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EDGE DOMINATION IN GRAPHS

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Abstract. Let G be a (p, q)-graph with edge domination number γ' and edge domatic number d'. In this paper we characterize connected graphs for which $\gamma' = p/2$ and graphs for which $\gamma' + d' = q + 1$. We also characterize trees and unicyclic graphs for which $\gamma' = \lfloor p/2 \rfloor$ and $\gamma' = q - \Delta'$, where Δ' denotes the maximum degree of an edge in G.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [3].

A subset S of V is called a *dominating set* of G if every vertex not in S is adjacent to some vertex in S. The *domination number* $\gamma(G)$ (or γ for short) of G is the minimum cardinality taken over all dominating sets of G. A dominating set S is called an *independent dominating set* if no two vertices of S are adjacent. The *independent domination number* $\gamma_i(G)$ (or γ_i for short) of G is the minimum cardinality taken over all independent dominating sets of G.

The concept of edge domination was introduced by Mitchell and Hedetniemi [5]. A subset X of E is called an *edge dominating set* of G if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G. The maximum order of a partition of E into edge dominating sets of G is called the *edge domatic number* of G and is denoted by $d'(G)$ (or d' for short). An edge dominating set X is called an *independent edge dominating set* if no two edges in X are adjacent. The *independent edge domination*

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number $\gamma_i'(G)$ (or γ_i' for short) of G is the minimum cardinality taken over all independent edge dominating sets of G. The *edge independence number* $\beta_1(G)$ (or β_1 for short) is defined to be the number of edges in a maximum independent set of edges of G.

A path with n vertices is denoted by P_n. The graph $S(G)$ obtained from G by subdividing each edge of G exactly once is called the *subdivision* of G. The *degree* of an edge $e = uv$ of G is defined by $\deg e = \deg u + \deg v - 2$. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. We need the following theorems.

Theorem 1.1. [1]. *If G is $K_{1,3}$-free, then $\gamma = \gamma_i$.*

Since $L(G)$ is $K_{1,3}$-free, we have the following.

Corollary 1.2. *For any graph $G, \gamma' = \gamma_i'$.*

Theorem 1.3. [4]. $\gamma'(C_p) = \lceil p/3 \rceil$ *for* $p \geq 3$.

Theorem 1.4. [2]. *For any (p, q)-graph $G, \gamma' \leq \lfloor p/2 \rfloor$.*

Theorem 1.5. [4]. *For any (p, q)-graph $G, \gamma' \leq q - \Delta'$ where Δ' denotes the maximum degree of an edge in G.*

Theorem 1.6. [4]. *For any (p, q)-graph $G, \gamma' \leq q - \beta_1 + q_0$ where q_0 is the number of isolated edges in G.*

Theorem 1.7. [4]. *For any (p, q)-graph $G, \gamma' + d' \leq q + 1$.*

In this paper we consider the problem of characterizing the class of graphs which attain the upper bounds given in Theorems 1.4, 1.5, 1.6 and 1.7.

2. Main Results

Theorem 2.1. *For any connected graph G of even order $p, \gamma' = p/2$ if and only if G is isomorphic to K_p or $K_{p/2,p/2}$.*

To prove this theorem, we need the following result.

Lemma 2.2. *A connected graph G is either a complete graph or a complete bipartite graph if G has the following property: Whenever any two vertices are joined by a path of length 3, then they are adjacent.*

Proof. By the hypothesis, distance between any two vertices is at most 2. Therefore $\mathrm{diam}(G) \leq 2$. If $\mathrm{diam}(G) = 1$, then G is a complete graph. So, assume that $\mathrm{diam}(G) = 2$.

Let a be any vertex with $\deg a < p-1$ where p is the order of G. Let us show that G is complete bipartite with the bipartition $\{V_1, V_2\}$ where $V_2 = N(a)$.

If x, y are any two distinct vertices in $V_1 \backslash \{a\}$, then x, y are non-adjacent for otherwise the non-adjacent vertices a and y would be joined by a path of length 3, contradicting the hypothesis.

Let b be any vertex in $V_1 \backslash \{a\}$. Then there exists a vertex x in V_2 such that x is adjacent to both a and b. If y is any other vertex in V_2, then $(bxay)$ is a path of length 3 and therefore by the hypothesis b and y are adjacent. Thus every vertex in V_1 is adjacent to all the vertices in V_2.

Next let us show that no two vertices in V_2 are adjacent. Suppose x, y are two distinct vertices in V_2. Let b be any vertex in $V_1 \backslash \{a\}$. (Note that $V_1 \backslash \{a\}$ is non-empty since $\deg a < p - 1$.) Then x, y are non-adjacent for otherwise the non-adjacent vertices a and b would be joined by a path of length 3.

Thus it follows that G is a complete bipartite graph. ■

Proof of Theorem 2.1. Suppose $\gamma'(G) = p/2$. Let us first show that the hypothesis of Lemma 2.2 holds. Suppose $(abcd)$ is a path of length 3 in G. Let S be an independent edge dominating set in $G \backslash \{a, b, c, d\}$. Then $S \cup \{ab, cd\}$ is an edge dominating set in G. Therefore $|S| + 2 \geq \gamma'(G) = p/2$. Since $S \cup \{ab, cd\}$ is an independent edge dominating set in G, it follows that $|S| = (p-4)/2$. Therefore a and d are adjacent for otherwise $S \cup \{bc\}$ would be an edge dominating set of cardinality $\gamma' - 1$.

Hence by Lemma 2.2, G is either complete or complete bipartite. Since for any complete bipartite graph $K_{r,s}, \gamma'(K_{r,s}) = \min\{r, s\}$. When the second possibility holds, G is isomorphic to $K_{p/2,p/2}$.

The second part of the theorem is obvious. ■

Theorem 2.3. *For any tree T of order $p \neq 2, \gamma' \leq (p-1)/2$; equality holds if and only if T is isomorphic to the subdivision of a star.*

Proof. First of all, by Theorem 1.4 and Theorem 2.1, $\gamma' \leq (p-1)/2$. It is clear that $\gamma' = (p-1)/2$ when T is the subdivision of a star. On the other hand, suppose T is a tree for which $\gamma' = (p-1)/2$. If $p \leq 5$ the result is trivial. Let $p = 2n + 1$, where $n \geq 3$. Since $\gamma' = n$, there exists an independent edge dominating set $S = \{e_1, e_2, \ldots, e_n\}$ such that the unique vertex v which is not covered by the edges in S is a non-pendant vertex of T; otherwise if v is a pendant vertex adjacent to u, then we may replace the edge e_i incident to u

with uv. Let $e_i = v_i u_i$ and suppose v is adjacent to u_1 and u_2. Since T is a tree, v is adjacent to at most one end of each e_i. Now suppose there exists an edge, say e_3, in S such that v is adjacent to neither u_3 nor v_3. Since T is connected, we may assume that u_3 is adjacent to v_1. Now $S\backslash\{e_1, e_2, e_3\} \cup \{v_1u_3, vu_2\}$ is an edge dominating set of cardinality $\gamma' - 1$, which is a contradiction. Hence, v is adjacent to exactly one end vertex of each e_i and so T is isomorphic to the subdivision of a star. ■

We now proceed to characterize connected unicyclic graphs with $\gamma' = \lfloor p/2 \rfloor$. We denote by $C_{3,n}$ the graph obtained from a C_3 and $n (\geq 0)$ copies of K_2 by joining one end of each K_2 with a fixed vertex of C_3. We denote by $C_{4,n}$ the graph obtained from C_4 by joining a vertex of C_4 with the center of $S(K_{1,n})$.

Theorem 2.4. *Let G be a connected unicyclic graph. Then $\gamma' = \lfloor p/2 \rfloor$ if and only if G is isomorphic to either $C_4, C_5, C_7, C_{3,n}$ or $C_{4,n}$ for some $n \geq 0$.*

Proof. Let G be a connected unicyclic graph with $\gamma' = \lfloor p/2 \rfloor$. If p is even then by Theorem 2.1, C_4 is the only connected unicyclic graph with $\gamma' = p/2$. Now let us assume that p is odd. Let C be the unique cycle of G. If $G = C$, it follows from Theorem 1.3 that G is isomorphic to either $C_{3,0} (= C_3), C_5$ or C_7. Suppose $G \neq C$. Let $T_1, T_2, \ldots, T_r (r \geq 1)$ be the components of $G\backslash V(C)$. Let $p_0, p_1, p_2, \ldots, p_r$ denote the number of vertices in $C, T_1, T_2, \ldots, T_r$ respectively. Then

$$\gamma'(G) \leq \left\lceil \frac{p_0}{2} \right\rceil + \sum_{i=1}^{r} \left\lfloor \frac{p_i}{2} \right\rfloor. \tag{1}$$

We consider the following cases.

Case i. p_0 is even.

If $\gamma'(T_i) < (p_i - 1)/2$ for some i or if $\gamma'(T_i) = (p_i - 1)/2$ for at least two components, then it follows from (1) that $\gamma'(G) < (p-1)/2$, which is a contradiction. Therefore, each component of $G\backslash V(C)$ is isomorphic to either K_2 or $S(K_{1,n})$ for some $n \geq 0$ and at most one component of $G\backslash V(C)$ is isomorphic to $S(K_{1,n})$. Since p is odd, it follows that exactly one component, say T_1, is isomorphic to $S(K_{1,n})$. Now if $r \geq 2$ then we choose an edge dominating set containing an edge e of G such that e dominates the edge of T_2 as well as two edges of C. Hence it follows from (1) that

$$\gamma'(G) \leq \frac{p_0 - 2}{2} + \frac{p_1 - 1}{2} + \sum_{i=2}^{r} \left\lfloor \frac{p_i}{2} \right\rfloor < \frac{p-1}{2},$$

which is a contradiction. Thus $r = 1$ and it follows from Theorem 1.3 that $p_0 = 4$. Hence G is isomorphic to $C_{4,n}$.

Case ii. p_0 is odd.

In this case, we claim that each T_i is isomorphic to K_2. Suppose not. Since p is odd, it follows that there exist at least two components, say T_1, T_2, each isomorphic to $S(K_{1,n})$ for some $n \geq 0$. If one more component, say T_3, is isomorphic to $S(K_{1,n})$ then

$$\gamma'(G) \leq \frac{p_0+1}{2} + \frac{p_1+p_2+p_3-3}{2} + \sum_{i=4}^{r} \left\lfloor \frac{p_i}{2} \right\rfloor < \frac{p-1}{2},$$

which is a contradiction. Hence there are only two components that are subdivisions of stars. So, there exists a vertex of degree two in C or a component, say T_3, isomorphic to K_2. Then the term $\lceil p_0/2 \rceil$ in the right side of (1) can be replaced by $(p_0-1)/2$ and hence

$$\gamma'(G) \leq \frac{p_0-1}{2} + \frac{p_1-1}{2} + \frac{p_2-1}{2} + \sum_{i=3}^{r} \left\lfloor \frac{p_i}{2} \right\rfloor < \frac{p-1}{2}$$

which is again a contradiction. Hence it follows that each T_i is isomorphic to K_2. Since $\gamma'(G) = (p-1)/2$, it follows that each component K_2 of $G \backslash V(C)$ has one end adjacent to a fixed vertex of C and $p_0 = 3$. Hence G is isomorphic to $C_{3,n}$ for some $n \geq 0$. The converse is obvious. ■

Theorem 2.5. *Let T be any tree and let $e = uv$ be an edge of maximum degree Δ'. Then $\gamma' = q - \Delta'$ if and only if* $\operatorname{diam}(T) \leq 4$ *and* $\deg w \leq 2$ *for every vertex $w \neq u, v$.*

Proof. Let T be a tree with $\gamma' = q - \Delta'$. Let A denote the set of all pendant edges of T. Since $E(T) - A$ is an edge dominating set of T, it follows that $|A| \leq \Delta'$ and hence $\deg w \leq 2$ for all vertices $w \neq u, v$. Also, if $\operatorname{diam}(T) \geq 5$, then there exists a non-pendant edge e of T such that $E(T) \backslash (A \cup \{e\})$ is an edge dominating set. Hence, $\gamma' \leq q - \Delta' - 1 < q - \Delta'$, which is a contradiction. Therefore, $\operatorname{diam}(T) \leq 4$. Conversely, let T be a tree with $\operatorname{diam}(T) \leq 4$ and $\deg w \leq 2$ for all vertices $w \neq u, v$, where $e = uv$ is an edge of maximum degree in T. If $\operatorname{diam}(T) = 2$ or 3, then $\gamma' = q - \Delta' = 1$. If $\operatorname{diam}(T) = 4$, then each non-pendant edge of T is adjacent to a pendant edge of T and hence the set S of all non-pendant edges of T forms a minimum edge dominating set and $\gamma' = |S| = q - \Delta'$. ■

Theorem 2.6. *For any connected unicyclic graph $G = (V, E)$ with cycle C, $\gamma' = q - \Delta'$ if and only if one of the following holds.*

(i). $G = C_3$.

(ii). $C = C_3 = (u_1u_2u_3u_1), \deg u_1 \geq 3, \deg u_2 = \deg u_3 = 2, d(u_1, w) \leq 2$ *for all vertices* w *not on* C *and* $\deg w \geq 3$ *for at most one vertex* w *not on* C.

(iii). $C = C_3 = (u_1u_2u_3u_1), \deg u_1 \geq 3, \deg u_2 \geq 3, \deg u_3 = 2,$ *all vertices not on* C *adjacent to* u_1 *have degree at most 2 and all vertices whose distance from* u_1 *is 2 are pendant vertices.*

(iv). $C = C_3, \deg u_1 = 3, \deg u_2 \geq 3, \deg u_3 \geq 3$ *and all vertices not on* C *are pendant vertices.*

(v). $G = C_4$.

(vi). $C = C_4$, *either exactly one vertex of* C *or two adjacent vertices of* C *have degree at least 3 and all vertices not on* C *are pendant vertices.*

Proof. Suppose $\gamma' = q - \Delta'$. Let S denote the set of all pendant edges of G and let $|S| = k$. Since $E \backslash (S \cup \{e_1\})$ is an edge dominating set for any edge e_1 of C, $\gamma' \leq q - k - 1$ so that $k \leq \Delta' - 1$. Let $e = uv$ be an edge of maximum degree Δ'. If both u and v are not on C then $k = \Delta' - 1$ and there exist edges e_1 and e_2 on C such that $E \backslash (S \cup \{e_1, e_2\})$ is an edge dominating set of cardinality $q - \Delta' - 1$ which is a contradiction. Hence, u or v lies on C and $k \geq \Delta' - 2$. We consider the following cases.

Case i. $k = \Delta' - 2$.

In this case, all the vertices other than u and v have degree either one or two. Hence $C = C_3$ or C_4 and G is isomorphic to one of the graphs described in (i), (ii), (iii), (v) or (vi).

Case ii. $k = \Delta' - 1$.

In this case, there exists a unique edge e on C such that $E \backslash (S \cup \{e\})$ is a minimum edge dominating set of G. It follows that $C = C_3$ and G is isomorphic to the graph described in (iv).

The converse is obvious. ■

Theorem 2.7. *For any connected graph* $G, \gamma' = q - \beta_1$ *if and only if* G *is isomorphic to* C_4 *or the subdivision graph of a star.*

Proof. Suppose $\gamma' = q - \beta_1$. Since $\gamma' \leq p/2$ and $\beta_1 \leq p/2$ we have $\gamma' + \beta_1 \leq p$ and hence $q \leq p$. If $q = p$, then p is even, $\gamma' = \beta_1 = p/2$ and G is unicyclic.

Hence it follows from Theorem 2.4 that $G = C_4$. If $q = p - 1$, then p is odd, $\gamma' = \beta_1 = (p-1)/2$ and G is a tree. Hence it follows from Theorem 2.3 that G is isomorphic to the subdivision of a star. The converse is obvious. ■

Theorem 2.8. *For any (p, q)-graph $G, \gamma' + d' = q + 1$ if and only if $G = C_3$ or $K_{1,p-1}$ or mK_2.*

Proof. Suppose $\gamma' + d' = q + 1$. Since $\gamma' d' \leq q$, we have $(d' - 1)(q - d') \leq 0$. Further, $d' \geq 1$ and $q \geq d'$. So, $(q - d')(d' - 1) = 0$. Hence $q = d'$ or $d' = 1$. If $d' = 1$, then G is isomorphic to mK_2. If $q = d'$, then $G = C_3$ or $K_{1,p-1}$. The converse is obvious. ∎

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