\ &= \inf_{(u,u)=1} (\boldsymbol{R}^{-1}\boldsymbol{Bu}, \boldsymbol{R}^{-1}\boldsymbol{Bu}). \end{aligned} \tag{3.84}$$

Next, using the standard Lagrange multiplier technique, we arrive at the eigenvalue problem

$$(3.85)\qquad \begin{cases} \text{Find } \boldsymbol{u} \in V, \gamma^2 & \text{such that} \\ (\boldsymbol{R}^{-1}\boldsymbol{B}\boldsymbol{u}, \boldsymbol{R}^{-1}\boldsymbol{B}\delta\boldsymbol{u}) = \gamma^2(\boldsymbol{u}, \delta\boldsymbol{u}) & \forall\delta\boldsymbol{u} \in V \\ (\boldsymbol{u}, \boldsymbol{u}) = 1. & \end{cases}$$

Finally, it is convenient to rewrite (3.85) as a system of two equations. Introducing an auxiliary variable

$$(3.86)\qquad \boldsymbol{u}^a = \boldsymbol{R}^{-1}\boldsymbol{B}\boldsymbol{u},$$

we get

$$(3.87)\qquad (\boldsymbol{R}^{-1}\boldsymbol{B}\boldsymbol{u}, \delta\boldsymbol{u}^a) - (\boldsymbol{u}^a, \delta\boldsymbol{u}^a) = 0 \quad \forall\delta\boldsymbol{u}^a,$$

$$(3.88)\qquad (\boldsymbol{u}^a, \boldsymbol{R}^{-1}\boldsymbol{B}\delta\boldsymbol{u}) - \gamma^2(\boldsymbol{u}, \delta\boldsymbol{u}) = 0 \quad \forall\delta\boldsymbol{u},$$

or, recalling the definitions of operators $\boldsymbol{B}$ and $\boldsymbol{R}$,

$$(3.89)\qquad b(\boldsymbol{u}, \delta\boldsymbol{u}^a) - (\boldsymbol{u}^a, \delta\boldsymbol{u}^a) = 0 \quad \forall\delta\boldsymbol{u}^a,$$

$$(3.90)\qquad \overline{b(\delta\boldsymbol{u}, \boldsymbol{u}^a)} - \gamma^2(\boldsymbol{u}, \delta\boldsymbol{u}) = 0 \quad \forall\delta\boldsymbol{u}.$$

Introducing a family of finite dimensional subspaces V_h of V, converging to V as $h \to 0$, we formulate the usual Bubnov-Galerkin approximation of (3.78)

$$(3.91)\qquad \begin{cases} \text{Find } \boldsymbol{u}_h \in V_h & \text{such that} \\ b(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) & \forall\boldsymbol{v}_h \in V_h. \end{cases}$$

Repeating exactly the same steps as for the continuous problem, we can introduce the *discrete LBB* constant γ_h which represents for the stability of the discrete problem. It has been proved in [5] that, under approximate assumptions, the discrete LBB constant γ_h converges to its continuous counterpart γ, as $h \to 0$. This motivated the study in [6] on the LBB constant for the acoustic scattering and other related problems, in context of the spherical geometry.

3-2. Forced Vibrations of a Viscoelastic Hollow Sphere by the Shell Theory

In the following we calculate the LBB constant for forced vibrations of a viscoelastic hollow sphere by the shell theory. We shall focus on the dependence of the LBB constant upon the wave number k and study the effect of damping.

Restricting ourselves to *axisymmetric vibrations* only, we describe the deformation of the shell with radial displacement $U_r(\theta)$ and transversial displacement $U_\theta(\theta)$ of the middle surface $r = a$, where r, θ, ϕ denote the usual spherical coordinates.

Referring to [10] for details, we recall here only the formulas for the sesquilinear form b and the inner product:

$$
\begin{aligned}
b((U_r,U_\theta),(\delta U_r,\delta U_\theta)) = {} & \frac{2\pi E^* t}{1-\nu^{*2}} \int_0^\pi \left\{ \left(\frac{\partial U_\theta}{\partial \theta} + U_r \right) \left(\frac{\partial \delta \bar{U}_\theta}{\partial \theta} + \delta \bar{U}_r \right) \right. \\
& + (\cot\theta U_\theta + U_r)(\cot\theta \delta\bar{U}_\theta + \delta\bar{U}_r) \\
& + \nu^* \left(\frac{\partial U_\theta}{\partial \theta} + U_r \right) (\cot\theta\delta\bar{U}_\theta + \delta\bar{U}_r) \\
& \left. + \nu^* (\cot\theta U_\theta + U_r) \left(\frac{\partial \delta\bar{U}_\theta}{\partial\theta} + \delta\bar{U}_r \right) \right\} \sin\theta d\theta \\
& + \frac{2\pi E^* t \beta^2}{1-\nu^{*2}} \int_0^\pi \left\{ \left(\frac{\partial U_\theta}{\partial \theta} - \frac{partial^2 U_r}{\partial \theta^2} \right) \left(\frac{\partial \delta\bar{U}_\theta}{\partial\theta} - \frac{\partial^2 \delta\bar{U}_r}{\partial\theta^2} \right) \right. \\
& + \cot^2\theta \left(U_\theta - \frac{\partial U_r}{\partial\theta} \right) \left(\delta\bar{U}_\theta - \frac{\partial\delta\bar{U}_r}{\partial\theta} \right) \\
& + \nu^* \cot\theta \left(\frac{\partial U_\theta}{\partial\theta} - \frac{\partial^2 U_r}{\partial\theta^2} \right) \left(\delta\bar{U}_\theta - \frac{\partial\delta U_r}{\partial\theta} \right) \\
& \left. + \nu^* \cot\theta \left(U_\theta - \frac{\partial U_r}{\partial\theta} \right) \left(\frac{\partial\delta\bar{U}_\theta}{\partial\theta} - \frac{\partial^2\delta\bar{U}_r}{\partial\theta^2} \right) \right\} \sin\theta d\theta \\
& - 2\pi\omega^2 \rho_s t a^2 \int_0^\pi (U_r \delta\bar{U}_r + U_\theta\delta\bar{U}_\theta) \sin\theta d\theta \\
& + \left[2\pi a^2 \int_0^\pi p(\eta)\delta\bar{U}_r \sin\theta d\theta \right],
\end{aligned} \tag{3.92}
$$

$$
\begin{aligned}
((U_r,U_\theta),(\delta U_r,\delta U_\theta)) = {} & \frac{2\pi E t}{1-\nu^2} \int_0^\pi \left\{ \left(\frac{\partial U_\theta}{\partial \theta} + U_r \right) \left(\frac{\partial \delta \bar{U}_\theta}{\partial \theta} + \delta \bar{U}_r \right) \right. \\
& + (\cot\theta U_\theta + U_r)(\cot\theta \delta\bar{U}_\theta + \delta\bar{U}_r) \\
& + \nu \left(\frac{\partial U_\theta}{\partial \theta} + U_r \right) (\cot\theta\delta\bar{U}_\theta + \delta\bar{U}_r) \\
& \left. + \nu (\cot\theta U_\theta + U_r) \left(\frac{\partial \delta U_\theta}{\partial\theta} + \delta\bar{U}_r \right) \right\} \sin\theta d\theta \\
& + \frac{2\pi E t \beta^2}{1-\nu^2} \int_0^\pi \left\{ \left(\frac{\partial U_\theta}{\partial \theta} - \frac{\partial^2 U_r}{\partial \theta^2} \right) \left(\frac{\partial \delta\bar{U}_\theta}{\partial\theta} - \frac{\partial^2 \delta\bar{U}_r}{\partial\theta^2} \right) \right.
\end{aligned} \tag{3.93}
$$

$$
\begin{aligned}
&+\cot^2\theta\left(U_\theta-\frac{\partial U_r}{\partial\theta}\right)\left(\frac{\partial\delta\bar{U}_r}{\partial\theta}\right)\\
&+\nu\cot\theta\left(\frac{\partial U_\theta}{\partial\theta}-\frac{\partial U_\theta}{\partial\theta}-\frac{\partial^2 U_r}{\partial\theta^2}\right)\left(\frac{\partial\delta\bar{U}_r}{\partial\theta}\right)\\
&+\nu\cot\theta\left(U_\theta-\frac{\partial U_r}{\partial\theta}\right)\left(\frac{\partial\delta\bar{U}_\theta}{\partial\theta}-\frac{\partial^2\delta\bar{U}_r}{\partial\theta^2}\right)\Bigg\}\sin\theta d\theta\\
&+2\pi t a^2 C\int_0^\pi(U_r\delta\bar{U}_r+U_\theta\delta\bar{U}_\theta)\sin\theta d\theta.
\end{aligned}
\tag{3.93}
$$

The following notation has been used

- a : the radius of the middle surface of the shell,
- E, ν : the Young modulus and Poisson ratio for the elastic material,
- E^*, ν^* : the corresponding viscoelastic moduli,
- t : the thickness of the shell,
- ω : the frequency,
- k : the wave number, $k = \frac{\omega}{c_w}$,
- c_w : sound speed in fluid,
- ρ_s : the density of the solid,
- ρ_w : the density of the fluid.

The terms in brackets [ ] correspond to the interaction with the fluid, and are set to zero in the case of the shell vibrating in vacuo.

Although the choice of the inner product is, up to a certain extent, arbitrary, selecting the elastic energy seems to be rather natural. Constant C in (3.93), however, controlling the mass contribution to the final norm, is certainly arbitrary and the final results will depend upon its choice. Here, we adopt the first natural frequency of the system, i.e., $C = \rho_s\omega_{21}^2$ (ω_{21} is natural frequency of the second mode in the first branch).

Problem (3.78) admits the usual spectral decomposition using the classical representation

$$
U_r(\eta) = \sum_{n=0}^{\infty} U_{rn} P_n(\eta); \tag{3.94}
$$

$$
U_\theta(\eta) = \sum_{n=1}^{\infty} U_{rn}[-P_n^1(\eta)], \tag{3.95}
$$

where $P_n(\eta)$ and $P_n^1(\eta)$ are Legendre polynomials and associated Legendre functions, and $\eta = \cos\theta$.

Substituting (3.94)-(3.95) and related formulas for δU_r and δU_θ into (3.92)-(3.93), we arrive at a sequence of modal eigenvalue problems of the form

$$\begin{bmatrix} -g_n & b_n^* \\ \bar{b}_n^{*T} & -\gamma^2 g_n \end{bmatrix} \begin{bmatrix} u^a \\ u \end{bmatrix} = 0. \tag{3.96}$$

Here, superscript $*$ refers to the viscoelasticity problem.
For $n = 0$,

$$\boldsymbol{b}_0^* = \boldsymbol{I}_0^* + A_2^* \left[-\Omega^{*^2} \tfrac{m_n}{\rho_s h} + i\Omega^* \tfrac{a}{h} \tfrac{r_n}{\rho_s c_p^*} \right], \tag{3.97}$$

$$\boldsymbol{g}_0 = \boldsymbol{J}_0 + A_1 C, \tag{3.98}$$

$$\boldsymbol{I}_0^* = A_2^* \left[2(1+\nu^*) - \Omega^{*^2} \right], \tag{3.99}$$

$$\boldsymbol{J}_0 = A_2 \left[2(1+\nu) \right], \tag{3.100}$$

where

$$A_1 = \frac{4\pi h a^2}{2n+1}, \tag{3.101}$$

$$A_2 = \frac{4\pi \rho_s h c_p^2}{2n+1}, \tag{3.102}$$

$$A_2^* = \frac{4\pi \rho_s h c_p^{*^2}}{2n+1}. \tag{3.103}$$

For $n > 0$,

$$\boldsymbol{b}_n^* = \boldsymbol{I}_n^* + A_2^* \begin{bmatrix} -\Omega^{*^2} \frac{m_n}{\rho_s h} + i\Omega^* \frac{a}{h} \frac{r_n}{\rho_s c_p^*} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.104}$$

$$\boldsymbol{g}_n = \boldsymbol{J}_n + A_1 C \begin{bmatrix} 1 & 0 \\ 0 & \kappa_n \end{bmatrix}, \tag{3.105}$$

$$\boldsymbol{I}_n^* = A_2^* \begin{bmatrix} -\Omega^{*^2} + 2(1+\nu^*) + \beta^2 \kappa_n (\nu^* + \kappa_n - 1) & \kappa_n [\beta^2 (\nu^* + \kappa_n - 1) + (1+\nu^*)] \\ \kappa_n \beta^2 (\nu^* + \kappa_n - 1) + (1+\nu^*) & \kappa_n [-\Omega^{*^2} + (1+\beta^2)(\nu^* + \kappa_n - 1)] \end{bmatrix}, \tag{3.106}$$

$$\boldsymbol{J}_n = A_2 \begin{bmatrix} 2(1+\nu) + \beta^2 \kappa_n (\nu + \kappa_n - 1) & \kappa_n [\beta^2 (\nu + \kappa_n - 1) + (1+\nu)] \\ \kappa_n \beta^2 (\nu + \kappa_n - 1) + (1+\nu) & \kappa_n (1+\beta^2)(\nu + \kappa_n - 1) \end{bmatrix}, \tag{3.107}$$

with $\boldsymbol{b}_n$ corresponding to the forced vibrations in vacuum or in water, $\boldsymbol{J}_n$ corresponding to the elastic strain energy in vacuum.

The following notation has been used

- $\kappa_n = n(n+1)$,
- $\eta = \cos\theta$,
- Ω : dimensionless frequency of the elastic shell,

$$\Omega = \frac{wa}{c_p} = \left(\frac{c}{c_p}\right) ka, \tag{3.108}$$

- Ω^* : dimensionless frequency of the viscoelastic shell,

$$\Omega^* = \frac{wa}{c_p^*} = \left(\frac{c}{c_p^*}\right) ka, \tag{3.109}$$

- c_w : wave velocity in water,
- c_p : the low frequency phase velocity of compressional waves in an elastic plate,
- c_p^* : the low frequency phase velocity of compressional waves in a viscoelastic plate.

The final LBB constant γ is equal to the infimum (minimum as a matter of fact) of the modal eigenvalues.

3-3. Forced Vibrations of a Viscoelastic Hollow Sphere by the 3-D Theory

We begin by recalling the formula for the sesquilinear form corresponding to the standard variational formulation

$$\begin{aligned} b(\boldsymbol{u}, \delta\boldsymbol{u}) = & \int_V \sigma_{ij}^*(\boldsymbol{u})\varepsilon_{ij}(\delta\bar{\boldsymbol{u}})dV - \rho_s\omega^2 \int_V \boldsymbol{u}\cdot\delta\bar{\boldsymbol{u}}dV \\ & + \left[2\pi r_o^2 \int_0^\pi p(\cos\theta)\delta\bar{u}_r \sin\theta d\theta\right]. \end{aligned} \tag{3.110}$$

Here, the term in the bracket is equal to zero for forced vibrations in vacuum.

As for the formulation based on the shell theory, we select the following inner product

$$(\boldsymbol{u}, \delta\boldsymbol{u}) = \int_V \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\delta\bar{\boldsymbol{u}})dV + C\int_V \boldsymbol{u}\cdot\delta\bar{\boldsymbol{u}}dV, \tag{3.111}$$

where the first term corresponds to the elastic energy and the mass term is scaled with constant C equal to $\rho_s\omega_{\min}^2$ ($\omega_{\min}$ is the smallest positive eigenfrequency).

As in the previous case, the main idea consists of using the spectral decomposition to reduce the calculation of the LBB constant to a sequence of finite-dimensional modal eigenvalue problems. Formally one represents fields $\boldsymbol{u}$ and $\boldsymbol{v}$ as sums of infinite series involving spectral components given by formulas (3.94)-(3.95) and uses the L^2-orthogonality of the Legendre polynomials on the unit circle. As the displacement field $\boldsymbol{u}$ satisfies the Navier equations, it is convenient to convert the volume integral in the sesquilinear form into two surface integrals on the outer sphere S_o and the inner sphere S_i

$$
\begin{aligned}
b(\boldsymbol{u},\delta\boldsymbol{u}) = & \int_{S_o} [\sigma_{rr}^*\delta\bar{u}_r + \sigma_{r\theta}^*\delta\bar{u}_\theta]\, dS_o \\
& - \int_{S_i} [\sigma_{rr}^*\delta\bar{u}_r + \sigma_{r\theta}^*\delta\bar{u}_\theta]\, dS_i \\
& + \left[2\pi r_o^2 \int_0^\pi p(\cos\theta)\delta\bar{u}_r \sin\theta d\theta\right].
\end{aligned}
\tag{3.112}
$$

Substituting now formulas (2.8), (2.11), (2.28), and (2.29) for the stresses σ_{rr}, $\sigma_{r\theta}$ and the displacements δu_r, δu_θ, one can arrive at modal eigenvalue problems involving 4 constants for $n = 0$ (A_n and C_n for both original and adjoint variables) and 8 constants for $n > 0$ (A_n, B_n, C_n, D_n for both original and adjoint variables). Each of the equations is obtained by setting one out of four constants in the formulas for δu_r, δu_θ to one and the remaining ones to zero at the inner and outer boundary for the test functions and the values of stresses.

From the point of view of the following analysis for multi-layered shells, it is more convenient to identify as new degrees of freedom values of r-dependent factors in the formulas for u_r and u_θ, at the inner and outer surfaces

$$
\frac{1}{r}\left[U_1^{(i)}(\alpha^* r) + U_3^{(i)}(\beta^* r)\right], \quad r = r_i, r_o,
\tag{3.113}
$$

$$
\frac{1}{r}\left[V_1^{(i)}(\alpha^* r) + V_3^{(i)}(\beta^* r)\right], \quad r = r_i, r_o.
\tag{3.114}
$$

We shall denote the corresponding vector of four degrees of freedom by $\boldsymbol{u}$. Note that for $n = 0$, the new variables include only the first of the two quantities defined in (3.113). The final sequence of the modal eigenproblems takes the form

$$
\begin{bmatrix} -g_n & b_n^* \\ \bar{b}_n^{*T} & -\gamma^2 g_n \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^a \\ \boldsymbol{u} \end{bmatrix} = 0.
\tag{3.115}
$$

For $n = 0$,

$$\boldsymbol{b}_0^* = \boldsymbol{S}_0^* \boldsymbol{Q}_0^{*-1}, \tag{3.116}$$

$$\boldsymbol{g}_0 = \boldsymbol{S}_0 \boldsymbol{Q}_0^{-1} + (\rho_s \omega^2 + C) \boldsymbol{V}_0, \tag{3.117}$$

$$\begin{aligned} \boldsymbol{S}_0^* &= c_0^* \begin{bmatrix} -T_{11}^{(1)}(\alpha^* r_i) & -T_{11}^{(2)}(\alpha^* r_i) \\ T_{11}^{(1)}(\alpha^* r_o) & T_{11}^{(2)}(\alpha^* r_o) \end{bmatrix} \\ &\quad + d_0 \begin{bmatrix} 0 & 0 \\ U_1^{(1)}(\alpha^* r_o) & U_1^{(2)}(\alpha^* r_o) \end{bmatrix}, \end{aligned} \tag{3.118}$$

$$d_0 = \begin{cases} 0 & \text{in vacuum;} \\ -\frac{r_0}{2\mu} i \omega z_0 & \text{in water,} \end{cases} \tag{3.119}$$

$$\boldsymbol{S}_0 = c_0 \begin{bmatrix} -T_{11}^{(1)}(\alpha r_i) & -T_{11}^{(2)}(\alpha r_i) \\ T_{11}^{(1)}(\alpha r_o) & T_{11}^{(2)}(\alpha r_o) \end{bmatrix}, \tag{3.120}$$

$$\boldsymbol{V}_0 = \boldsymbol{Q}_0^{-T} \boldsymbol{W}_0 \boldsymbol{Q}_0^{-1}, \tag{3.121}$$

$$\boldsymbol{Q}_0^* = \begin{bmatrix} U_1^{(1)}(\alpha^* r_i)/r_i & U_1^{(2)}(\alpha^* r_i)/r_i \\ U_1^{(1)}(\alpha^* r_o)/r_o & U_1^{(2)}(\alpha^* r_o)/r_o \end{bmatrix}, \tag{3.122}$$

$$\boldsymbol{Q}_0 = \begin{bmatrix} U_1^{(1)}(\alpha r_i)/r_i & U_1^{(2)}(\alpha r_i)/r_i \\ U_1^{(1)}(\alpha r_o)/r_o & U_1^{(2)}(\alpha r_o)/r_o \end{bmatrix}, \tag{3.123}$$

$$\boldsymbol{W}_0 = 4\pi \int_{r_i}^{r_o} \left[\boldsymbol{\chi}_1^T \boldsymbol{\chi}_1 \right] dr, \tag{3.124}$$

$$\boldsymbol{\chi}_1 = \left[U_1^{(1)}(\alpha r) \;\; U_1^{(2)}(\alpha r) \right], \tag{3.125}$$

with

$$c_0^* = 8\pi\mu^*, \tag{3.126}$$

$$c_0 = 8\pi\mu, \tag{3.127}$$

and $T_{11}^{(1)}$, $T_{11}^{(2)}$, $U_1^{(1)}$ and $U_1^{(2)}$ are given by (2.13) and (2.31).
For $n > 0$,

$$\boldsymbol{b}_n^* = \boldsymbol{S}_n^* \boldsymbol{Q}_n^{*-1}, \tag{3.128}$$

$$\boldsymbol{g}_n = \boldsymbol{S}_n \boldsymbol{Q}_n^{-1} + (\rho_s \omega^2 + C) \boldsymbol{V}_n, \tag{3.129}$$

$$\boldsymbol{S}_n^* = c_n^* \begin{bmatrix} -T_{11}^{(1)}(\alpha^* r_i) & -T_{13}^{(1)}(\beta^* r_i) & -T_{11}^{(2)}(\alpha^* r_i) & -T_{13}^{(2)}(\beta^* r_i) \\ -\kappa_n T_{41}^{(1)}(\alpha^* r_i) & -\kappa_n T_{43}^{(1)}(\beta^* r_i) & -\kappa_n T_{41}^{(2)}(\alpha^* r_i) & -\kappa_n T_{43}^{(2)}(\beta^* r_i) \\ T_{11}^{(1)}(\alpha^* r_o) & T_{13}^{(1)}(\beta^* r_o) & T_{11}^{(2)}(\alpha^* r_o) & T_{13}^{(2)}(\beta^* r_o) \\ \kappa_n T_{41}^{(1)}(\alpha^* r_o) & \kappa_n T_{43}^{(1)}(\beta^* r_o) & \kappa_n T_{41}^{(2)}(\alpha^* r_o) & \kappa_n T_{43}^{(2)}(\beta^* r_o) \end{bmatrix} + d_n \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ U_1^{(1)}(\alpha^* r_o) & U_3^{(1)}(\beta^* r_o) & U_1^{(2)}(\alpha^* r_o) & U_3^{(2)}(\beta^* r_o) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.130}$$

$$d_n = \begin{cases} 0 & \text{in vacuum;} \\ -\frac{r_0}{2\mu} i\omega z_n & \text{in water,} \end{cases} \tag{3.131}$$

$$\boldsymbol{S}_n = c_n \begin{bmatrix} -T_{11}^{(1)}(\alpha r_i) & -T_{13}^{(1)}(\beta r_i) & -T_{11}^{(2)}(\alpha r_i) & -T_{13}^{(2)}(\beta r_i) \\ -\kappa_n T_{41}^{(1)}(\alpha r_i) & -\kappa_n T_{43}^{(1)}(\beta r_i) & -\kappa_n T_{41}^{(2)}(\alpha r_i) & -\kappa_n T_{43}^{(2)}(\beta r_i) \\ T_{11}^{(1)}(\alpha r_o) & T_{13}^{(1)}(\beta r_o) & T_{11}^{(2)}(\alpha r_o) & T_{13}^{(2)}(\beta r_o) \\ \kappa_n T_{41}^{(1)}(\alpha r_o) & \kappa_n T_{43}^{(1)}(\beta r_o) & \kappa_n T_{41}^{(2)}(\alpha r_o) & \kappa_n T_{43}^{(2)}(\beta r_o) \end{bmatrix}, \tag{3.132}$$

$$\boldsymbol{V}_n = \boldsymbol{Q}_n^{-T} \boldsymbol{W}_n \boldsymbol{Q}_n^{-1}, \tag{3.133}$$

$$\boldsymbol{Q}_n^* = \begin{bmatrix} U_1^{(1)}(\alpha^* r_i)/r_i & U_3^{(1)}(\beta^* r_i)/r_i & U_1^{(2)}(\alpha^* r_i)/r_i & U_3^{(2)}(\beta^* r_i)/r_i \\ V_1^{(1)}(\alpha^* r_i)/r_i & V_3^{(1)}(\beta^* r_i)/r_i & V_1^{(2)}(\alpha^* r_i)/r_i & V_3^{(2)}(\beta^* r_i)/r_i \\ U_1^{(1)}(\alpha^* r_o)/r_o & U_3^{(1)}(\beta^* r_o)/r_o & U_1^{(2)}(\alpha^* r_o)/r_o & U_3^{(2)}(\beta^* r_o)/r_o \\ V_1^{(1)}(\alpha^* r_o)/r_o & V_3^{(1)}(\beta^* r_o)/r_o & V_1^{(2)}(\alpha^* r_o)/r_o & V_3^{(2)}(\beta^* r_o)/r_o \end{bmatrix}, \tag{3.134}$$

$$\boldsymbol{Q}_n = \begin{bmatrix} U_1^{(1)}(\alpha r_i)/r_i & U_3^{(1)}(\beta r_i)/r_i & U_1^{(2)}(\alpha r_i)/r_i & U_3^{(2)}(\beta r_i)/r_i \\ V_1^{(1)}(\alpha r_i)/r_i & V_3^{(1)}(\beta r_i)/r_i & V_1^{(2)}(\alpha r_i)/r_i & V_3^{(2)}(\beta r_i)/r_i \\ U_1^{(1)}(\alpha r_o)/r_o & U_3^{(1)}(\beta r_o)/r_o & U_1^{(2)}(\alpha r_o)/r_o & U_3^{(2)}(\beta r_o)/r_o \\ V_1^{(1)}(\alpha r_o)/r_o & V_3^{(1)}(\beta r_o)/r_o & V_1^{(2)}(\alpha r_o)/r_o & V_3^{(2)}(\beta r_o)/r_o \end{bmatrix}, \tag{3.135}$$

$$\boldsymbol{W}_n = \frac{4\pi}{2n+1} \int_{r_i}^{r_o} \left[\chi_{\mathbf{1}}^{\mathbf{T}} \chi_{\mathbf{1}} + \kappa_{\mathbf{n}} \chi_{\mathbf{2}}^{\mathbf{T}} \chi_{\mathbf{2}}\right] dr, \tag{3.136}$$

$$\chi_{\mathbf{1}} = \left[U_1^{(1)}(\alpha r) \;\; U_3^{(1)}(\beta r) \;\; U_1^{(2)}(\alpha r) \;\; U_3^{(2)}(\beta r)\right], \tag{3.137}$$

$$\chi_{\mathbf{2}} = \left[V_1^{(1)}(\alpha r) \;\; V_3^{(1)}(\beta r) \;\; V_1^{(2)}(\alpha r) \;\; V_3^{(2)}(\beta r)\right], \tag{3.138}$$

with

$$c_n^* = \frac{8\pi\mu^*}{2n+1}, \tag{3.139}$$

$$c_n = \frac{8\pi\mu}{2n+1}, \tag{3.140}$$

and $T_{11}^{(i)}$, $T_{13}^{(i)}$, $T_{41}^{(i)}$, $T_{43}^{(i)}$, $U_1^{(i)}$,$U_3^{(i)}$, $V_1^{(i)}$ and $V_3^{(i)}$ are given by (2.13), (2.14), (2.23), (2.24) and (2.31)-(2.34).

Remark: In all reported numerical experiments, integrals in matrices $\boldsymbol{W}_n$ are evaluated numerically using Romberg's method. The AMOS portable package [2] is used to evaluate Bessel's functions of complex argument.

3-4. Forced Vibrations of a Multilayered Viscoelastic Hollow Sphere

We assume that the shell is composed of M viscoelastic layers bounded by spheres with radii

$$r_1 < r_2 < \ldots < r_M < r_{M+1} \tag{3.141}$$

and the displacements are continuous at all interfaces. The sesquilinear form is now obtained by summing up the contributions corresponding to the layers

$$\begin{aligned} b(\boldsymbol{u}, \delta\boldsymbol{u}) = & \sum_{m=1}^{M} \int_{V_m} \sigma_{ij}^{*}(\boldsymbol{u}|_{V_m})\varepsilon_{ij}(\delta\bar{\boldsymbol{u}}|_{V_m})dV_m - \rho_s^{(m)}\omega^2 \int_{V_m} \boldsymbol{u}|_{V_m} \cdot \delta\bar{\boldsymbol{u}}|_{V_m} dV_m \\ & + \left[2\pi r_{M+1}^2 \int_0^{\pi} p(\cos\theta)\delta\bar{u}_r|_{r=r_{M+1}} \sin\theta d\theta\right]. \end{aligned} \tag{3.142}$$

Here V_m denotes the mth layer, $\boldsymbol{u}|_{V_m}$ is the restriction of the displacement field $\boldsymbol{u}$ to the layer V_m and index m indicates correspondence to the mth layer. As previously, the term in brackets corresponds to the interaction with the fluid. Taking advantage of the fact that the displacement field (2.1)-(2.3) satisfies the Navier equations, we obtain

$$\begin{aligned} b(\boldsymbol{u}, \delta\boldsymbol{u}) = & \sum_{m=1}^{M} \left\{ \int_{S_{m+1}} (\sigma_{rr}^{*}\delta\bar{u}_r + \sigma_{r\theta}^{*}\delta\bar{u}_\theta)dS_{m+1} - \int_{S_m} (\sigma_{rr}^{*}\delta\bar{u}_r + \sigma_{r\theta}^{*}\delta\bar{u}_\theta)dS_m \right\} \\ & + \left[2\pi r_{M+1}^2 \int_0^{\pi} p(\cos\theta)\delta\bar{u}_r|_{r=r_{M+1}} \sin\theta d\theta\right]. \end{aligned} \tag{3.143}$$

Following the spectral analysis for a single layer shell from the previous section, each of the terms within the curly braces can be represented in terms of new "nodal" unknowns identified as values of factors (3.113)-(3.114) at $r = r_m, r_{m+1}$ and corresponding to the same values of the test functions. Consequently, identifying values of factors (3.113)-(3.114) at $r = r_1, \ldots, r_{M+1}$ as the vector of unknowns $\boldsymbol{u}$, one can assemble the corresponding modal matrix in the same way as it is done in the usual FE discretization of 1-D problems. The same assembly procedure will apply to the inner product.

Symbolically writing, the nth modal eigenvalue problem has the same form as before with matrices defined as

$$\boldsymbol{b}_n^{*} = \text{Assembly}_{m=1}^{M}\boldsymbol{b}_n^{(m)*}, \tag{3.144}$$

$$\boldsymbol{g}_n = \text{Assembly}_{m=1}^{M}\boldsymbol{g}_n^{(m)}, \tag{3.145}$$

where $\boldsymbol{b}_n^{(m)*}$ and $\boldsymbol{g}_n^{(m)}$ correspond to layer V_n, and the unknowns at $r = r_m, r_{m+1}$. As before, for $n = 0$, the number of unknowns is decreased by half.

4. Numerical Results

4-1. Forced Vibrations of a Thin Hollow Sphere by the Kirchhoff-Love Shell Theory

All tests involving the hollow spherical shell were run with the following data:

Water density	ρ_w	=	$1000\ \mathrm{kg/m^3}$
Steel density	ρ_s	=	$7800\ \mathrm{kg/m^3}$
Sound speed in water	c_w	=	$1524\ \mathrm{m/sec}$
Elastic Young′s modulus	E	=	$2.1 \times 10^{11}\ \mathrm{N/m^2}$
Fictitious Young′s modulus	E^*	=	$2.1 \times 10^{11} \times (1 - \eta i)\,\mathrm{N/m^2}$, (Kelvin′s model)
Loss factor	η	=	$0,\ 5 \times 10^{-3} ka$
Poisson Ratio	ν	=	0.3
Fictitious Poisson Ratio	ν^*	=	0.3
Midsurface radius	a	=	$1\ \mathrm{m}$
Thickness of the shell	t	=	$0.01\ \mathrm{m}$
Constant for the chosen norm	C	=	$\rho_s \omega_{21}^2 = 1.1 \times 10^{11}$.

A quantitative study to indicate the effect of damping on the LBB constant is summarized in Figs. 1-4. The results are clear. As expected, increasing the loss factor results in an increase of the LBB constant and therefore the overall stability of the problem, both in vacuo and in water. We emphasize the quantitative character of presented results, as the "real" loss factor η is a function of frequency ω and it depends upon the choice of the viscoelastic constitutive law.

4-2. Forced Vibrations of a Thin Hollow Sphere by the 3-D Theory

In order to compare the 3-D theory with the Kirchhoff-Love model, the same case was studied using the 3-D results. Due to the instability in evaluation of Bessel's functions for small ka, the wave number range was restricted to $0.03 < ka < 10$ in this study. Practically speaking, the results summarized in Figs. 5-8 are indistinguishable from the ones obtained using the shell theory.

In order to verify and illustrate the theoretical investigations, the classical problem of scattering of a plane wave on a viscoelastic hollow sphere was solved. Two cases were considered:

Fig. 1 Vibrations of a thin spherical shell in vacuum by the shell theory.

The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 0\%$.

Fig. 2 Vibrations of a thin spherical shell in vacuum by the shell theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 5 \times 10^{-3} ka$.

Fig. 3 Vibrations of a thin spherical shell in water by the shell theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 0\%$.

Fig. 4 Vibrations of a thin spherical shell in water by the shell theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 5 \times 10^{-3} ka$.

Fig 5. Vibrations of a thin spherical shell in vacuum by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 0\%$.

Fig 6. Vibrations of a thin spherical shell in vacuum by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 5 \times 10^{-3} ka$.

Fig. 7 Vibrations of a thin spherical shell in water by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 0\%$.

Fig. 8 Vibrations of a thin spherical shell in water by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 5 \times 10^{-3} ka$.

- $k = 1.56$ (near the first local minimum of LBB constants for mode 4, $\gamma = 7.9 \times 10^{-2}$, see Fig. 8)
- $k = 1.66237$ (the first local minimum of LBB constants for mode 4, $\gamma = 5.7 \times 10^{-3}$, see Fig. 8).

With the same physical data as in the previous section, the problem was solved using a BE/FE approximation based on the Burton-Miller integral equation coupled with the standard 3-D viscoelasticity formulation. For all numerical details we refer to [7]. Figs. 9-12 display the real part of the pressure along a cross section of the sphere compared with the 3-D exact pressure distribution derived in Section 2. Four uniform meshes of $8 \times m^2$ quadratic triangular elements, with $m = 1, 2, 4, 6$ were used. A stable convergence can be observed. The next four Figs. 13-16 illustrate the same experiment for the second value of wave number k. The effect of one order of magnitude smaller value of the LBB constant γ is clearly visible, as the approximation becomes unstable and the method does not converge. All calculations have been carried out on a DEC 3000 workstation.

4-3. Forced Vibrations of a Thick Hollow Sphere by the 3-D Theory

The calculations described in Section 4.2 were repeated for a thick sphere with thickness $t = 0.1$ m (all other data remained the same). The results are summarized in Figs. 17-20. The effect of the increased thickness on spreading the resonant wave numbers is clearly visible.

Fig. 9 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for $k = 1.56$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 1$.

Fig. 10 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for $k = 1.56$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 2$.

Fig. 11 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for $k = 1.56$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 4$.

Fig. 12 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for $k = 1.56$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 6$.

Fig. 13 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for $k = 1.66237$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 1$.

Fig. 14 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for $k = 1.66237$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 2$.

Fig. 15 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for

$k = 1.66237$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 4$.

Fig. 16 Acoustic scattering of a plane wave on a viscoelastic hollow sphere for $k = 1.66237$, comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with $m = 6$.

Fig. 17 Vibrations of a thick spherical shell in vacuum by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 0\%$.

Fig. 18 Vibrations of a thick spherical shell in vacuum by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 5 \times 10^{-3} ka$.

Fig. 19 Vibrations of a thick spherical shell in water by the 3-D theory.

The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 0\%$.

Fig. 20 Vibrations of a thick spherical shell in water by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $\eta = 5 \times 10^{-3} ka$.

4-4. Forced Vibrations of a Rubber-Coated Steel Shell

Finally, we investigate the effect of damping on the LBB constant for the forced vibrations of a silicon rubber-coated spherical steel shell. The data used in the simulation are as follows [1, 9]:

Water density	ρ_w	$=$	$1000\ \mathrm{kg/m^3}$
Sound speed in water	c_w	$=$	$1524\ \mathrm{m/sec}$
Steel density	ρ_s	$=$	$7800\ kg/m^3$
Steel elastic Young′s modulus	E_s	$=$	$2.1 \times 10^{11}\ \mathrm{N/m^2}$
Steel Poisson Ratio	ν_s	$=$	0.3
Rubber density	ρ_b	$=$	$1020\ \mathrm{kg/m^3}$
Rubber elastic Lame′s modulus	λ_e	$=$	$1.19 \times 10^9\ \mathrm{N/m^2}$
	μ_e	$=$	$4.40 \times 10^6\ \mathrm{N/m^2}$
Rubber viscous Lame′s modulus	λ_v	$=$	$6.3 \times 10^6/\omega\ \mathrm{N/m^2}$
	μ_v	$=$	$1.49 \times 10^{-1}\ \mathrm{N/m^2}$
Inner radius	r_1	$=$	$0.995\ \mathrm{m}$
Middle radius	r_2	$=$	$1.005\ \mathrm{m}$
Outer radius	r_3	$=$	$1.010\ \mathrm{m}$
Constant for the chosen norm	C	$=$	1.1×10^{11}

Fig. 21 Vibrations of a coated thin spherical shell in vacuum by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $r_3 = 1.010$ m.

Fig. 22 Vibrations of a coated thin spherical shell in water by the 3-D theory. The pointwise infimum of the curves shown represents dependence of LBB constant γ upon the wave number k for the case $r_3 = 1.010$ m.

Here, Kelvin's model for rubber has been used

$$\lambda^* = \lambda_e - i\omega\lambda_v, \tag{4.146}$$

$$\mu^* = \mu_e - i\omega\mu_v. \tag{4.147}$$

Figs. 21-22 show the effect of the coating thickness on the LBB constants. It is very clear that, compared with Fig. 5 and Fig. 7, coating does not improve the stability.

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