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INTEGRABILITY, MEAN CONVERGENCE, AND PARSEVAL'S FORMULA FOR DOUBLE TRIGONOMETRIC SERIES

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Abstract. Consider the double trigonometric series whose coefficients satisfy conditions of bounded variation of order $(p,0)$, $(0,p)$, and (p,p) with the weight $(\overline{|j|}\,\overline{|k|})^{p-1}$ for some $p>1$. The following properties concerning the rectangular partial sums of this series are obtained: (a) regular convergence; (b) uniform convergence; (c) weighted L^r-integrability and weighted L^r-convergence; and (d) Parseval's formula. Our results generalize Bary [1, p. 656], Boas [2, 3], Chen [6, 7], Kolmogorov [9], Marzug [10], Móricz [11, 12, 13, 14], Ul'janov [15], Young [16], and Zygmund [17, p. 4].

0. Introduction

Let $\{c_{jk} : j,k\in\mathbb{Z}\}$ be a double sequence of complex numbers satisfying the following conditions for some $p\in I\!N$:

$$|c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\longrightarrow 0 \qquad \text{as} \quad \max\{|j|,|k|\}\to\infty, \tag{0.1}$$

$$\lim_{|k|\to\infty}\sum_{j=-\infty}^{\infty}|\Delta_{p0}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}=0, \tag{0.2}$$

$$\lim_{|j|\to\infty}\sum_{k=-\infty}^{\infty}|\Delta_{0p}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}=0, \tag{0.3}$$

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(0.4) $$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|\Delta_{pp}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}<\infty,$$

where $\bar{\xi}\equiv\max\{\xi,1\}$ and the finite-order differences $\Delta_{pq}c_{jk}$ are defined by

$$\begin{aligned}
\Delta_{00}c_{jk}&=c_{jk};\\
\Delta_{pq}c_{jk}&=\Delta_{p-1,q}c_{jk}-\Delta_{p-1,q}c_{j+1,k}\qquad (p\ge 1);\\
\Delta_{pq}c_{jk}&=\Delta_{p,q-1}c_{jk}-\Delta_{p,q-1}c_{j,k+1}\qquad (q\ge 1).
\end{aligned}$$

Conditions $(0.2)-(0.4)$ are known as conditions of bounded variation of order $(p,0),(0,p)$, and (p,p) with the weight $(\overline{|j|}\,\overline{|k|})^{p-1}$, respectively. They generalize the concept of monotone sequences. Any double sequence $\{c_{jk}\}$ satisfying (0.4) with $p=2$ is called a quasiconvex sequence (cf. [4, 9, 12]). For $p=1$, conditions (0.2) and (0.3) can be derived from (0.1) and (0.4). Moreover, (0.1) and (0.4) reduce to

(0.1*) $$c_{jk}\longrightarrow 0\qquad\text{as}\quad \max\{|j|,|k|\}\longrightarrow\infty,$$

and

(0.4*) $$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|\Delta_{11}c_{jk}|<\infty.$$

Let $\mathbb{T}=[-\pi,\pi]$ and denote by $s_{mn}(x,y)$ the rectangular partial sums of the double trigonometric series

(0.5) $$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}c_{jk}\,e^{i(jx+ky)}\qquad\qquad (x,y\in\mathbb{T});$$

that is,

$$s_{mn}(x,y)=\sum_{|j|\le m}\sum_{|k|\le n}c_{jk}\,e^{i(jx+ky)}.$$

We say that the series (0.5) *converges in Pringsheim's sense* to $f(x,y)$ if $s_{mn}(x,y)\to f(x,y)$ as $\min\{m,n\}\to\infty$. In addition, if the row series $\sum_{j=-\infty}^{\infty}c_{jk}e^{i(jx+ky)}$ converges for each fixed value of k, and the column series $\sum_{k=-\infty}^{\infty}c_{jk}e^{i(jx+ky)}$ converges for each fixed value of j, then series (0.5) is said to *converge regularly* to $f(x,y)$ (cf. [8]). For $E\subseteq\mathbb{T}^2$, the series (0.5) is said to *converge uniformly* on E to $f(x,y)$ if $s_{mn}(x,y)\to f(x,y)$ uniformly on E as

$\min\{m, n\} \to \infty$. Set

$$\|f\|_{r,\phi} = \left(\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x,y)|^r |\phi(x,y)| \, dxdy \right)^{1/r},$$

$$\|f(\cdot, y)\|_{r,\phi} = \left(\int_{-\pi}^{\pi} |f(x,y)|^r |\phi(x,y)| \, dx \right)^{1/r},$$

$$\|f(x, \cdot)\|_{r,\phi} = \left(\int_{-\pi}^{\pi} |f(x,y)|^r |\phi(x,y)| \, dy \right)^{1/r}.$$

It is well known that $\|f\|_{r,\phi}$, $\|f(\cdot, y)\|_{r,\phi}$, and $\|f(x, \cdot)\|_{r,\phi}$ define norms for $r \geq 1$, while $\|f\|_{r,\phi}^r$, $\|f(\cdot, y)\|_{r,\phi}^r$, and $\|f(x, \cdot)\|_{r,\phi}^r$ induce metrics for $0 < r < 1$.

The purpose of this paper is to investigate the validity of the following statements for suitable r and ϕ:

(0.6) $s_{mn}(x,y)$ converges uniformly to $f(x,y)$ on $\{\alpha \leq |x| \leq \pi\} \times \{\beta \leq |y| \leq \pi\}$ for all $0 < \alpha, \beta \leq \pi$;

(0.7) $s_{mn}(x,y)$ converges regularly to $f(x,y)$ on $(\mathbb{T} \setminus \{0\})^2$;

(0.8) $|f(x,y)|^r \phi(x,y) \in L^1(\mathbb{T}^2)$;

(0.9) $$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |s_{mn}(x,y) - f(x,y)|^r |\phi(x,y)| \, dxdy \longrightarrow 0 \quad \text{as} \quad \min\{m,n\} \to \infty;$$

(0.10) $$\lim_{\epsilon, \delta \downarrow 0} \iint\limits_{\substack{\epsilon \leq |x| \leq \pi \\ \delta \leq |y| \leq \pi}} f(x,y)\phi(x,y) \, dxdy = (4\pi^2) \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} \, \hat{\phi}^*(-j,-k),$$

where

$$\hat{\phi}^*(j,k) = \lim_{\epsilon, \delta \downarrow 0} \frac{1}{4\pi^2} \iint\limits_{\substack{\epsilon \leq |x| \leq \pi \\ \delta \leq |y| \leq \pi}} \phi(x,y) e^{-i(jx+ky)} \, dxdy.$$

Formula (0.10) is known as *Parseval's formula*. The definition for $\hat{\phi}^*(j,k)$ is a generalization of the Fourier coefficient $\hat{\phi}(j,k)$. These problems have been investigated by Bary [1], Boas [2,3], Kolmogorov [9], Ul'janov [15], Young [16], Zygmund [17] for one-dimensional case, and by Chen [4,5,6], Marzug [10], Móricz [11,12,13,14] for higher dimensions. All of them discussed the case of $p = 1$ or $p = 2$ only. Our goal in this paper is to extend the above results from $p = 1$ to general cases. A detailed argument on these problems will be given in the next five sections. Throughout this paper C, C_p, and C_{pr} denote constants, which are not necessarily the same at each occurrence.

1. The Family $\{\Psi_j^k(t)\}$

Let $\Psi_{0^+}^0(t) = \Psi_{0^-}^0(t) = \frac{1}{2}$, $\Psi_j^0(t) = e^{ijt}$ for $j \geq 1$, and $\Psi_{-j}^0(t) = \Psi_j^0(-t) = e^{-ijt}$ for $j \geq 1$. Denote by $\{\Psi_j^k(t)\}$ the Cesàro sums of order k of the sequence $\{\Psi_j^0(t)\}$ (cf. [17] for this terminology). By this we mean

$$\begin{aligned} \Psi_j^k(t) &= \sum_{s=0^+}^{j} \Psi_s^{k-1}(t) \qquad && (k \geq 1, j \geq 0^+), \\ \Psi_{-j}^k(t) &= \sum_{s=0^+}^{j} \Psi_{-s}^{k-1}(t) \qquad && (k \geq 1, j \geq 0^+). \end{aligned}$$

Here $-0^+ \equiv 0^-$. Obviously, $\Psi_{-j}^k(t) = \Psi_j^k(-t)$ for all $k \geq 0$ and all $j \geq 0^+$. As given in [6], we introduce the following finite-order differences

$$\begin{aligned} \Delta_{00}^* c_{jk} &= c_{jk}; \\ \Delta_{pq}^* c_{jk} &= \Delta_{p-1,q}^* c_{jk} - \Delta_{p-1,q}^* c_{\tau(j),k} \qquad && (p \geq 1); \\ \Delta_{pq}^* c_{jk} &= \Delta_{p,q-1}^* c_{jk} - \Delta_{p,q-1}^* c_{j,\tau(k)} \qquad && (q \geq 1). \end{aligned}$$

Here $c_{0^+,k} = c_{0^-,k} = c_{0k}$ and $c_{j,0^+} = c_{j,0^-} = c_{j0}$. The function $\tau(j)$ is defined by $\tau(0^+) = 1, \tau(0^-) = -1, \tau(j) = j+1$ for $j \geq 1$, and $\tau(j) = j-1$ for $j \leq -1$. After applying a double summation by parts, we obtain

$$\begin{aligned} s_{mn}(x,y) = & \sum_{|j|=0^+}^{m} \sum_{|k|=0^+}^{n} (\Delta_{pp}^* c_{jk}) \Psi_j^p(x) \Psi_k^p(y) \\ & + \sum_{t=0}^{p-1} \sum_{|j|=0^+}^{m} \sum_{|k|=n} (\Delta_{pt}^* c_{j,\tau(k)}) \Psi_j^p(x) \Psi_k^{t+1}(y) \\ & + \sum_{s=0}^{p-1} \sum_{|j|=m} \sum_{|k|=0^+}^{n} (\Delta_{sp}^* c_{\tau(j),k}) \Psi_j^{s+1}(x) \Psi_k^p(y) \\ & + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{|j|=m} \sum_{|k|=n} (\Delta_{st}^* c_{\tau(j),\tau(k)}) \Psi_j^{s+1}(x) \Psi_k^{t+1}(y), \end{aligned} \tag{1.1}$$

where $m, n > 0$, $x, y \in \mathbb{T}$, and $\sum_{|j|=0^+}^{m} \equiv \sum_{0^+ \leq j \leq m} + \sum_{-m \leq j \leq 0^-}$. It is important to get an estimate of $|\Psi_j^k(t)|$ and an upper bound for $\|\Psi_j^k\|_{r,\phi}$. For the latter, we introduce the concept of pairs of type I_r below. We say that (ϕ, θ) is a pair of *type* I_r if there is an absolute constant C such that

$$\rho \Big(\int_{|t| \leq \pi/\rho} |\phi(t)|\, dt \Big)^{1/r} + \Big(\int_{\pi/\rho \leq |t| \leq \pi} \frac{|\phi(t)|}{|t|^r}\, dt \Big)^{1/r} \leq C\theta(\rho) \quad \text{for all } \rho \geq 1.$$

This generalizes the concept of pairs of type I given in [6]. The main result of this section reads as follows.

Theorem 1.1.

(i) *For all $t \in \mathbb{T}$, all j, and all $k \geq 1$,*

$$|\Psi_j^k(t)| \leq 2^{k(k+1)/2}\pi \cdot \min\{(\overline{|j|})^k, (\overline{|j|})^{k-1}|t|^{-1}\}.$$

(ii) *Let $p \geq 1$, $0 < r < \infty$, and (ϕ, θ) be of type I_r. Then*

$$\left(\int_{-\pi}^{\pi} |\Psi_j^k(t)|^r |\phi(t)|\, dt\right)^{1/r} \leq C_{pr}\, (\overline{|j|})^{p-1}\theta(\overline{|j|})$$

for all j and all $0 \leq k \leq p$.

Proof. An elementary calculation gives

$$|\Psi_j^1(t)| \leq \min\{|j| + 1/2, \pi/|t|\}.$$

This proves (i) for $k = 1$. For $j \geq 0^+$, we have

$$|\Psi_j^{k+1}(t)| \leq \sum_{s=0^+}^{j} |\Psi_s^k(t)| \qquad \text{and} \qquad |\Psi_{-j}^{k+1}(t)| \leq \sum_{s=0^+}^{j} |\Psi_{-s}^k(t)|.$$

By induction on k, we get (i). As for (ii), it follows from (i) that

$$|\Psi_j^k(t)| \leq 2^{p(p+1)/2}\pi \cdot \min\{(\overline{|j|})^p, (\overline{|j|})^{p-1}|t|^{-1}\}$$

for all $t \in \mathbb{T}$, all j, and all $0 \leq k \leq p$. Let $\rho = \overline{|j|}$. We obtain

$$\begin{aligned}
&\left(\int_{-\pi}^{\pi} |\Psi_j^k(t)|^r |\phi(t)|\, dt\right)^{1/r} \\
&\quad \leq 2^{p(p+1)/2+1/r}\pi \left\{ (\overline{|j|})^p \left(\int_{|t|\leq\pi/\rho} |\phi(t)|\, dt\right)^{1/r} \right. \\
&\qquad\qquad \left. + (\overline{|j|})^{p-1}\left(\int_{\pi/\rho\leq|t|\leq\pi} \frac{|\phi(t)|}{|t|^r}\, dt\right)^{1/r} \right\} \\
&\quad \leq C_{pr}\, (\overline{|j|})^{p-1}\theta(\overline{|j|}).
\end{aligned}$$

2. Convergence for $\phi(x, y) = O(|xy|^{r-1+\epsilon})$

The following theorem confirms (0.6)–(0.9) for the case that $\phi(x, y)/|xy|^r \in L^1(\mathbb{T}^2)$, where $r > 0$. In particular, it will apply to the case that $\phi(x, y) = O(|xy|^\sigma)$, where $\sigma > r - 1$. Our result for $\sigma = 0$ generalizes [11], [15], and

[17, p. 4]. Since any quasiconvex null-sequence is of bounded variation, the following result also includes [9] as a special case.

Theorem 2.1. *Assume that conditions* $(0.1)-(0.4)$ *are satisfied for some* $p \geq 1$. *Then series* (0.5) *converges regularly to some measurable function* $f(x, y)$ *for all* $x, y \in \mathbb{T} \setminus \{0\}$, *and the convergence is uniform on the rectangle* $\{\alpha \leq |x| \leq \pi, \beta \leq |y| \leq \pi\}$ *for all* $0 < \alpha, \beta \leq \pi$. *In addition, let* $r > 0$ *and* $x_0, y_0 \in \mathbb{T} \setminus \{0\}$.

(i) *If* $\phi(x, y_0)/|x|^r \in L^1(\mathbb{T})$, *then* $|f(x, y_0)|^r \phi(x, y_0) \in L^1(\mathbb{T})$ *and*

$$\|s_{mn}(\cdot, y_0) - f(\cdot, y_0)\|_{r,\phi} \to 0 \text{ as } \min\{m, n\} \to \infty.$$

(ii) *If* $\phi(x_0, y)/|y|^r \in L^1(\mathbb{T})$, *then* $|f(x_0, y)|^r \phi(x_0, y) \in L^1(\mathbb{T})$ *and*

$$\|s_{mn}(x_0, \cdot) - f(x_0, \cdot)\|_{r,\phi} \to 0 \text{ as } \min\{m, n\} \to \infty.$$

(iii) *If* $\phi(x, y)/|xy|^r \in L^1(\mathbb{T}^2)$, *then* $|f(x, y)|^r \phi(x, y) \in L^1(\mathbb{T}^2)$ *and*

$$\|s_{mn} - f\|_{r,\phi} \to 0 \text{ as } \min\{m, n\} \to \infty.$$

Moreover, the conclusions (i)$-$(iii) *still hold provided the corresponding* $L^1(\mathbb{T})$ *and* $L^1(\mathbb{T}^2)$ *are replaced by* $C(\mathbb{T})$ *and* $C(\mathbb{T}^2)$.

Proof. By Theorem 1.1 (i), we get the following estimates:

$$\begin{aligned}
&\sum_{|j|=0^+}^{m} \sum_{|k|=0^+}^{n} |(\Delta_{pp}^* c_{jk}) \Psi_j^p(x) \Psi_k^p(y)| \\
&\qquad \leq C_p \left(\sum_{|j|=0^+}^{m} \sum_{|k|=0^+}^{n} |\Delta_{pp}^* c_{jk}| \, (\overline{|j|}\, \overline{|k|})^{p-1} \right) |xy|^{-1}
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&\sum_{t=0}^{p-1} \sum_{|j|=0^+}^{m} \sum_{|k|=n} |(\Delta_{pt}^* c_{j,\tau(k)}) \Psi_j^p(x) \Psi_k^{t+1}(y)| \\
&\qquad \leq C_p \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \left(\sum_{|j|=0^+}^{m} \sum_{|k|=n+v+1} |\Delta_{p0}^* c_{jk}| \, (\overline{|j|}\, \overline{|k|})^{p-1} \right) |xy|^{-1} \\
&\qquad \leq C_p \left(\sup\nolimits_{|k|>n} \sum_{|j|=0^+}^{m} |\Delta_{p0}^* c_{jk}| \, (\overline{|j|}\, \overline{|k|})^{p-1} \right) |xy|^{-1}.
\end{aligned} \tag{2.2}$$

Similarly, we have

$$\begin{aligned}(2.3)\qquad &\sum_{s=0}^{p-1}\sum_{|j|=m}\sum_{|k|=0^+}^{n}|(\Delta^*_{sp}c_{\tau(j),k})\Psi_j^{s+1}(x)\Psi_k^p(y)|\\ &\qquad \le C_p\left(\sup\nolimits_{|j|>m}\sum_{|k|=0^+}^{n}|\Delta^*_{0p}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\right)|xy|^{-1}\end{aligned}$$

and

$$\begin{aligned}(2.4)\qquad &\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{|j|=m}\sum_{|k|=n}|(\Delta^*_{st}c_{\tau(j),\tau(k)})\Psi_j^{s+1}(x)\Psi_k^{t+1}(y)|\\ &\quad \le C_p\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}\sum_{|j|=m+u+1}\sum_{|k|=n+v+1}|\Delta^*_{00}c_{jk}|\\ &\qquad\qquad \times(\overline{|j|}\,\overline{|k|})^{p-1}|xy|^{-1}\\ &\quad \le C_p\Big(\sup\nolimits_{|j|>m,|k|>n}|c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\Big)|xy|^{-1}.\end{aligned}$$

Obviously, conditions $(0.2)-(0.4)$ are equivalent to

$$\lim_{|k|\to\infty}\sum_{|j|=0^+}^{\infty}|\Delta^*_{p0}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}=0, \tag{2.5}$$

$$\lim_{|j|\to\infty}\sum_{|k|=0^+}^{\infty}|\Delta^*_{0p}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}=0, \tag{2.6}$$

$$\sum_{|j|=0^+}^{\infty}\sum_{|k|=0^+}^{\infty}|\Delta^*_{pp}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}<\infty. \tag{2.7}$$

Putting these with $(0.1), (1.1)$, $(2.1)-(2.4)$ together, we infer that $s_{mn}(x,y)$ converges to some measurable function $f(x,y)$ for $x,y\in\mathbb{T}\setminus\{0\}$, and the convergence is uniform on the rectangle $\{\alpha\le|x|\le\pi,\beta\le|y|\le\pi\}$ for all $0<\alpha,\beta\le\pi$. Moreover,

$$f(x,y)=\sum_{|j|=0^+}^{\infty}\sum_{|k|=0^+}^{\infty}(\Delta^*_{pp}c_{jk})\Psi_j^p(x)\Psi_k^p(y). \tag{2.8}$$

A modified proof also shows that series (0.5) converges regularly to $f(x,y)$ for $x,y\in\mathbb{T}\setminus\{0\}$. Assume that $r>0$ and $\phi(x,y)/|xy|^r\in L^1(\mathbb{T}^2)$. Then, by

$(2.1), (2.7)$, and (2.8), we get

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^r|\phi(x,y)|\,dxdy\\&\qquad\le C_p\|\phi(x,y)/|xy|^r\|_1\left\{\sum_{|j|=0^+}^{\infty}\sum_{|k|=0^+}^{\infty}|\Delta_{pp}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\right\}^r\\&\qquad<\infty.\end{aligned}$$

This says that $|f(x,y)|^r\phi(x,y)\in L^1(\mathbb{T}^2)$. Set $M=\|\phi(x,y)/|xy|^r\|_1$ and $\Lambda_{mn}\equiv\{(j,k)\in\mathbb{Z}\times\mathbb{Z}:|j|>m \text{ or } |k|>n\}$. By (1.1), (2.1) – (2.4), and (2.8), we infer that, for $0<r<1$,

$$\begin{aligned}&\|s_{mn}-f\|_{r,\phi}^r\\&\le C_pM\Bigg\{\left(\sum_{\Lambda_{mn}}|\Delta_{pp}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\right)^r+\left(\sup_{|k|>n}\sum_{|j|=0^+}^{m}|\Delta_{p0}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\right)^r\\&\quad+\left(\sup\nolimits_{|j|>m}\sum_{|k|=0^+}^{n}|\Delta_{0p}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\right)^r+\left(\sup_{|j|>m,|k|>n}|c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\right)^r\Bigg\}\\&\longrightarrow 0\qquad\text{as}\quad\min\{m,n\}\to\infty,\end{aligned}$$

and, for $r\ge1$,

$$\begin{aligned}&\|s_{mn}-f\|_{r,\phi}\\&\le C_pM^{1/r}\Bigg\{\sum_{\Lambda_{mn}}|\Delta_{pp}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}+\sup_{|k|>n}\sum_{|j|=0^+}^{m}|\Delta_{p0}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\\&\quad+\sup\nolimits_{|j|>m}\sum_{|k|=0^+}^{n}|\Delta_{0p}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}+\sup_{|j|>m,|k|>n}|c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\Bigg\}\\&\longrightarrow 0\qquad\text{as}\quad\min\{m,n\}\to\infty,\end{aligned}$$

which proves (iii). For $x,y\in\mathbb{T}\setminus\{0\}$, it follows from (2.8) that

$$xyf(x,y)=\sum_{|j|=0^+}^{\infty}\sum_{|k|=0^+}^{\infty}\left(\Delta_{pp}^*c_{jk}\right)\left(x\Psi_j^p(x)\right)\left(y\Psi_k^p(y)\right).$$

We have $\|(x\Psi_j^p(x))(y\Psi_k^p(y))\|_\infty\le C_p(\overline{|j|}\,\overline{|k|})^{p-1}$ for all j and k. By (2.7), we find that $xyf(x,y)\in C(\mathbb{T}^2)$. If $\phi(x,y)/|xy|^r\in C(\mathbb{T}^2)$, then $|f(x,y)|^r\phi(x,y)=|xyf(x,y)|^r\left(\phi(x,y)/|xy|^r\right)\in C(\mathbb{T}^2)$. The proofs of (i) and (ii) are similar, and we leave them to the reader.

Set $\phi_1(x) = O(|x|^\sigma)$, $\phi_2(y) = O(|y|^\lambda)$, and $\phi_3(x,y) = O(|x|^\sigma|y|^\lambda)$, where $\sigma, \lambda > r-1$. Then $\phi_1(x)/|x|^r \in L^1(\mathbb{T})$, $\phi_2(y)/|y|^r \in L^1(\mathbb{T})$, and $\phi_3(x,y)/|xy|^r \in L^1(\mathbb{T}^2)$. Thus, Theorem 2.1 can apply to this case.

Corollary 2.2. *Assume that conditions* (0.1) − (0.4) *are satisfied for some* $p \geq 1$.*Then series* (0.5) *converges regularly to some measurable function* $f(x,y)$ *for all* $x, y \in \mathbb{T} \setminus \{0\}$, *and the convergence is uniform on the rectangle* $\{\alpha \leq |x| \leq \pi, \beta \leq |y| \leq \pi\}$ *for all* $0 < \alpha, \beta \leq \pi$. *Moreover, let* $r > 0$ *and* $\sigma, \lambda > r-1$. *The following statements are true.*

(i) *For all* $y \in \mathbb{T}\setminus\{0\}$, $|f(x,y)|^r|x|^\sigma \in L^1(\mathbb{T})$ *and* $\|s_{mn}(\cdot,y) - f(\cdot,y)\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where* $\phi(x,y) = O(|x|^\sigma)$.

(ii) *For all* $x \in \mathbb{T}\setminus\{0\}$, $|f(x,y)|^r|y|^\lambda \in L^1(\mathbb{T})$ *and* $\|s_{mn}(x,\cdot) - f(x,\cdot)\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where* $\phi(x,y) = O(|y|^\lambda)$.

(iii) $|f(x,y)|^r|x|^\sigma|y|^\lambda \in L^1(\mathbb{T}^2)$ *and* $\|s_{mn} - f\|_{r,\phi} \longrightarrow 0$ *as* $\min\{m,n\} \to \infty$, *where* $\phi(x,y) = O(|x|^\sigma|y|^\lambda)$.

Whenever $\sigma, \lambda \geq r$, $|f(x,y)|^r|x|^\sigma \in C(\mathbb{T})$ *for all* $y \in \mathbb{T}\setminus\{0\}$, $|f(x,y)|^r|y|^\lambda \in C(\mathbb{T})$ *for all* $x \in \mathbb{T}\setminus\{0\}$, *and* $|f(x,y)|^r|x|^\sigma|y|^\lambda \in C(\mathbb{T}^2)$.

As indicated in [5, 6, 12], Corollary 2.2 may not hold for the case that $\sigma = \lambda = r-1$. For this case, consider $\phi_1(x) = O(|x|^{r-1}(\overline{\log 1/|x|})^{-\epsilon})$, $\phi_2(y) = O(|y|^{r-1}(\overline{\log 1/|y|})^{-\delta})$, and $\phi_3(x,y) = O(|xy|^{r-1}(\overline{\log 1/|x|})^{-\epsilon}(\overline{\log 1/|y|})^{-\delta})$, where $\epsilon, \delta > 1$. We have $\phi_1(x)/|x|^r \in L^1(\mathbb{T})$, $\phi_2(y)/|y|^r \in L^1(\mathbb{T})$, and $\phi_3(x,y)/|xy|^r \in L^1(\mathbb{T}^2)$. Thus, we are led to the following result.

Corollary 2.3. *Assume that conditions* (0.1) − (0.4) *are satisfied for some* $p \geq 1$. *Then series* (0.5) *converges regularly to some measurable function* $f(x,y)$ *for all* $x, y \in \mathbb{T} \setminus \{0\}$, *and the convergence is uniform on* $\{\alpha \leq |x| \leq \pi, \beta \leq |y| \leq \pi\}$ *for all* $0 < \alpha, \beta \leq \pi$. *Moreover, let* $r > 0$ *and* $\epsilon, \delta > 1$. *The following statements are true.*

(i) *For all* $y \in \mathbb{T} \setminus \{0\}$, $|f(x,y)|^r|x|^{r-1}(\overline{\log 1/|x|})^{-\epsilon} \in L^1(\mathbb{T})$ *and* $\|s_{mn}(\cdot,y) - f(\cdot,y)\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where*
$$\phi(x,y) = O(|x|^{r-1}(\overline{\log 1/|x|})^{-\epsilon}).$$

(ii) *For all* $x \in \mathbb{T} \setminus \{0\}$, $|f(x,y)|^r|y|^{r-1}(\overline{\log 1/|y|})^{-\delta} \in L^1(\mathbb{T})$ *and* $\|s_{mn}(x,\cdot) - f(x,\cdot)\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where*
$$\phi(x,y) = O(|y|^{r-1}(\overline{\log 1/|y|})^{-\delta}).$$

(iii) $|f(x,y)|^r |xy|^{r-1}(\overline{\log 1/|x|})^{-\epsilon}(\overline{\log 1/|y|})^{-\delta} \in L^1(\mathbb{T}^2)$ *and* $\|s_{mn} - f\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where*

$$\phi(x,y) = O(|xy|^{r-1}(\overline{\log 1/|x|})^{-\epsilon}(\overline{\log 1/|y|})^{-\delta}).$$

3. Convergence for $\phi(x,y) = O(|xy|^{r-1-\epsilon})$

Theorem 2.1(iii) excludes the case $\phi(x,y) = |xy|^{\sigma}$, where $\sigma \le r-1$. In this section, we investigate the case. To do so, we shall assume the following conditions, which are stronger than (0.1) – (0.4).

$$|c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\theta(\overline{|j|})\vartheta(\overline{|k|}) \longrightarrow 0 \text{ as } \max\{|j|,|k|\} \to \infty, \tag{0.1$'$}$$

$$\lim_{|k|\to\infty} \sum_{j=-\infty}^{\infty} |\Delta_{p0} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\theta(\overline{|j|})\vartheta(\overline{|k|}) = 0, \tag{0.2$'$}$$

$$\lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |\Delta_{0p} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\theta(\overline{|j|})\vartheta(\overline{|k|}) = 0, \tag{0.3$'$}$$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\Delta_{pp} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\theta(\overline{|j|})\vartheta(\overline{|k|}) < \infty, \tag{0.4$'$}$$

where θ and ϑ are two positive increasing functions defined on $[1,\infty)$. For $p=1$, conditions (0.2′) and (0.3′) can be derived from (0.1*) and (0.4′), and conditions (0.1*) and (0.4′) together imply (0.1′). The main result of this section is the following theorem, which extends [6, Theorem 1] from $p=1$ to general cases. As indicated in [6], our result also generalizes Boas [3], Marzug [10], Móricz [12], and Young [16]. A detailed argument on these will be given later.

Theorem 3.1. *Assume that θ and ϑ are two positive increasing functions defined on $[1,\infty)$ such that (0.1′) – (0.4′) are satisfied for some $p \ge 1$. Then series (0.5) converges regularly to some measurable function $f(x,y)$ for all $x,y \in \mathbb{T}\setminus\{0\}$, and the convergence is uniform on the rectangle $\{\alpha \le |x| \le \pi, \beta \le |y| \le \pi\}$ for all $0 < \alpha,\beta \le \pi$. In addition, let $r \ge 1$.*

(i) *If (ϕ,θ) is of type I_r, then for all $y \in \mathbb{T}\setminus\{0\}$, $|f(x,y)|^r\phi(x) \in L^1(\mathbb{T})$ and $\|s_{mn}(\cdot,y) - f(\cdot,y)\|_{r,\phi} \to 0$ as $\min\{m,n\}\to\infty$, where $\phi(x,y) = \phi(x)$.*

(ii) *If (ψ,ϑ) is of type I_r, then for all $x \in \mathbb{T}\setminus\{0\}$, $|f(x,y)|^r\psi(y) \in L^1(\mathbb{T})$ and $\|s_{mn}(x,\cdot) - f(x,\cdot)\|_{r,\phi} \to 0$ as $\min\{m,n\}\to\infty$, where $\phi(x,y) = \psi(y)$.*

(iii) *If* (ϕ,θ) *and* (ψ,ϑ) *are of type* I_r, *then* $|f(x,y)|^r\phi(x)\psi(y)\in L^1(\mathbb{T}^2)$ *and* $\|s_{mn}-f\|_{r,\phi}\to 0$ *as* $\min\{m,n\}\to\infty$, *where* $\phi(x,y)=\phi(x)\psi(y)$.

Proof. Since $(0.1')-(0.4')$ imply $(0.1)-(0.4)$, it follows from Theorem 2.1 that series (0.5) converges regularly to some measurable function $f(x,y)$ for all $x,y\in\mathbb{T}\setminus\{0\}$, and the convergence is uniform on the rectangle $\{\alpha\le|x|\le\pi,\beta\le|y|\le\pi\}$ for all $0<\alpha,\beta\le\pi$. Let $r\ge 1$ and assume that (ϕ,θ) and (ψ,ϑ) are of type I_r. Set

$$\alpha_j^k=\left(\int_{-\pi}^{\pi}|\Psi_j^k(x)|^r|\phi(x)|\,dx\right)^{1/r},\quad \beta_j^k=\left(\int_{-\pi}^{\pi}|\Psi_j^k(y)|^r|\psi(y)|\,dy\right)^{1/r}.$$

Then Theorem 1.1(ii) says that $\alpha_j^k\le C_{pr}\overline{|j|}^{p-1}\theta(\overline{|j|})$ and $\beta_j^k\le C_{pr}\overline{|j|}^{p-1}\vartheta(\overline{|j|})$ for all j and all $0\le k\le p$. By $(0.4')$, (2.8), Fatou's lemma, and Minkowski's inequality, we infer that

$$\begin{aligned}&\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^r|\phi(x)\psi(y)|\,dxdy\right)^{1/r}\\ &\quad\le C_{pr}\left\{\liminf_{m\to\infty}\sum_{|j|=0^+}^{m}\sum_{|k|=0^+}^{m}|\Delta_{pp}^*c_{jk}|\,\alpha_j^p\beta_k^p\right\}\\ &\quad\le C_{pr}\left\{\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|\Delta_{pp}c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\,\theta(\overline{|j|})\vartheta(\overline{|k|})\right\}\\ &\quad<\infty.\end{aligned}$$

Let Λ_{mn} consist of all (j,k) with $|j|>m$ or $|k|>n$. By (1.1), (2.8), and $(0.1')-(0.4')$, we get

$$\begin{aligned}&\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|s_{mn}(x,y)-f(x,y)|^r|\phi(x)\psi(y)|\,dxdy\right)^{1/r}\\ &\le C_{pr}\Bigg\{\sum_{\Lambda_{mn}}|\Delta_{pp}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\,\theta(\overline{|j|})\vartheta(\overline{|k|})\\ &\qquad+\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\left(\sum_{|j|=0^+}^{m}\sum_{|k|=n+v+1}|\Delta_{p0}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\theta(\overline{|j|})\vartheta(\overline{|k|})\right)\\ &\qquad+\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\left(\sum_{|j|=m+u+1}\sum_{|k|=0^+}^{n}|\Delta_{0p}^*c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\theta(\overline{|j|})\vartheta(\overline{|k|})\right)\\ &\qquad+\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}\sum_{|j|=m+u+1}\sum_{|k|=n+v+1}|\Delta_{00}^*c_{jk}|\\ &\qquad\qquad\times(\overline{|j|}\,\overline{|k|})^{p-1}\,\theta(\overline{|j|})\vartheta(\overline{|k|})\Bigg\}\end{aligned}$$

$$\begin{aligned}
&\le C_{pr}\Bigg\{ \sum_{\Lambda_{mn}} |\Delta^*_{pp} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\,\theta(\overline{|j|})\vartheta(\overline{|k|}) \\
&\qquad +2^p\Big(\sup\nolimits_{|k|>n} \sum_{|j|=0^+}^{m} |\Delta^*_{p0} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\,\theta(\overline{|j|})\vartheta(\overline{|k|})\Big) \\
&\qquad +2^p\Big(\sup\nolimits_{|j|>m} \sum_{|k|=0^+}^{n} |\Delta^*_{0p} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\,\theta(\overline{|j|})\vartheta(\overline{|k|})\Big) \\
&\qquad +2^{2p}\Big(\sup\nolimits_{|j|>m,|k|>n} |c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}\,\theta(\overline{|j|})\vartheta(\overline{|k|})\Big)\Bigg\} \\
&\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty.
\end{aligned}$$

This completes the proof of (iii). For (i) and (ii), we leave them to the reader.

We now go back to discuss applications of Theorem 3.1. By definition, we find that if $(\phi, 1)$ is a pair of type I_r, then $\phi(t)/|t|^r \in L^1(\mathbb{T})$. Thus, the case $\theta(\rho) = \vartheta(\rho) = 1$ has been dealt with in Theorem 2.1. If we consider $\theta(\rho) = (\overline{\log \rho})^{(1-\epsilon)/r}$ and $\vartheta(\rho) = (\overline{\log \rho})^{(1-\delta)/r}$ with $0 \le \epsilon, \delta < 1$, then conditions $(0.1') - (0.4')$ become

$(0.1'')$

$$|c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}(\overline{\log |j|})^{(1-\epsilon)/r}(\overline{\log |k|})^{(1-\delta)/r} \to 0 \quad \text{as } \max\{|j|,|k|\} \to \infty,$$

$$(0.2'') \quad \lim_{|k|\to\infty} \sum_{j=-\infty}^{\infty} |\Delta_{p0} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}(\overline{\log |j|})^{(1-\epsilon)/r}(\overline{\log |k|})^{(1-\delta)/r} = 0,$$

$$(0.3'') \quad \lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |\Delta_{0p} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}(\overline{\log |j|})^{(1-\epsilon)/r}(\overline{\log |k|})^{(1-\delta)/r} = 0,$$

$$(0.4'') \quad \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\Delta_{pp} c_{jk}|\,(\overline{|j|}\,\overline{|k|})^{p-1}(\overline{\log |j|})^{(1-\epsilon)/r}(\overline{\log |k|})^{(1-\delta)/r} < \infty.$$

Conditions $(0.2'')$ and $(0.3'')$ are not necessary for $p = 1$, and condition $(0.1'')$ can be replaced by (0.1^*) for this case. An elementary calculation says that (ϕ, θ) and (ψ, ϑ) are of type I_r, where $\phi(t) = O(|t|^{r-1}(\overline{\log 1/|t|})^{-\epsilon})$ and $\psi(t) = O(|t|^{r-1}(\overline{\log 1/|t|})^{-\delta})$. As a consequence of Theorem 3.1, we get the following extension of [6, Corollary 2]. As indicated in [6], it generalizes [12, Theorems 2, 4, and 5] and [16].

Corollary 3.2. *Let $p, r \geq 1$ and $0 \leq \epsilon, \delta < 1$. Assume that conditions* $(0.1'') - (0.4'')$ *are satisfied for $p, r, \epsilon,$ and δ. Then series* (0.5) *converges regularly to some measurable function $f(x,y)$ for all $x, y \in \mathbb{T} \setminus \{0\}$, and the convergence is uniform on the rectangle $\{\alpha \leq |x| \leq \pi, \beta \leq |y| \leq \pi\}$ for all $0 < \alpha, \beta \leq \pi$. Moreover,*

(i) *for all $y \in \mathbb{T} \setminus \{0\}$, $|f(x,y)|^r |x|^{r-1} (\overline{\log 1/|x|})^{-\epsilon} \in L^1(\mathbb{T})$ and $\|s_{mn}(\cdot, y) - f(\cdot, y)\|_{r,\phi} \to 0$ as $\min\{m,n\} \to \infty$, where*
$$\phi(x,y) = O(|x|^{r-1} (\overline{\log 1/|x|})^{-\epsilon});$$

(ii) *for all $x \in \mathbb{T} \setminus \{0\}$, $|f(x,y)|^r |y|^{r-1} (\overline{\log 1/|y|})^{-\delta} \in L^1(\mathbb{T})$ and $\|s_{mn}(x, \cdot) - f(x, \cdot)\|_{r,\phi} \to 0$ as $\min\{m,n\} \to \infty$, where*
$$\phi(x,y) = O(|y|^{r-1} (\overline{\log 1/|y|})^{-\delta});$$

(iii) *$|f(x,y)|^r |xy|^{r-1} (\overline{\log 1/|x|})^{-\epsilon} (\overline{\log 1/|y|})^{-\delta} \in L^1(\mathbb{T}^2)$ and $\|s_{mn} - f\|_{r,\phi} \to 0$ as $\min\{m,n\} \to \infty$, where*
$$\phi(x,y) = O(|xy|^{r-1} (\overline{\log 1/|x|})^{-\epsilon} (\overline{\log 1/|y|})^{-\delta}).$$

The third case we investigate is $\theta(\rho) = \rho^{1-(\sigma+1)/r}$ and $\vartheta(\rho) = \rho^{1-(\lambda+1)/r}$, where $-1 < \sigma, \lambda < r-1$. In this case, conditions $(0.1') - (0.4')$ take the forms:

$$(0.1''') \qquad |c_{jk}| \, (\overline{|j|})^{p-(\sigma+1)/r} (\overline{|k|})^{p-(\lambda+1)/r} \to 0 \quad \text{as } \max\{|j|, |k|\} \to \infty,$$

$$(0.2''') \qquad \lim_{|k| \to \infty} \sum_{j=-\infty}^{\infty} |\Delta_{p0} c_{jk}| \, (\overline{|j|})^{p-(\sigma+1)/r} (\overline{|k|})^{p-(\lambda+1)/r} = 0,$$

$$(0.3''') \qquad \lim_{|j| \to \infty} \sum_{k=-\infty}^{\infty} |\Delta_{0p} c_{jk}| \, (\overline{|j|})^{p-(\sigma+1)/r} (\overline{|k|})^{p-(\lambda+1)/r} = 0,$$

$$(0.4''') \qquad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{pp} c_{jk}| \, (\overline{|j|})^{p-(\sigma+1)/r} (\overline{|k|})^{p-(\lambda+1)/r} < \infty.$$

It is known that (ϕ, θ) and (ψ, ϑ) are of type I_r, where $\phi(t) = O(|t|^\sigma)$ and $\psi(t) = O(|t|^\lambda)$. This leads us to the following extension of [6, Corollary 6], which generalizes [10, Theorem 4] and [3, Theorems 4.1 and 4.2].

Corollary 3.3. *Assume that conditions* $(0.1''') - (0.4''')$ *are satisfied for some $p, r \geq 1$ and some $-1 < \sigma, \lambda < r-1$. Then series* (0.5) *converges regularly to some measurable function $f(x,y)$ for all $x, y \in \mathbb{T} \setminus \{0\}$, and the convergence is uniform on the rectangle $\{\alpha \leq |x| \leq \pi, \beta \leq |y| \leq \pi\}$ for all $0 < \alpha, \beta \leq \pi$. Moreover,*

(i) *for all* $y \in \mathbb{T} \setminus \{0\}$, $|f(x,y)|^r |x|^\sigma \in L^1(\mathbb{T})$ *and* $\|s_{mn}(\cdot, y) - f(\cdot, y)\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where* $\phi(x,y) = O(|x|^\sigma)$;

(ii) *for all* $x \in \mathbb{T} \setminus \{0\}$, $|f(x,y)|^r |y|^\lambda \in L^1(\mathbb{T})$ *and* $\|s_{mn}(x, \cdot) - f(x, \cdot)\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where* $\phi(x,y) = O(|y|^\lambda)$;

(iii) $|f(x,y)|^r |x|^\sigma |y|^\lambda \in L^1(\mathbb{T}^2)$ *and* $\|s_{mn} - f\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where* $\phi(x,y) = O(|x|^\sigma |y|^\lambda)$.

4. Parseval's Formula

To ensure the validity of (0.10), we shall assume conditions (0.1) – (0.4) for some $p \geq 1$. Under these conditions, Theorem 2.1 guarantees the existence of the limiting function f of series (0.5). To derive (0.10), we shall also assume that

$$\sup_{\substack{-\infty < j,k < \infty \\ 0 < \epsilon, \delta \leq \pi}} \left| \iint_{\substack{\epsilon \leq |x| \leq \pi \\ \delta \leq |y| \leq \pi}} \phi(x,y) \Psi_j^p(x) \Psi_k^p(y) \, dxdy \right| < \infty; \tag{4.1}$$

in other words,

$$\sup_{\substack{-\infty < j,k < \infty \\ 0 < \epsilon, \delta \leq \pi}} |\Phi_{jk}^{pp}(\epsilon, \delta)| < \infty, \tag{4.2}$$

where

$$\Phi_{jk}^{st}(\epsilon, \delta) \equiv \iint_{\substack{\epsilon \leq |x| \leq \pi \\ \delta \leq |y| \leq \pi}} \phi(x,y) \Psi_j^s(x) \Psi_k^t(y) \, dxdy. \tag{4.3}$$

Let $\phi_{\epsilon\delta}(x,y) = \phi(x,y)$ for $\epsilon \leq |x| \leq \pi$ and $\delta \leq |y| \leq \pi$, and 0 otherwise. The definition of $\Psi_j^k(t)$ tells us that

$$\Phi_{jk}^{00}(\epsilon, \delta) = \begin{cases} 4\pi^2 \hat{\phi}_{\epsilon\delta}(-j, -k) & \text{if} \quad |j|, |k| \geq 1, \\ 2\pi^2 \hat{\phi}_{\epsilon\delta}(-j, 0) & \text{if} \quad |j| \geq 1, |k| = 0^+, \\ 2\pi^2 \hat{\phi}_{\epsilon\delta}(0, -k) & \text{if} \quad |j| = 0^+, |k| \geq 1, \\ \pi^2 \hat{\phi}_{\epsilon\delta}(0, 0) & \text{if} \quad |j| = |k| = 0^+. \end{cases}$$

Moreover, for $s, t \geq 1$ and $j, k \geq 0^+$,

$$\begin{aligned} \Phi_{jk}^{st}(\epsilon, \delta) &= \sum_{u=0^+}^{j} \Phi_{uk}^{s-1,t}(\epsilon, \delta) = \sum_{v=0^+}^{k} \Phi_{jv}^{s,t-1}(\epsilon, \delta) \\ &= \sum_{u=0^+}^{j} \sum_{v=0^+}^{k} \Phi_{uv}^{s-1,t-1}(\epsilon, \delta). \end{aligned}$$

This says that $\Phi^{st}_{jk}(\epsilon,\delta)$ with $j,k\geq 0^+$ can be regarded as the two -dimensional Cesàro sums of order (s,t) of the double sequence $\{\Phi^{00}_{jk}(\epsilon,\delta) : j,k \geq 0^+\}$. Similarly, $\Phi^{st}_{-j,-k}(\epsilon,\delta)$ are the two-dimensional Cesàro sums of order (s,t) of the double sequence $\{\Phi^{00}_{-j,-k}(\epsilon,\delta) : j,k\geq 0^+\}$. Based on these, we see that condition (4.1) is equivalent to the existence of the constant C such that

$$\sup_{\substack{-\infty<j,k<\infty\\ 0<\epsilon,\delta\leq\pi}} |\Phi^{st}_{jk}(\epsilon,\delta)| \leq C < \infty \qquad \text{for all} \quad 0\leq s,t\leq p. \tag{4.4}$$

Moreover, condition (4.1) with $p=1$ is equivalent to

$$\sup_{0<\epsilon,\delta\leq\pi} s^{\omega}\phi_{\epsilon\delta}(0,0)<\infty, \qquad (\omega=(\omega_1,\omega_2), |\omega_1|=|\omega_2|=1),$$

where s^ω denotes the one-sided maximal operator defined below:

$$s^{\omega}\phi(x,y) \equiv \sup_{m,n\geq 0} |\sum_{j=0}^{m}\sum_{k=0}^{n}\hat{\phi}^*(j,k)e^{i(jx+ky)}| \qquad (\omega=(1,1));$$
$$s^{\omega}\phi(x,y) \equiv \sup_{m,n\geq 0} |\sum_{j=0}^{m}\sum_{k=0}^{n}\hat{\phi}^*(-j,k)e^{i(-jx+ky)}| \qquad (\omega=(-1,1));$$
$$s^{\omega}\phi(x,y) \equiv \sup_{m,n\geq 0} |\sum_{j=0}^{m}\sum_{k=0}^{n}\hat{\phi}^*(j,-k)e^{i(jx-ky)}| \qquad (\omega=(1,-1));$$
$$s^{\omega}\phi(x,y) \equiv \sup_{m,n\geq 0} |\sum_{j=0}^{m}\sum_{k=0}^{n}\hat{\phi}^*(-j,-k)e^{i(-jx-ky)}| \quad (\omega=(-1,-1)).$$

Hence, the following theorem and Theorem 2.1 together extend [7, Theorem 1] from $p=1$ to the general case. As explained in [7], our result generalizes [1, p. 656], [2], and [13, 14].

Theorem 4.1. *Assume that conditions* (0.1) $-$ (0.4) *are satisfied for some* $p\geq 1$. *Then series* (0.5) *converges regularly to some measurable function* $f(x,y)$ *for all* $x,y\in\mathbb{T}\setminus\{0\}$, *and the convergence is uniform on* $\{\alpha\leq|x|\leq\pi, \beta\leq|y|\leq\pi\}$ *for all* $0<\alpha,\beta\leq\pi$. *Moreover, if* $\phi:\mathbb{T}^2\mapsto\mathbb{C}$ *is measurable and locally bounded in* $(\mathbb{T}\setminus\{0\})^2$, $\hat{\phi}^*(j,k)$ *exists for all* (j,k), *and condition* (4.1) *is satisfied, then formula* (0.10) *holds and the following three statements remain true for* $r\geq 1$.

(i) *If* $\phi(x,y)/|x|\in L^r(\mathbb{T})$ *for almost all* $y\in\mathbb{T}$, *then*

$$\lim_{\epsilon,\delta\downarrow 0}\iint_{\substack{\epsilon\leq|x|\leq\pi\\ \delta\leq|y|\leq\pi}} f(x,y)\phi(x,y)\,dxdy = \lim_{\delta\downarrow 0}\int_{\delta\leq|y|\leq\pi}\int_{-\pi}^{\pi} f(x,y)\phi(x,y)\,dxdy.$$

(ii) *If* $\phi(x,y)/|y| \in L^r(\mathbb{T})$ *for almost all* $x \in \mathbb{T}$, *then*

$$\lim_{\epsilon,\delta\downarrow 0} \iint\limits_{\substack{\epsilon\le|x|\le\pi\\ \delta\le|y|\le\pi}} f(x,y)\phi(x,y)\,dxdy = \lim_{\epsilon\downarrow 0}\int_{-\pi}^{\pi}\int_{\epsilon\le|x|\le\pi} f(x,y)\phi(x,y)\,dxdy.$$

(iii) *If* $\phi(x,y)/|xy| \in L^r(\mathbb{T}^2)$, *then*

$$\lim_{\epsilon,\delta\downarrow 0} \iint\limits_{\substack{\epsilon\le|x|\le\pi\\ \delta\le|y|\le\pi}} f(x,y)\phi(x,y)\,dxdy = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\phi(x,y)\,dxdy.$$

Proof. By Theorem 2.1, we find that it suffices to prove the validity of (0.10) and the statements (i)–(iii). The proof of Theorem 2.1 indicates that, for $x, y \in \mathbb{T}\setminus\{0\}$,

$$\sum_{|j|=0^+}^{m}\sum_{|k|=0^+}^{n} (\Delta^*_{pp}c_{jk})\,\Psi^p_j(x)\Psi^p_k(y) \longrightarrow f(x,y) \qquad \text{as} \quad \min\{m,n\}\to\infty,$$

and the convergence is uniform on $\{\epsilon \le |x| \le \pi, \delta \le |y| \le \pi\}$ for all $0<\epsilon,\delta\le\pi$. Since ϕ is locally bounded in $(\mathbb{T}\setminus\{0\})^2$, it follows that, as $\min\{m,n\}\to\infty$,

$$\sum_{|j|=0^+}^{m}\sum_{|k|=0^+}^{n} (\Delta^*_{pp}c_{jk})\Phi^{pp}_{jk}(\epsilon,\delta) \longrightarrow \iint\limits_{\substack{\epsilon\le|x|\le\pi\\ \delta\le|y|\le\pi}} f(x,y)\phi(x,y)\,dxdy, \tag{4.5}$$

where $\Phi^{st}_{jk}(\epsilon,\delta)$ is defined by (4.3). We have

$$\lim_{\epsilon,\delta\downarrow 0}\Phi^{00}_{jk}(\epsilon,\delta) = \begin{cases} 4\pi^2\hat{\phi}^*(-j,-k) & \text{if} \quad |j|,|k|\ge 1,\\ 2\pi^2\hat{\phi}^*(-j,0) & \text{if} \quad |j|\ge 1, |k|=0^+,\\ 2\pi^2\hat{\phi}^*(0,-k) & \text{if} \quad |j|=0^+, |k|\ge 1,\\ \pi^2\hat{\phi}^*(0,0) & \text{if} \quad |j|=|k|=0^+,\end{cases}$$

and $\hat{\phi}^*(j,k)$ exists for all (j,k). Therefore, the limit $\zeta^{st}_{jk} \equiv \lim_{\epsilon,\delta\downarrow 0}\Phi^{st}_{jk}(\epsilon,\delta)$ exists for all s,t,j,k, and (4.4) implies that

$$|\zeta^{st}_{jk}| \le C \qquad (|j|,|k|\ge 0^+, \quad 0\le s,t\le p). \tag{4.6}$$

Putting (0.4), (4.2), and (4.5) together, we infer that the following limit of double integral exists, the double series at the right is absolutely convergent, and

$$\lim_{\epsilon,\delta\downarrow 0} \iint\limits_{\substack{\epsilon\le|x|\le\pi\\ \delta\le|y|\le\pi}} f(x,y)\phi(x,y)\,dxdy = \sum_{|j|=0^+}^{\infty}\sum_{|k|=0^+}^{\infty}(\Delta^*_{pp}c_{jk})\zeta^{pp}_{jk}.$$

For $m, n > 0$, we have

$$\begin{aligned}\lambda_{mn} &\equiv (4\pi^2) \sum_{|j|\le m} \sum_{|k|\le n} c_{jk}\, \hat{\phi}^*(-j,-k) \\ &= \lim_{\epsilon,\delta\downarrow 0} \iint\limits_{\substack{\epsilon\le|x|\le\pi \\ \delta\le|y|\le\pi}} s_{mn}(x,y)\phi(x,y)\,dxdy.\end{aligned}$$

It follows from (1.1) that

$$\begin{aligned}\lambda_{mn} = &\sum_{|j|=0^+}^{m} \sum_{|k|=0^+}^{n} (\Delta^*_{pp} c_{jk})\, \zeta^{pp}_{jk} + \sum_{t=0}^{p-1} \sum_{|j|=0^+}^{m} \sum_{|k|=n} (\Delta^*_{pt} c_{j,\tau(k)})\, \zeta^{p,t+1}_{jk} \\ &+ \sum_{s=0}^{p-1} \sum_{|j|=m} \sum_{|k|=0^+}^{n} (\Delta^*_{sp} c_{\tau(j),k})\, \zeta^{s+1,p}_{jk} \\ &+ \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{|j|=m} \sum_{|k|=n} (\Delta^*_{st} c_{\tau(j),\tau(k)})\, \zeta^{s+1,t+1}_{jk}.\end{aligned}$$

By (0.4) and (4.4), the series $\sum_{|j|=0^+}^{\infty} \sum_{|k|=0^+}^{\infty} (\Delta^*_{pp} c_{jk})\, \zeta^{pp}_{jk}$ converges absolutely. On the other hand, (0.1) $-$ (0.3) and (4.4) imply

$$\begin{aligned}\sum_{t=0}^{p-1} \sum_{|j|=0^+}^{m} \sum_{|k|=n} |\Delta^*_{pt} c_{j,\tau(k)}||\zeta^{p,t+1}_{jk}| &\le C \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{|j|=0^+}^{m} \sum_{|k|=n+v+1} |\Delta^*_{p0} c_{jk}| \\ &\le C_p \Big(\sup_{|k|>n} \sum_{|j|=0^+}^{m} |\Delta^*_{p0} c_{jk}| \Big) \\ &\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty,\end{aligned}$$

$$\begin{aligned}\sum_{s=0}^{p-1} \sum_{|j|=m} \sum_{|k|=0^+}^{n} |\Delta^*_{sp} c_{\tau(j),k}||\zeta^{s+1,p}_{jk}| &\le C \sum_{s=0}^{p-1} \sum_{u=0}^{s} \binom{s}{u} \sum_{|j|=m+u+1} \sum_{|k|=0^+}^{n} |\Delta^*_{0p} c_{jk}| \\ &\le C_p \Big(\sup_{|j|>m} \sum_{|k|=0^+}^{n} |\Delta^*_{0p} c_{jk}| \Big) \\ &\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty,\end{aligned}$$

and

$$\begin{aligned}&\sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{|j|=m} \sum_{|k|=n} |\Delta^*_{st} c_{\tau(j),\tau(k)}||\zeta^{s+1,t+1}_{jk}| \\ &\quad\le C \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{u=0}^{s} \sum_{v=0}^{t} \binom{s}{u} \binom{t}{v} \sum_{|j|=m+u+1} \sum_{|k|=n+v+1} |\Delta^*_{00} c_{jk}| \\ &\quad\le C_p \Big(\sup_{|j|>m, |k|>n} |c_{jk}| \Big) \\ &\quad\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty.\end{aligned}$$

Hence, as $\min\{m,n\} \to \infty$, λ_{mn} tends to $\sum_{|j|=0^+}^{\infty}\sum_{|k|=0^+}^{\infty}(\Delta_{pp}^* c_{jk})\zeta_{jk}^{pp}$ and, consequently, (0.10) follows. Let $r \geq 1$. Then $L^r(\mathbb{T}) \subset L^1(\mathbb{T})$ and $L^r(\mathbb{T}^2) \subset L^1(\mathbb{T}^2)$. With the help of Theorem 2.1, we find that (i) – (iii) follow from the Lebesgue dominated convergence theorem. This finishes the proof.

5. Another Type of Conditions of Bounded Variation

The rectangular partial sum $s_{mn}(x,y)$ can be rewritten in the following form:

$$\begin{aligned} s_{mn}(x,y) = &\sum_{|j|=0^+}^{m}\sum_{|k|\leq n}(\Delta_{p0}^* c_{jk})\Psi_j^p(x)e^{iky} \\ &+\sum_{s=0}^{p-1}\sum_{|j|=m}\sum_{|k|\leq n}(\Delta_{s0}^* c_{\tau(j),k})\Psi_j^{s+1}(x)e^{iky}. \end{aligned}$$

Consider the following two conditions instead of (0.1) – (0.4):

$$\lim_{|j|\to\infty}\sum_{k=-\infty}^{\infty}|c_{jk}|\,(\overline{|j|})^{p-1} = 0, \tag{5.1}$$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|\Delta_{p0}c_{jk}|\,(\overline{|j|})^{p-1} < \infty. \tag{5.2}$$

Obviously, condition (5.1) implies that $c_{jk}(\overline{|j|})^{p-1} \to 0$ as $\max\{|j|,|k|\} \to \infty$, and they are equivalent for $p = 1$ under the condition (5.2). Employing the same proofs as those given in §2, we obtain

Theorem 5.1. *Assume that conditions* (5.1) – (5.2) *are satisfied for some* $p \geq 1$. *Then series* (0.5) *converges regularly to some measurable function* $f(x,y)$ *for all* $x \in \mathbb{T}\setminus\{0\}$ *and all* $y \in \mathbb{T}$, *and the convergence is uniform on* $\{\alpha \leq |x| \leq \pi\}\times\mathbb{T}$ *for all* $0 < \alpha \leq \pi$. *In addition, let* $r > 0$, $x_0 \in \mathbb{T}\setminus\{0\}$, *and* $y_0 \in \mathbb{T}$.

(i) *If* $\phi(x,y_0)/|x|^r \in L^1(\mathbb{T})$, *then* $|f(x,y_0)|^r\phi(x,y_0) \in L^1(\mathbb{T})$ *and*
$$\|s_{mn}(\cdot,y_0) - f(\cdot,y_0)\|_{r,\phi} \to 0 \quad \text{as} \quad \min\{m,n\}\to\infty.$$

(ii) *If* $\phi(x_0,y) \in L^1(\mathbb{T})$, *then* $|f(x_0,y)|^r\phi(x_0,y) \in L^1(\mathbb{T})$ *and*
$$\|s_{mn}(x_0,\cdot) - f(x_0,\cdot)\|_{r,\phi} \to 0 \quad \text{as} \quad \min\{m,n\}\to\infty.$$

(iii) *If* $\phi(x,y)/|x|^r \in L^1(\mathbb{T}^2)$, *then* $|f(x,y)|^r\phi(x,y) \in L^1(\mathbb{T}^2)$ *and*
$$\|s_{mn} - f\|_{r,\phi} \to 0 \quad \text{as} \quad \min\{m,n\}\to\infty.$$

Moreover, the conclusions (i)–(iii) still hold provided the corresponding $L^1(\mathbb{T})$ and $L^1(\mathbb{T}^2)$ are replaced by $C(\mathbb{T})$ and $C(\mathbb{T}^2)$.

Apply Theorem 5.1 to the cases: $\phi(x,y) = O(|x|^\sigma)$, $\phi(x,y) = O(|y|^\lambda)$, and $\phi(x,y) = O(|x|^\sigma|y|^\lambda)$, where $r > 0$, $\sigma > r-1$, and $\lambda > -1$. Then we get an analogue of Corollary 2.2. The difference between them is the range of λ. We need to change λ from "$\lambda > r-1$" to "$\lambda > -1$". Similarly, consider $\phi(x,y) = O(|x|^{r-1}(\overline{\log 1/|x|})^{-\epsilon})$, $\phi(x,y) = O(|y|^{-1}(\overline{\log 1/|y|})^{-\delta})$, and $\phi(x,y) = O(|x|^{r-1}|y|^{-1}(\overline{\log 1/|x|})^{-\epsilon}(\overline{\log 1/|y|})^{-\delta})$, where $\epsilon, \delta > 1$. Then we obtain an analogue of Corollary 2.3. For this case, we only replace $|y|^{r-1}$ in (ii) – (iii) of Corollary 2.3 by $|y|^{-1}$.

To get an analogue of Theorem 3.1, we replace (5.1) – (5.2) by the following stronger conditions:

$$\lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |c_{jk}|\,(\overline{|j|})^{p-1}\theta(\overline{|j|}) = 0, \tag{5.1$'$}$$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\Delta_{p0}c_{jk}|\,(\overline{|j|})^{p-1}\theta(\overline{|j|}) < \infty. \tag{5.2$'$}$$

Theorem 5.2. *Assume that θ is a positive increasing function defined on $[1,\infty)$ such that $(5.1')$ – $(5.2')$ are satisfied for some $p \geq 1$. Then series (0.5) converges regularly to some measurable function $f(x,y)$ for all $x \in \mathbb{T}\setminus\{0\}$ and all $y \in \mathbb{T}$, and the convergence is uniform on $\{\alpha \leq |x| \leq \pi\} \times \mathbb{T}$ for all $0 < \alpha \leq \pi$. In addition, let $r \geq 1$.*

- (i) *If (ϕ,θ) is of type I_r, then for all $y \in \mathbb{T}$, $|f(x,y)|^r\phi(x) \in L^1(\mathbb{T})$ and $\|s_{mn}(\cdot,y) - f(\cdot,y)\|_{r,\phi} \to 0$ as $\min\{m,n\} \to \infty$, where $\phi(x,y) = \phi(x)$.*
- (ii) *If $\psi \in L^1(\mathbb{T})$, then for all $x \in \mathbb{T}\setminus\{0\}$, $|f(x,y)|^r\psi(y) \in L^1(\mathbb{T})$ and $\|s_{mn}(x,\cdot) - f(x,\cdot)\|_{r,\phi} \to 0$ as $\min\{m,n\} \to \infty$, where $\phi(x,y) = \psi(y)$.*
- (iii) *If (ϕ,θ) is of type I_r and $\psi \in L^1(\mathbb{T})$, then $|f(x,y)|^r\phi(x)\psi(y) \in L^1(\mathbb{T}^2)$ and $\|s_{mn} - f\|_{r,\phi} \to 0$ as $\min\{m,n\} \to \infty$, where $\phi(x,y) = \phi(x)\psi(y)$.*

Consider the following conditions with $p, r \geq 1$ and $0 \leq \epsilon < 1$:

$$\lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |c_{jk}|\,(\overline{|j|})^{p-1}(\overline{\log|j|})^{(1-\epsilon)/r} = 0, \tag{5.1$''$}$$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\Delta_{p0}c_{jk}|\,(\overline{|j|})^{p-1}(\overline{\log|j|})^{(1-\epsilon)/r} < \infty. \tag{5.2$''$}$$

These correspond to $(5.1')$ – $(5.2')$ with $\theta(\rho) = (\overline{\log\rho})^{(1-\epsilon)/r}$. Choose $\phi(t) = O(|t|^{r-1}(\overline{\log 1/|t|})^{-\epsilon})$ and $\psi(t) = O(|t|^{-1}(\overline{\log 1/|t|})^{-\delta})$, where $\delta > 1$. Then

(ϕ, θ) is of type I_r and $\psi \in L^1(\mathbb{T})$, and therefore, an analogue of Corollary 3.2 occurs. The only change is to replace $|y|^{r-1}$ in (ii) – (iii) of Corollary 3.2 by $|y|^{-1}$. If we consider $\theta(\rho) = \rho^{1-(\sigma+1)/r}$ with $r \geq 1$ and $-1 < \sigma < r-1$, then $(5.1') - (5.2')$ reduce to

$$(5.1''') \qquad \lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |c_{jk}|\,(\overline{|j|})^{p-(\sigma+1)/r} = 0,$$

$$(5.2''') \qquad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{p0} c_{jk}|\,(\overline{|j|})^{p-(\sigma+1)/r} < \infty.$$

Here we assume $p \geq 1$. It is known that (ϕ, θ) is of type I_r, where $\phi(t) = O(|t|^{\sigma})$. Hence, as a consequence of Theorem 5.2, an analogue of Corollary 3.3 is established. The only change is to replace $|y|^{\lambda}$ in (ii) – (iii) of Corollary 3.3 by $|y|^{-1}(\overline{\log 1/|y|})^{-\delta}$ with $\delta > 1$.

To correspond to condition (4.1), we assume

$$(5.3) \qquad \sup_{\substack{-\infty<j,k<\infty \\ 0<\epsilon,\delta\leq\pi}} \left| \iint_{\substack{\epsilon\leq|x|\leq\pi \\ \delta\leq|y|\leq\pi}} \phi(x,y)\Psi_j^p(x)e^{iky}\,dxdy \right| < \infty,$$

or equivalently,

$$\sup_{\substack{-\infty<j,k<\infty \\ 0<\epsilon,\delta\leq\pi}} |\Phi_{jk}^{p0}(\epsilon,\delta)| < \infty,$$

where $\Phi_{jk}^{st}(\epsilon,\delta)$ is defined by (4.3). For $p = 1$, it is the same as the condition

$$\sup_{0<\epsilon,\delta\leq\pi} s^{\omega}\phi_{\epsilon\delta}(0,0) < \infty \qquad (\omega = (\omega_1, 0), |\omega_1| = 1).$$

The maximal operator s^{ω} is defined by

$$s^{\omega}\phi(x,y) \equiv \sup_{\substack{m\geq 0 \\ -\infty<k<\infty}} \left| \sum_{j=0}^{m} \hat{\phi}^*(j,k)e^{ijx} \right| \qquad (\omega = (1,0)),$$

$$s^{\omega}\phi(x,y) \equiv \sup_{\substack{m\geq 0 \\ -\infty<k<\infty}} \left| \sum_{j=0}^{m} \hat{\phi}^*(-j,k)e^{-ijx} \right| \qquad (\omega = (-1,0))$$

(cf. [7]). Like Theorem 4.1, we have

Theorem 5.3. *Assume that conditions* $(5.1)-(5.2)$ *are satisfied for some* $p \geq 1$. *Then series* (0.5) *converges regularly to some measurable function* $f(x,y)$ *for all* $x \in \mathbb{T}\setminus\{0\}$ *and all* $y \in \mathbb{T}$, *and the convergence is uniform on* $\{\alpha \leq |x| \leq \pi\} \times \mathbb{T}$ *for all* $0 < \alpha \leq \pi$. *Moreover, if* $\phi : \mathbb{T}^2 \mapsto \mathbb{C}$ *is measurable and locally bounded in* $(\mathbb{T}\setminus\{0\}) \times \mathbb{T}$, $\hat{\phi}^*(j,k)$ *exists for all* (j,k), *and condition* (5.3) *is satisfied, then formula* (0.10) *holds and the following three statements remain true for* $r \geq 1$.

(i) *If* $\phi(x,y)/x \in L^r(\mathbb{T})$ *for almost all* $y \in \mathbb{T}$*, then*

$$\lim_{\epsilon,\delta\downarrow 0} \iint_{\substack{\epsilon\le|x|\le\pi \\ \delta\le|y|\le\pi}} f(x,y)\phi(x,y)\,dxdy = \lim_{\delta\downarrow 0} \int_{\delta\le|y|\le\pi} \int_{-\pi}^{\pi} f(x,y)\phi(x,y)\,dxdy.$$

(ii) *If* $\phi(x,y) \in L^r(\mathbb{T})$ *for almost all* $x \in \mathbb{T}$*, then*

$$\lim_{\epsilon,\delta\downarrow 0} \iint_{\substack{\epsilon\le|x|\le\pi \\ \delta\le|y|\le\pi}} f(x,y)\phi(x,y)\,dxdy = \lim_{\epsilon\downarrow 0} \int_{-\pi}^{\pi} \int_{\epsilon\le|x|\le\pi} f(x,y)\phi(x,y)\,dxdy.$$

(iii) *If* $\phi(x,y)/x \in L^r(\mathbb{T}^2)$*, then*

$$\lim_{\epsilon,\delta\downarrow 0} \iint_{\substack{\epsilon\le|x|\le\pi \\ \delta\le|y|\le\pi}} f(x,y)\phi(x,y)\,dxdy = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y)\phi(x,y)\,dxdy.$$

The theory developed here also works for the case:

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_{jk}| < \infty.$$

We refer the reader to [7, Theorem 3] for details.

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