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INTRODUCTION TO SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. This paper provides a careful and accessible exposition of static bifurcation theory for *degenerate* boundary value problems for semilinear second-order elliptic differential operators. The purpose of this paper is twofold. The first purpose is to prove that the first eigenvalue of the linearized boundary value problem is *simple* and its associated eigenfunction is positive. The second purpose is to discuss the changes that occur in the structure of the solutions as a parameter varies near the first eigenvalue of the linearized problem.

Introduction

Let D be a bounded domain of Euclidean space $\mathbb{R}^N$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real C^∞ coefficients on $\overline{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x), x \in \overline{D}, 1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \overline{D}, \xi \in \mathbb{R}^N.$$

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(2) $c(x) \geq 0$ on $\overline{D}$.

(I) First we consider the following linear boundary value problem: Given function $f(x)$ defined in D, find a function $u(x)$ in D such that

$$\begin{cases} Au = f & \text{in } D, \\ Bu = a\frac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D. \end{cases} \tag{0.1}$$

Here:

(1) $a \in C^\infty(\partial D)$ and $a(x') \geq 0$ on ∂D.

(2) $b \in C^\infty(\partial D)$ and $b(x') \geq 0$ on ∂D.

(3) $\partial/\partial\boldsymbol{\nu}$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i=1}^{N} a^{ij} n_j \frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary ∂D (see Figure 1 below).

It is worth pointing out that, under the condition $a(x') \geq 0$ on ∂D, problem (0.1) becomes a degenerate boundary value problem from an analytical point of view. This is due to the fact that the well-known Shapiro-Lopatinskii complementary condition is violated at the points $x' \in \partial D$ where $a(x') = 0$. More precisely it is easy to see that problem (0.1) is non-degenerate (or coercive) if and only if either $a(x') \neq 0$ on ∂D or $a(x') \equiv 0$ and $b(x') \neq 0$ on ∂D. In particular, if $a(x') \equiv 1$ and $b(x') \equiv 0$ on ∂D (resp. $a(x') \equiv 0$ and $b(x') \equiv 1$ on ∂D), then the boundary condition B is the so-called Neumann (resp. Dirichlet) condition.

In this paper we consider the boundary value problem (0.1) in the framework of L^2 spaces, and prove that its first eigenvalue is positive and *simple* with positive eigenfunction in D.

FIG. 1.

First we state our fundamental hypotheses on the functions a, b and c:

(H.1) $a(x') + b(x') > 0$ on ∂D.

(H.2) $c(x) > 0$ in D.

We remark that an intuitive meaning of condition (H.1) is that, for the diffusion process described by problem (0.1), either the reflection phenomenon or the absorption phenomenon occurs at each point of the boundary ∂D (cf. [20]).

We associate with problem (0.1) an unbounded linear operator $\mathcal{U}$ from the Hilbert space $L^2(D)$ into itself as follows:

(a) The domain of definition $D(\mathcal{U})$ is the space

$$D(\mathcal{U}) = \{u \in H^{2,2}(D) : Bu = 0\}.$$

(b) $\mathcal{U}\, u = Au, \ u \in D(\mathcal{U})$.

The first purpose of this paper is to prove the following:

Theorem 0. *If conditions (H.1) and (H.2) are satisfied, then the first eigenvalue* λ_1 *of* $\mathcal{U}$ *is positive and* simple, *and its associated eigenfunction* $\psi_1(x)$ *is* positive *everywhere in* D. *Moreover no other eigenvalues have positive eigenfunctions.*

(II) Now, as an application of Theorem 0, we consider local static *bifurcation* problems for the following semilinear elliptic boundary value problem:

$$\left\{ \begin{array}{ll} Au - \lambda u + G(\lambda, u) = 0 & \text{in} \ \ D, \\ Bu = a\frac{\partial u}{\partial \nu} + bu = 0 & \text{on} \ \ \partial D. \end{array} \right. \tag{0.2}$$

Here $G(\lambda, u)$ is a nonlinear operator, depending on a real parameter λ, which operates on the unknown function u. The word "bifurcation" means a "splitting", and in the context of nonlinear boundary value problems it connotes a situation wherein at some critical value of λ the number of solutions of the equation changes.

The second purpose of this paper is to discuss those aspects of static bifurcation theory for the semilinear boundary value problem (0.2). We shall only restrict ourselves to some aspects which have been discussed in the papers Taira [22], Taira-Umezu [23] and [24]. Our approach is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of linear partial differential equations.

We associate with problem (0.2) a nonlinear mapping $F(\lambda, u)$ of $\mathbb{R} \times C_B^{2+\theta}(\overline{D})$ into $C^\theta(\overline{D})$ as follows:

$$\begin{aligned} F : \mathbb{R} \times C_B^{2+\theta}(\overline{D}) &\longrightarrow C^\theta(\overline{D}) \\ (\lambda, u) &\longmapsto Au - \lambda u + G(\lambda, u). \end{aligned}$$

Here

$$C_B^{2+\theta}(\overline{D}) = \{u \in C^{2+\theta}(\overline{D}) : Bu = 0 \text{ on } \partial D\}.$$

It may happen that, as the parameter λ varies, there exists a family of solutions which splits into several branches at some critical value λ_0.

Suppose that there is a curve Γ in the space $\mathbb{R} \times C_B^{2+\theta}(\overline{D})$ given by the formula $\Gamma = \{w(t) : t \in I\}$, where I is an interval, such that $F(w) = 0$ for all $w \in \Gamma$. If there exists a number $\tau_0 \in I$ such that every neighborhood of $w(\tau_0)$ contains zeros of F not lying on Γ, then the point $w(\tau_0)$ is called a *bifurcation point* for the equation $F(w) = 0$ with respect to the curve Γ. In many situations the curve Γ is of the form $\{(\lambda, 0) : \lambda \in \mathbb{R}, 0 \in C_B^{2+\theta}(\overline{D})\}$. The basic problem of bifurcation theory is that of finding the bifurcation points for $F(w) = 0$ with respect to Γ and studying the structure of $F^{-1}\{0\}$ near such points.

The next theorem asserts that if λ_1 is the first eigenvalue of $\mathcal{U}$, then the point $(\lambda_1, 0)$ is a bifurcation point for the equation $F(\lambda, u) = 0$:

Theorem 1. *Let λ_1 be the first eigenvalue of $\mathcal{U}$ and let $G(\lambda, u)$ be a C^k map, $k \geq 3$, of a neighborhood of $(\lambda_1, 0)$ in the space $\mathbb{R} \times C_B^{2+\theta}(\overline{D})$ into the space $C^\theta(\overline{D})$. Assume that the following four conditions are satisfied:*

(i) $G(\lambda_1, 0) = 0, G_\lambda(\lambda_1, 0) = 0.$

(ii) $G_u(\lambda_1, 0) = 0.$

(iii) *The function $G_{\lambda\lambda}(\lambda_1, 0)$ belongs to the range $R(\mathcal{U}\text{-}\lambda_1 I)$:*

$$\int_D G_{\lambda\lambda}(\lambda_1, 0) \cdot \psi_1 dx = 0.$$

(iv) *The function $G_{\lambda u}(\lambda_1, 0)\psi_1 - \psi_1$ does not belong to the range $R(\mathcal{U}\text{-}\lambda_1 I)$:*

$$\int_D (G_{\lambda u}(\lambda_1, 0)\psi_1 - \psi_1) \cdot \psi_1 dx \neq 0.$$

Then the point $(\lambda_1, 0)$ is a bifurcation point for the equation $F(\lambda, u) = 0$. In fact, the set of solutions of $F(\lambda, u) = 0$ near $(\lambda_1, 0)$ consists of two C^{k-2} curves Γ_1 and Γ_2 intersecting only at the point $(\lambda_1, 0)$. Furthermore the curve Γ_1 is tangent to the λ-axis at $(\lambda_1, 0)$ and may be parametrized by λ as

$$\Gamma_1 = \{(\lambda, u_1(\lambda)) : |\lambda - \lambda_1| < \varepsilon\},$$

while the curve Γ_2 may be parametrized by a variable s as

$$\Gamma_2 = \{(\lambda_2(s), s\psi_1 + u_2(s)) : |s| < \varepsilon\}.$$

Here

$$u_2(0) = \frac{du_2}{ds}(0) = 0, \quad \lambda_2(0) = \lambda_1.$$

The conditions in Theorem 1 are based on the linear approximation, and are independent of the nonlinearities. The following two corollaries analyze in detail the nonlinear nature of the problem; it is essential to know some properties of the nonlinearities in u in the operator $G(\lambda, u)$.

The first corollary deals with local bifurcation theory under generic conditions on the *quadratic* term:

Corollary 1. *Assume that the following four conditions are satisfied*:

(i) $G(\lambda, 0) = 0$ *for all* $|\lambda - \lambda_1|$ *sufficiently small.*

(ii) $G_u(\lambda_1, 0) = 0$.

(iii) *The function* $G_{\lambda u}(\lambda_1, 0)\psi_1 - \psi_1$ *does not belong to the range* $R(\mathcal{U}\text{-}\lambda_1 I)$.

(iv) *The function* $G_{uu}(\lambda_1, 0)(\psi_1, \psi_1)$ *does not belong to the range* $R(\mathcal{U}\text{-}\lambda_1 I)$:

$$\int_D G_{uu}(\lambda_1, 0)\ (\psi_1,\, \psi_1) \cdot \psi_1 dx \neq 0.$$

Then the set of solutions of $F(\lambda, u) = 0$ *near* $(\lambda_1, 0)$ *consists of two* C^{k-2} *curves* Γ_1 *and* Γ_2 *which may be parametrized respectively by* λ *and* s *as follows* (*see Figure* 2 *below*) :

$$\Gamma_1 = \{(\lambda, 0) : |\lambda - \lambda_1| < \varepsilon\},$$
$$\Gamma_2 = \{(\lambda_2(s), s\psi_1 + u_2(s)) : |s| < \varepsilon\}.$$

Here

$$u_2(0) = \frac{du_2}{ds}(0) = 0, \quad \lambda_2(0) = \lambda_1, \quad \frac{d\lambda_2}{ds}(0) \neq 0.$$

Example 1. For Corollary 1, we give a simple example of $F(\lambda, u)$:

$$F(\lambda, u) = Au - \lambda u \pm u^2.$$

The second corollary deals with local bifurcation theory under generic conditions on the *cubic* term:

Corollary 2. *Assume that the following five conditions are satisfied:*

(i) $G(\lambda, 0) = 0$ *for all* $|\lambda - \lambda_1|$ *sufficiently small.*

(ii) $G_u(\lambda_1, 0) = 0$.

(iii) *The function* $G_{\lambda u}(\lambda_1, 0)\psi_1 - \psi_1$ *does not belong to the range* $R(\mathcal{U}\text{-}\lambda_1 I)$.

FIG. 2.

(iv) *The function* $G_{uu}(\lambda_1, 0)(\psi_1,\, \psi_1)$ *belongs to the range* $R(\mathcal{U}\text{-}\lambda_1 I)$*:*

$$\int_D G_{uu}(\lambda_1, 0)(\psi_1,\, \psi_1) \cdot \psi_1 dx = 0.$$

(v) *The function* $G_{uuu}(\lambda_1, 0)(\psi_1,\, \psi_1,\, \psi_1)$ *does not belong to the range* $R(\mathcal{U}\text{-}\lambda_1 I)$*:*

$$\int_D G_{uuu}(\lambda_1, 0)(\psi_1,\, \psi_1,\, \psi_1) \cdot \psi_1 dx \neq 0.$$

Then the set of solutions of $F(\lambda, u) = 0$ *near* $(\lambda_1, 0)$ *consists of a pitchfork. More precisely, the two* C^{k-2} *curves* Γ_1 *and* Γ_2 *may be parametrized respectively by* λ *and* s *as follows* (*see Figure* 3 *below*) :

$$\Gamma_1 = \{(\lambda, 0) : |\lambda - \lambda_1| < \varepsilon\},$$
$$\Gamma_2 = \{(\lambda_2(s), s\psi_1 + u_2(s)) : |s| < \varepsilon\}.$$

Here

$$u_2(0) = \frac{du_2}{ds}(0) = 0, \quad \lambda_2(0) = \lambda_1, \quad \frac{d\lambda_2}{ds}(0) = 0, \quad \frac{d^2\lambda_2}{ds^2}(0) \neq 0.$$

Example 2. For Corollary 2, we give a simple example of $F(\lambda, u)$:

$$F(\lambda, u) = Au - \lambda u \pm u^3.$$

(III) Now we consider *global* static bifurcation problems for the following semilinear elliptic boundary value problem:

FIG. 3.

$$
(0.3) \qquad \begin{cases} Au - \lambda u + g(u) = 0 & \text{in } D, \\ Bu = a\frac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D. \end{cases}
$$

Here λ is a real parameter and $g(t)$ is a real-valued function on $\mathbb{R}$, not depending explicitly on x.

By Theorem 1, we know that there exist precisely two nontrivial branches of solutions of problem (0.3) bifurcating at the point $(\lambda_1, 0)$. The forthcoming two theorems characterize them *globally*.

The first theorem is a generalization of Szulkin [18, Theorem 1.3] to the degenerate case:

Theorem 2. *Let λ_1 be the first eigenvalue of $\mathcal{U}$, and let $g(t)$ be a function of class C^1 on $\mathbb{R}$ such that*

$$
g(0) = g'(0) = 0.
$$

Assume that the derivative $g'(t)$ is strictly decreasing for $t < 0$ and strictly increasing for $t > 0$, and that there exist constants $k_- > 0$ and $k_+ > 0$ such that

$$
\lim_{t \to -\infty} g'(t) = k_-, \quad \lim_{t \to +\infty} g'(t) = k_+.
$$

Then the point $(\lambda_1, 0)$ is a bifurcation point of problem (0.3). More precisely, the set of nontrivial solutions of problem (0.3) consists of two C^1 curves Γ_- and Γ_+ parametrized respectively by λ as follolus (see Figure 4 below):

$$
\Gamma_- = \{(\lambda, u_-(\lambda)) \in \mathbb{R} \times C(\overline{D}) : \lambda_1 \le \lambda < \lambda_1 + k_-\},
$$

$$
\Gamma_+ = \{(\lambda, u_+(\lambda)) \in \mathbb{R} \times C(\overline{D}) : \lambda_1 \le \lambda < \lambda_1 + k_+\}.
$$

FIG. 4.

The branch Γ_- *is negative and the branch* Γ_+ *is positive except at* $(\lambda_1, 0)$, *and the uniform norms* $\|u_-(\lambda)\|$ *and* $\|u_+(\lambda)\|$ *tend to* $+\infty$ *as* $\lambda \to \lambda_1 + k_-$ *and as* $\lambda \to \lambda_1 + k_+$, *respectively. Furthermore problem* (0.3) *has no other positive or negative solutions for all* $\lambda \geq \lambda_1$.

Example 3. For Theorem 2, we give an example of the function $g(t)$:

$$g(t) = \begin{cases} k_+ \ \left(t + \frac{1}{2t} - \frac{4}{3}\right) & \text{for } t > 1, \\ \frac{k_+}{6}\ t^3 & \text{for } 0 \leq t \leq 1, \\ \frac{k_-}{6}\ t^3 & \text{for } -1 \leq t \leq 0, \\ k_- \ \left(t + \frac{1}{2t} + \frac{4}{3}\right) & \text{for } t < -1. \end{cases}$$

The second theorem asserts that if the function $g(t)$ is bounded, then the bifurcation curves "turn back" towards λ_1. More precisely we have the following generalization of [18, Theorem 5.2] to the degenerate case:

Theorem 3. *Let* λ_1, λ_2 *be the first and second eigenvalues of* $\mathcal{U}$, *respectively, and let* $g(t)$ *be a function of class* C^1 *on* $\mathbb{R}$ *such that*

$$g(0) = g'(0) = 0.$$

Assume that $g(t)$ *is bounded and that there exists a constant* $k > 0$ *such that*

$$0 \leq g'(t) \leq k < \lambda_2 - \lambda_1 \quad \text{for all } t \in \mathbb{R}.$$

Then the set of nontrivial solutions of problem (0.3), *bifurcating at* $(\lambda_1, 0)$, *consists of two* C^1 *branches* Γ_1 *and* Γ_2. *The branches* Γ_1 *and* Γ_2 *may be parametrized respectively by* s *as follows* (*see Figure* 5 *below*) :

FIG. 5.

$$\Gamma_1 = \{(\lambda^1(s), u^1(s)) \in \mathbb{R} \times C(\overline{D}) : 0 \le s < \infty\},$$
$$\Gamma_2 = \{(\lambda^2(s), u^2(s)) \in \mathbb{R} \times C(\overline{D}) : 0 \le s < \infty\}.$$

Here $(\lambda^i(0),\ u^i(0)) = (\lambda_1, 0)$ *and* $\lambda^i(s) \to \lambda_1$ *as* $s \to \infty$ $(i = 1, 2)$.

Example 4. For Theorem 3, we give an example of the function $g(t)$:

$$g(t) = \begin{cases} k\left(-\frac{1}{2t} + \frac{2}{3}\right) & \text{for } t > 1, \\ \frac{k}{6}t^3 & \text{for } -1 \le t \le 1, \\ k\left(-\frac{1}{2t} - \frac{2}{3}\right) & \text{for } t < -1. \end{cases}$$

(IV) Finally we consider the following general nonlinear elliptic boundary value problem: Given function $f(x, \xi)$ defined on $\overline{D} \times [0, \infty)$, find a nonnegative function $u(x)$ in D such that

$$\text{(0.4)} \qquad \begin{cases} Au = f(x, u) & \text{in } D, \\ Bu = a\frac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D. \end{cases}$$

In order to state our existence theorem of positive solutions of problem (0.4), we introduce a fundamental condition (*slope condition*) on the nonlinear term $f(x, \xi)$:

For a positive number σ, there exists a constant $\omega = \omega(\sigma) > 0$, independent of $x \in \overline{D}$, such that

$$(R)_\sigma) \qquad f(x, \xi) - f(x, \eta) > -\omega(\xi - \eta), \quad x \in \overline{D},\ 0 \le \eta < \xi \le \sigma.$$

A nonnegative function $\psi \in C^2(\overline{D})$ is said to be a *supersolution* of problem (0.4) if it satisfies the conditions

$$\begin{cases} A\psi - f(x,\psi) \geq 0 & \text{in } D, \\ B\psi \geq 0 & \text{on } \partial D. \end{cases}$$

Similarly, a nonnegative function $\phi \in C^2(\overline{D})$ is said to be a *subsolution* of problem (0.4) if it satisfies the conditions

$$\begin{cases} A\phi - f(x,\phi) \leq 0 & \text{in } D, \\ B\phi \leq 0 & \text{on } \partial D. \end{cases}$$

Now we can state our existence theorem for problem (0.4) which is a generalization of [1, Theorem 9.4] to the degenerate case:

Theorem 4. *Assume that the function $f(x,\xi)$ belongs to $C^\theta(\overline{D}\times[0,\sigma])$, $0<\theta<1$, and satisfies condition $(R)_\sigma$ for some $\sigma>0$. If $\psi(x)$ and $\phi(x)$ are respectively super- and subsolutions of problem (0.4) satisfying $0 \leq \phi(x) \leq \psi(x) < \sigma$ on $\overline{D}$, then there exists a solution $u \in C^{2+\theta}(\overline{D})$ of problem (0.4) such that $\phi(x) \leq u(x) \leq \psi(x)$ on $\overline{D}$.*

In order to state our uniqueness theorem of positive solutions of problem (0.4), we introduce another fundamental condition (*sublinearity*) on the nonlinear term $f(x,\xi)$:

We have for all $0<\tau<1$

$$f(x,\tau\xi) \geq \tau f(x,\xi), \quad x \in \overline{D}, \quad \xi > 0, \tag{S1}$$

and

$$f(x,0) \geq 0, \quad x \in \overline{D}. \tag{S2}$$

Our uniqueness theorem for problem (0.4) is the following:

Theorem 5. *Assume that the function $f(x,\xi)$ belongs to $C^\theta(\overline{D}\times[0,\sigma])$, $0<\theta<1$, and satisfies condition $(R)_\sigma$, for every $\sigma>0$, and also satisfies condition (S). Then problem (0.4) has at most one positive solution.*

As an application of Theorems 4 and 5, we consider the following semilinear elliptic boundary value problem:

$$\begin{cases} Au - \lambda u + h(x)u^p = 0 & \text{in } D, \\ Bu = a\frac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D, \end{cases} \tag{0.5}$$

where $p > 1$ and $h(x)$ is a real-valued function on $\overline{D}$. It is worth pointing out here that the equation: $Au - \lambda u + h(x)u^p = 0$ in D originates from the so-called Yamabe problem which is a basic problem in Riemannian geometry if we take $p = (N+2)/(N-2) > 1$ where $N \geq 3$ (cf. [11], [14]).

Assume that $h(x)$ is a function in the Hölder space $C^\theta(\overline{D}), 0 < \theta < 1$, such that

$$h(x) \geq 0 \quad \text{on } \overline{D}.$$

We let

$$D_0(h) = \text{the interior of the set } \{x \in D : h(x) = 0\}.$$

We consider the case where $h(x) > 0$ on the boundary ∂D. More precisely, our fundamental hypothesis on the function h is the following (see Figure 6 below):

(Z) The open set $D_0(h)$ consists of a finite number of connected components with smooth boundary, say $D_i(h), 1 \leq i \leq l$, which are bounded away from ∂D.

We consider the Dirichlet eigenvalue problem in each connected component $D_i(h)$:

$$\begin{cases} A\varphi = \lambda\varphi & \text{in } D_i(h), \\ \varphi = 0 & \text{on } D_i(h). \end{cases} \tag{0.6}$$

We let

$$\lambda_1(D_i(h)) = \text{the first eigenvalue of problem (0.6)},$$

and

$$\tilde{\lambda}_1(D_0(h)) = \min_{1 \leq i \leq l} \lambda_1(D_i(h)).$$

FIG. 6.

We remark (cf. [4]) that the minimal eigenvalue $\tilde{\lambda}_1(D_0(h))$ is monotone decreasing with respect to the set $D_0(h)$; more precisely it tends to $+\infty$ if $D_0(h) \to \emptyset$ and tends to $\lambda_1(D)$ if $D_0(h) \to D$, where $\lambda_1(D)$ is the first eigenvalue of the Dirichlet problem in the whole domain D.

The next theorem is a generalization of [14, Theorems 2 and 3] to the degenerate case:

Theorem 6. *Assume that $h(x)$ is a function in $C^\theta(\overline{D})$, $0 < \theta < 1$, such that $h(x) \geq 0$ on $\overline{D}$ and that condition (Z) is satisfied. Then problem* (0.5) *has a unique positive solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ for every $\lambda_1 < \lambda < \tilde{\lambda}_1(D_0(h))$. For any $\lambda \geq \tilde{\lambda}_1(D_0(h))$, there exists no positive solution of problem* (0.5). *Furthermore the uniform norm $\|u(\lambda)\|_{C(\overline{D})}$ tends to $+\infty$ as $\lambda \to \lambda_1(D_0(h))$.*

The situation may be represented schematically by the following bifurcation diagram:

FIG. 7.

The rest of this paper is organized as follows.

The first section, Section 1, is devoted to the proof of Theorem 0. There is a standard method of reducing problem (0.1) to an equivalent integral equation on the boundary in an appropriate function space. More precisely, by using the Green and Poisson operators for problem (0.1) we transform problem (0.1) to the study of a pseudo-differential operator T on the boundary (Proposition 1.2), which may be considered as a generalization of the classical potential approach. The main difficulty in this approach lies in the fact that we have to establish *a priori* estimates for problem (0.1). In doing so, we use the theory of pseudo-differential operators to prove that conditions (H.1) and (H.2) are sufficient for the existence of a parametrix for the operator T (Lemma 1.3). Next the maximum principle, which stems from a second-order equation, gives us various *a priori* information about the possible solutions of problem (0.1). In this way we can prove an existence and uniqueness theorem for problem (0.1) in the framework of Hölder spaces (Theorem 1.1).

Furthermore the maximum principle tells us that the resolvent K of problem (0.1) is a positive operator in the ordered Banach space $C(\overline{D})$ (Proposition 1.6). In order to obtain an abstract formulation of this fact, we introduce an ordered Banach subspace $C_e(\overline{D})$ of $C(\overline{D})$ which combines the good properties of the resolvent K with the good properties of the natural ordering of $C(\overline{D})$. Here the function $e(x)$ is the unique solution of the linear boundary value problem

$$\begin{cases} Ae = 1 & \text{in } D, \\ Be = 0 & \text{on } \partial D, \end{cases}$$

and the ordered Banach space $C_e(\overline{D})$ is defined by the formula

$$C_e(\overline{D}) = \{u \in C(\overline{D}) : \textit{there exists a constant } c > 0 \textit{ such that } -ce \leq u \leq ce\},$$

with norm

$$\|u\|_e = \inf\{c > 0 : -ce \leq u \leq ce\}.$$

This setting has the advantages that it takes into consideration in an optimal way the *a priori* information given by the maximum principle and that it is amenable to the methods of abstract functional analysis (cf. [1], [9]). Theorem 1.1 is an immediate consequence of a sharper version of the well-known Kreĭn-Rutman theorem for strongly positive, compact linear operators (Theorem 1.7). We recall that Taira [22] proved Theorem 0 by using the theory of Feller semigroups in functional analysis.

In Section 2 we prove Theorem 1. A general class of semilinear second-order elliptic boundary value problems satisfies the maximum principle. Roughly speaking, this additional information means that the operators associated with the boundary value problems are compatible with the natural ordering of the underlying function spaces. Consequently we are led to the study of nonlinear equations in the framework of ordered Banach spaces. Theorem 1 follows by applying local static bifurcation theory from a simple eigenvalue due to Crandall-Rabinowitz [5] (Theorem 2.1).

Section 3 is devoted to the proof of Theorems 2 and 3. We transpose the nonlinear problem (0.3) into an equivalent fixed point equation for the resolvent K in an appropriate ordered Banach space. More precisely, by applying the resolvent K for problem (0.1) we transform problem (0.3) into a nonlinear operator equation in the ordered Banach space $C(\overline{D})$

$$u = K(F(u)) = K(f(\cdot, u(\cdot)))$$

in such a way that as much information as possible is carried over to the abstract setting. The proof of Theorems 2 and 3 is essentially the same as that of Szulkin [18].

Section 4 is devoted to the proof of Theorems 4 and 5. By condition $(R)_\sigma$, it follows that the map H, defined by $H(u) = K(F(u))$, leaves invariant the ordering of the space $C(\overline{D})$ (Lemma 4.1). In the case of an increasing map it suffices to verify that H maps *two* points of a bounded, closed and convex set into itself in order to apply Schauder's fixed point theorem (Lemma 4.2). This is a much easier task than to verify the standard hypotheses for an application of the same theorem.

The fact that the resolvent K is strongly positive has important consequences. Namely, if $u(x) \geq v(x)$ and $u(x) \not\equiv v(x)$ on $\overline{D}$, then the function $H(u) - H(v)$ is an interior point of the positive cone P_e of the ordered Banach space $C_e(\overline{D})$. This implies that the map H is a strongly increasing self-map of $C_e(\overline{D})$ (Lemma 4.6). The proof of Theorem 5 is based on a uniqueness theorem of fixed points of strongly increasing and strongly sublinear mappings in ordered Banach spaces (Theorem 4.4).

The final section, Section 5, is devoted to the proof of Theorem 6. First, by using Green's formula we prove that if there exists a positive solution $u(\lambda) \in C^2(\overline{D})$ of problem (0.5), then we have $\lambda_1 < \lambda < \tilde{\lambda}_1(D_0(h))$ (Lemmas 5.1 and 5.4). Next, by making use of the implicit function theorem we prove that there exists a critical value $\overline{\lambda}(h) \in (\lambda_1, \tilde{\lambda}_1(D_0(h))]$ such that problem (0.5) has a positive solution $u(\lambda)$ for all $\lambda \in (\lambda_1, \overline{\lambda}(h))$ (Lemma 5.3), and further that the solution $u(\lambda)$ is monotone increasing.

The formula $\overline{\lambda}(h) = \tilde{\lambda}_1(D_0(h))$ follows from an application of Theorem 4 by constructing explicitly super- and subsolutions of problem (0.5) for every $\lambda \in (\lambda_1(D), \tilde{\lambda}_1(D_0(h)))$ (Lemma 5.5). First, by using the positive eigenfunction $\psi_1(x)$ of problem (0.1) we have a subsolution $\phi_\varepsilon(x) = \varepsilon\psi_1(x)$ for $\varepsilon > 0$ sufficiently small. On the other hand, in order to construct a supersolution we make good use of the semilinear Dirichlet and Neumann boundary value problems, that is, the case where $a \equiv 0$ and $b \equiv 1$ on ∂D and the case where $a \equiv 1$ and $b \equiv 0$ on ∂D in problem (0.5).

1. Proof of Theorem 0

The proof of Theorem 0 is carried out by making use of the theory of positive mappings in ordered Banach spaces (cf. [1], [9]).

1.1. Existence and uniqueness theorem for problem (0.1).

In this subsection we prove an existence and uniqueness theorem for the linearized boundary value problem (0.1) in the framework of Hölder spaces which will play an important role in the proof of Theorem 0 in Subsections 1.2 and 1.4.

First we introduce a subspace of the Hölder space $C^{1+\theta}(\partial D), 0 < \theta < 1$, which is associated with the boundary condition

$$Bu = a\frac{\partial u}{\partial \boldsymbol{\nu}} + bu$$

in the following way: We let

$$C_*^{1+\theta}(\partial D) = \{\varphi = a\varphi_1 + b\varphi_2 : \varphi_1 \in C^{1+\theta}(\partial D),\ \varphi_2 \in C^{2+\theta}(\partial D)\},$$

and define a norm

$$|\varphi|_{C_*^{1+\theta}(\partial D)} = \inf\{|\varphi_1|_{C^{1+\theta}(\partial D)} + |\varphi_2|_{C^{2+\theta}(\partial D)} : \varphi = a\varphi_1 + b\varphi_2\}.$$

Then it is easy to verify that the space $C_*^{1+\theta}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{C_*^{1+\theta}(\partial D)}$. We remark that the space $C_*^{1+\theta}(\partial D)$ is an "interpolation space" between the spaces $C^{2+\theta}(\partial D)$ and $C^{1+\theta}(\partial D)$; more precisely, it is easy to see that

$$\begin{cases} C_*^{1+\theta}(\partial D) = C^{2+\theta}(\partial D) & \text{if } a(x') \equiv 0 \text{ on } \partial D, \\ C_*^{1+\theta}(\partial D) = C^{1+\theta}(\partial D) & \text{if } a(x') > 0 \text{ on } \partial D. \end{cases}$$

The purpose of this subsection is to prove the following:

Theorem 1.1. *If hypotheses* (H.1) *and* (H.2) *are satisfied, then the mapping*

$$(A, B) : C^{2+\theta}(\overline{D}) \to C^{\theta}(\overline{D}) \oplus C_*^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism *for all* $0 < \theta < 1$.

Proof. The proof is divided into four steps.

(i) Let g be an arbitrary element of $C^{\theta}(\overline{D})$, and φ an arbitrary element of $C_*^{1+\theta}(\partial D)$ such that

$$\varphi = a\varphi_1 + b\varphi_2, \quad \varphi_1 \in C^{1+\theta}(\partial D), \quad \varphi_2 \in C^{2+\theta}(\partial D).$$

First we show that the boundary value problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = \varphi & \text{on } \partial D, \end{cases} \tag{1.1}$$

can be reduced to the study of an operator on the boundary.

To do so, we consider the Neumann problem

$$
(1.2) \qquad \begin{cases} Av = g & \text{in } D, \\ \frac{\partial v}{\partial \nu} = \varphi_1 & \text{on } \partial D. \end{cases}
$$

By [7, Theorem 6.31], one can find a unique solution v in the space $C^{2+\theta}(\overline{D})$ of problem (1.2). Then it is easy to see that a function u in $C^{2+\theta}(\overline{D})$ is a solution of problem (1.1) if and only if the function $w = u - v \in C^{2+\theta}(\overline{D})$ is a solution of the problem

$$
\begin{cases} Aw = 0 & \text{in } D, \\ Bw = \varphi - Bv & \text{on } \partial D. \end{cases}
$$

Here we remark that

$$
Bv = a\frac{\partial v}{\partial \boldsymbol{\nu}} + bv = a\varphi_1 + bv,
$$

so that

$$
Bw = \varphi - Bv = b(\varphi_2 - v) \in C^{2+\theta}(\partial D).
$$

However we know that every solution $w \in C^{2+\theta}(\overline{D})$ of the homogeneous equation: $Aw = 0$ in D can be expressed as follows (cf. [20, Theorem 8.2.4]):

$$
w = \mathcal{P}\psi, \quad \psi \in C^{2+\theta}(\partial D).
$$

Here the operator $\mathcal{P} : C^{2+\theta}(\partial D) \to C^{2+\theta}(\overline{D})$ is the Poisson operator, that is, the function $w = \mathcal{P}\psi$ is the unique solution of the Dirichlet problem

$$
\begin{cases} Aw = 0 & \text{in } D, \\ w = \psi & \text{on } \partial D. \end{cases}
$$

Thus we have the following:

Proposition 1.2. *For given functions $g \in C^{\theta}(\overline{D})$ and $\varphi = a\varphi_1 + b\varphi_2 \in C^{1+\theta}_{*}(\partial D)$, there exists a solution $u \in C^{2+\theta}(\overline{D})$ of problem (1.1) if and only if there exists a solution $\psi \in C^{2+\theta}(\partial D)$ of the equation*

$$
(1.3) \qquad T\psi := B\mathcal{P}\psi = b(\varphi_2 - v) \quad \text{on } \partial D.
$$

Furthermore the solutions $u(x)$ and $\psi(x')$ are related as follows:

$$
u = v + \mathcal{P}\ \psi,
$$

where $v \in C^{2+\theta}(\overline{D})$ is the unique solution of problem (1.2).

We remark that equation (1.3) is a generalization of the classical Fredholm integral equation.

(ii) We study the operator T in question. It is known (cf. [8, Chapter XX], [16, Chapter 3]) that the operator

$$T\psi = B\mathcal{P}\psi = a\frac{\partial}{\partial \boldsymbol{\nu}}(\mathcal{P}\psi) + b\psi$$

is a first-order, *pseudo-differential operator* on the boundary ∂D.

The next proposition is an essential step in the proof of Theorem 1.1:

Lemma 1.3. *If hypothesis* (H.1) *is satisfied, then there exists* a parametrix *E in the Hörmander class $L^0_{1,1/2}(\partial D)$ for T which maps $C^{k+\theta}(\partial D)$ continuously into itself for all nonnegative integers k.*

Proof. By making use of [8, Theorem 22.1.3] just as in the proof of [21, Lemma 5.2], one can construct a parametrix $E \in L^0_{1,1/2}(\partial D)$ for T:

$$ET \equiv TE \equiv I \quad \text{mod } L^{-\infty}(\partial D).$$

The boundedness of $E : C^{k+\theta}(\partial D) \to C^{k+\theta}(\partial D)$ follows from an application of [3, Theorem 1], since we have $C^{k+\theta}(\partial D) = B^{k+\theta}_{\infty,\infty}(\partial D)$. ∎

(iii) We consider problem (1.1) in the framework of Sobolev spaces of L^p style, and prove an L^p version of Theorem 1.1.

If k is a positive integer and $1 < p < \infty$, we define the Sobolev space

$$\begin{aligned} H^{k,p}(D) = {} & \text{the space of (equivalence classes of) functions} \\ & u \in L^p(D) \text{ whose derivatives } D^\alpha u, |\alpha| \leq k, \text{ in the} \\ & \text{sense of distributions are in } L^p(D), \end{aligned}$$

and let

$$\begin{aligned} B^{k-1/p,p}(\partial D) = {} & \text{the space of the boundary values } \varphi \text{ of functions} \\ & u \in H^{k,p}(D). \end{aligned}$$

In the space $B^{k-1/p,p}(\partial D)$, we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial D)} = \inf\{\|u\|_{H^{k,p}(D)} : u \in H^{k,p}(D), u|_{\partial D} = \varphi\}.$$

The space $B^{k-1/p,p}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B^{k-1/p,p}(\partial D)}$; more precisely it is a Besov space (cf. [2], [25]).

We introduce a subspace of $B^{1-1/p,p}(\partial D)$ which is an L^p version of $C_*^{l+\theta}(\partial D)$. We let

$$B_*^{1-1/p,p}(\partial D) = \{\varphi = a\varphi_1 + b\varphi_2 : \varphi_1 \in B^{1-1/p,p}(\partial D),\ \varphi_2 \in B^{2-1/p,p}(\partial D)\},$$

and define a norm

$$|\varphi|_{B_*^{1-1/p,p}(\partial D)} = \inf\{|\varphi_1|_{B^{1-1/p,p}(\partial D)} + |\varphi_2|_{B^{2-1/p,p}(\partial D)} : \varphi = a\varphi_1 + b\varphi_2\}.$$

It is easy to verify that the space $B_*^{1-1/p,p}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B_*^{1-1/p,p}(\partial D)}$.

Then we can obtain the following L^p version of Theorem 1.1 (cf. [21, Theorem 1]):

Theorem 1.4. *If hypotheses* (H.1) *and* (H.2) *are satisfied, then the mapping*

$$(A, B) : H^{2,p}(D) \to L^p(D) \oplus B_*^{1-1/p,p}(\partial D)$$

is an algebraic and topological isomorphism *for all* $1 < p < \infty$.

(iv) Now we remark that

$$\begin{cases} C^\theta(\overline{D}) \subset L^p(D), \\ C_*^{1+\theta}(\partial D) \subset B_*^{1-1/p,p}(\partial D). \end{cases}$$

Thus we find from Theorem 1.4 that problem (1.1) has a unique solution $u \in H^{2,p}(D)$ for any $g \in C^\theta(\overline{D})$ and any $\varphi = a\varphi_1 + b\varphi_2 \in C_*^{1+\theta}(\partial D)$. Furthermore, by virtue of Proposition 1.2 it follows that the solution u can be written in the form

$$u = v + \mathcal{P}\psi, \quad v \in C^{2+\theta}(\overline{D}), \quad \psi \in B^{2-1/p,p}(\partial D).$$

However, Lemma 1.3 tells us that

$$\psi \in C^{2+\theta}(\partial D),$$

since we have by equation (1.3)

$$\psi \equiv E(T\psi) = E(b(\varphi_2 - v)) \mod C^\infty(\partial D).$$

Therefore we obtain that

$$u = v + \mathcal{P}\psi \in C^{2+\theta}(\overline{D}).$$

The proof of Theorem 1.1 is complete. ∎

1.2. Proof of Theorem 0 -(1)-.

First we let

$$H_B^{2,p}(D) = \{u \in H^{2,p}(D) : Bu = 0 \text{ on } \partial D\}.$$

By Theorem 1.4, we can introduce a continuous linear operator

$$K : L^p(D) \to H_B^{2,p}(D)$$

as follows: For any $g \in L^p(D)$, the function $u = Kg \in H^{2,p}(D)$ is the unique solution of the problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Furthermore, by the Ascoli-Arzelà theorem it follows that the operator K, considered as

$$K : C(\overline{D}) \to C^1(\overline{D}),$$

is *compact*. Indeed we find from an application of Sobolev's imbedding theorem that $H^{2,p}(D)$ is continuously imbedded into $C^{2-N/p}(\overline{D})$ for all $N < p < \infty$.

Then it follows from an application of regularity theorem for problem (1.1) ([21, Theorem 5.1]) that $u \in L^p(D), 1 < p < \infty$, is a solution of the problem

$$\begin{cases} Au = \lambda u & \text{in } D, \\ Bu = 0 & \text{on } \partial D \end{cases}$$

if and only if it satisfies the operator equation

$$u = \lambda K u \quad \text{in } \; C(\overline{D}). \tag{1.4}$$

1.3. Theory of positive mappings in ordered Banach spaces.

We shall make use of the theory of positive operators in ordered Banach spaces to study nontrivial solutions of equation (1.4).

Let X be a nonempty set. An ordering $\leq$ in X is a relation in X which is reflexive, transitive and antisymmetric. A nonempty set together with an ordering is called an ordered set.

Let V be a real vector space. An ordering $\leq$ in V is said to be *linear* if the following two conditions are satisfied:

(i) If $x, y \in V$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in V$.

(ii) If $x, y \in V$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A real vector space together with a linear ordering is called an *ordered vector space*.

If $x, y \in V$ and $x \leq y$, then the set $[x, y] = \{z \in X : x \leq z \leq y\}$ is called an *order interval*.

If we let

$$Q = \{x \in V : x \geq 0\},$$

then it is easy to verify that Q has the following two conditions:

(iii) If $x, y \in P$, then $\alpha x + \beta y \in Q$ for all $\alpha, \beta \geq 0$.

(iv) If $x \neq 0$, then at least one of x and $-x$ does not belong to Q.

The set Q is called the *positive cone* of the ordering $\leq$.

Let E be a Banach space E with a linear ordering $\leq$. The Banach space E is called an *ordered Banach space* if the positive cone Q is closed in E.

For two functions $u, v \in C(\overline{D})$, we write $u \leq v$ if $u(x) \leq v(x)$ for all $x \in \overline{D}$. Then it is easy to verify that the space $C(\overline{D})$ is an ordered Banach space with the linear ordering $\leq$ and the positive cone

$$P = \{u \in C(\overline{D}) : u \geq 0 \text{ on } \overline{D}\}.$$

Now we introduce an ordered Banach space which is associated with the operator $K : C(\overline{D}) \to C^1(\overline{D})$. To do so, we need the following:

Lemma 1.5. *Assume that hypotheses* (H.1) *and* (H.2) *are satisfied. If $v(x) \in C^\theta(\overline{D})$ with $0 < \theta < 1$ and if $v(x) \geq 0$ but $v(x) \not\equiv 0$ on $\overline{D}$, then the function $u = Kv \in C^{2+\theta}(\overline{D})$ satisfies the following conditions:*

(a) *$u(x') = 0$ on $M = \{x' \in \partial D : a(x') = 0\}$.*

(b) *$u(x) > 0$ on $\overline{D} \setminus M$.*

(c) *For the conormal derivative $\partial u / \partial \boldsymbol{\nu}$ of u, we have*

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x') < 0 \quad \text{on } M.$$

Furthermore the operator $K : C(\overline{D}) \to C(\overline{D})$ is positive, that is, $K(P) \subset P$.

Proof. (1) First, since the function $u = Kv \in C^{2+\theta}(\overline{D})$ satisfies the condition

$$Au = v \geq 0 \quad \text{in } D,$$

it follows from an application of the weak maximum principle (see Appendix, Theorem A.1) that the function u may take its negative minimum only on the boundary ∂D.

However we have the following:

Claim 1. *The function $u = Kv$ does not take its negative minimum on the boundary ∂D. In other words, the function u is nonnegative on $\overline{D}$.*

Proof. Assume to the contrary that there exists a point $x_0' \in \partial D$ such that

$$u(x_0') < 0.$$

If $a(x_0') = 0$, then we have by condition (H.1)

$$0 = Bu(x_0') = a(x_0')\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') + b(x_0')u(x_0') = b(x_0')u(x_0') < 0.$$

This is a contradiction.

If $a(x_0') > 0$, then it follows that

$$\begin{cases} Au(x) = v(x) \geq 0 \quad \text{in } D, \\ u(x_0') = \min_{x \in \overline{D}} u(x) < 0, \\ u(x) > u(x_0'), \ x \in D. \end{cases}$$

Thus it follows from an application of the boundary point lemma (see Theorem A.3) that

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') < 0,$$

so that

$$0 = Bu(x_0') = a(x_0')\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') + b(x_0')u(x_0') \leq a(x_0')\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') < 0.$$

This is also a contradiction. ∎

(2) Furthermore we have the following:

Claim 2. *The function $u = Kv$ is strictly positive in D.*

Proof. Assume to the contrary that there exists a point $x_0 \in D$ such that

$$u(x_0) = 0.$$

Then we obtain from the strong maximum principle (see Theorem A.2) that

$$u(x) \equiv 0 \quad \text{in } D,$$

so that

$$v(x) = Ku(x) \equiv 0 \quad \text{in } D.$$

This contradicts the condition that v is not the zero function in D. ■

(3) *Proof of Lemma 1.5.* If there exists a point $x_0' \in \partial D$ such that

$$u(x_0') = 0,$$

then we have

$$\begin{cases} Au(x) = v(x) \geq 0 & \text{in } D, \\ u(x_0') = \min_{x \in \overline{D}} u(x) = 0, \\ u(x) > 0, \ x \in D. \end{cases}$$

Thus it follows from an application of the boundary point lemma (see Theorem A.3) that

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') < 0,$$

so that

$$a(x_0') = 0,$$

since we have

$$0 = Bu(x_0') = a(x_0') \frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') = 0.$$

Conversely, if $a(x_0') = 0$, then we have, by condition (H.1), $b(x_0') > 0$, and so

$$u(x_0') = 0,$$

since $0 = Bu(x_0') = a(x_0') \partial u / \partial \boldsymbol{\nu}(x_0') + b(x_0') u(x_0') = b(x_0') u(x_0')$.

Summing up, we have proved that

$$u(x') = 0 \Longleftrightarrow a(x') = 0;$$

$$u(x) > 0 \Longleftrightarrow x \in \overline{D} \setminus M.$$

Assertion (c) is an immediate consequence of the boundary point lemma, since the function u attains its minimum 0 at the set M.

Finally, in order to prove the positivity of $K : C(\overline{D}) \to C(\overline{D})$, let v be an arbitrary function in $C(\overline{D})$ such that $v(x) > 0$ and $v(x) \not\equiv 0$ on $\overline{D}$. Then, by using Friedrichs' mollifiers we can find a sequence $\{v_j\} \subset C^1(\overline{D})$ satisfying the conditions

$$\begin{cases} v_j \geq 0 & \text{on } \overline{D}, \\ v_j \to v & \text{in } C(\overline{D}). \end{cases}$$

Hence we have by assertions (a) and (b)

$$\begin{cases} Kv_j \in C^2(\overline{D}), \\ Kv_j \geq 0 & \text{on } \overline{D}, \\ Kv_j \to Kv & \text{in } C(\overline{D}), \end{cases}$$

and so

$$Kv \geq 0 \text{ on } \overline{D}.$$

The proof of Lemma 1.5 is complete. ■

If we let

$$e = K1,$$

it follows from an application of Lemma 1.5 that the function $e \in C^{2+\theta}(\overline{D})$ satisfies the conditions

$$\begin{cases} e(x') = 0 & \text{on } M, \\ e(x) > 0 & \text{on } \overline{D} \setminus M, \\ \frac{\partial e}{\partial \nu}(x') < 0 & \text{on } M. \end{cases}$$

Further we let

$$C_e(\overline{D}) = \{u \in C(\overline{D}) : \text{there is a constant } c > 0 \text{ such that } -ce \leq u \leq ce\}.$$

Then the space $C_e(\overline{D})$ is given a norm by the formula

$$\|u\|_e = \inf\{c > 0 : -ce \leq u \leq ce\}.$$

If we let

$$P_e = C_e(\overline{D}) \cap P,$$

it is easy to verify that the space $C_e(\overline{D})$ is an ordered Banach space having the positive cone P_e with nonempty interior.

The next proposition will play an important role in the proof of Theorem 5 in Subsection 4.2:

Proposition 1.6. *The operator K maps $C(\overline{D})$* compactly *into $C_e(\overline{D})$. Moreover, K is* strongly positive, *that is, if $v \in P$ and $v \not\equiv 0$ on $\overline{D}$, then the function Kv is an interior point of P_e.*

Proof. (i) First, by the positivity of K we find that K maps $C(\overline{D})$ into $C_e(\overline{D})$. Indeed, since we have $-\|v\| \leq v(x) \leq \|v\|$ on $\overline{D}$ for all $v \in C(\overline{D})$, it follows that

$$-\|v\| K1(x) \leq Kv(x) \leq \|v\| K1(x) \quad \text{on } \overline{D}.$$

This proves that $-ce \leq Kv \leq ce$ with $c = \|v\|$.

(ii) Next we prove that $K : C(\overline{D}) \to C_e(\overline{D})$ is compact. To do so, we let

$$C^1_B(\overline{D}) = \{u \in C^1(\overline{D}) : Bu = 0 \text{ on } \partial D\}.$$

Since K maps $C(\overline{D})$ compactly into $C^1_B(\overline{D})$, it suffices to show that the inclusion mapping

$$\iota : C^1_B(\overline{D}) \longrightarrow C_e(\overline{D}) \tag{1.5}$$

is continuous.

(ii-a) We verify that ι maps $C^1_B(\overline{D})$ into $C_e(\overline{D})$.

Let u be an arbitrary function in $C^1_B(\overline{D})$. Since we have for some neighborhood ω of M in ∂D

$$\begin{cases} b > 0 & \text{in } \omega, \\ \frac{\partial e}{\partial \nu} < 0 & \text{in } \omega, \end{cases}$$

it follows that

$$\frac{u}{e} = \frac{\left(-\frac{a}{b}\right)\frac{\partial u}{\partial \nu}}{\left(-\frac{a}{b}\right)\frac{\partial e}{\partial \nu}} = \frac{\frac{\partial u}{\partial \nu}}{\frac{\partial e}{\partial \nu}} \quad \text{in } \omega \backslash M.$$

Hence there exists a constant $c_1 > 0$ such that

$$|u(x')| \leq c_1 e(x') \quad \text{in} \quad \omega.$$

Thus, by using Taylor's formula we can find a neighborhood W of ω in D and a constant $c_2 > 0$ such that

$$|u(x')| \leq c_2 e(x) \quad \text{in } W.$$

On the other hand, since we have for some constant $\alpha > 0$

$$e(x) \geq \alpha \quad \text{on } \overline{D} \backslash W,$$

we can find a constant $c_3 > 0$ such that

$$\left| \frac{u(x)}{e(x)} \right| \leq c_3 \quad \text{on } \overline{D} \backslash W.$$

Therefore there exists a constant $c > 0$ such that

$$-ce(x) \leq u(x) \leq ce(x) \quad \text{on} \quad \overline{D}.$$

This proves that $u \in C_e(\overline{D})$.

(ii-b) Now assume that

$$\begin{cases} u_j \in C^1_B(\overline{D}), \\ u_j \to u \quad \text{in } C^1_B(\overline{D}), \\ u_j \to v \quad \text{in } C_e(\overline{D}). \end{cases}$$

Then there exists a sequence $\{c_j\}, c_j \to 0$, such that

$$\|u_j - v\| \le c_j \|e\|.$$

This implies that $u_j \to v$ in $C(\overline{D})$. Hence we have $u = v$. By the closed graph theorem, it follows that the mapping ι is continuous.

(iii) It remains to prove the strong positivity of K.

(iii-a) We show that, for any $v(x) \ge 0$ but $v(x) \not\equiv 0$ on $\overline{D}$, there exist constants $\beta > 0$ and $\gamma > 0$ such that

$$\beta e(x) \le Kv(x) \le \gamma e(x) \quad \text{on } \overline{D}. \tag{1.6}$$

By the positivity of K, one may modify the function v in such a way that $v \in C^1(\overline{D})$. Furthermore, since the functions $u = Kv$ and $e = K1$ vanish only on the set M, it suffices to prove that there exists a neighborhood W of M in D such that

$$\beta e(x) \le u(x) \quad \text{in } W. \tag{1.7}$$

We recall that in a neighborhood ω of M in ∂D

$$\begin{cases} u = \left(-\frac{a}{b}\right) \frac{\partial u}{\partial \nu} & \text{in } \omega \\ \frac{\partial u}{\partial \nu} < 0 & \text{in } \omega, \end{cases}$$

and

$$\begin{cases} e = \left(-\frac{a}{b}\right) \frac{\partial e}{\partial \nu} & \text{in } \omega, \\ \frac{\partial e}{\partial \nu} < 0 & \text{in } \omega. \end{cases}$$

Thus we have for β sufficiently small

$$\begin{cases} u(x') - \beta e(x') \ge 0 & \text{in } \omega, \\ \frac{\partial}{\partial \nu}(u - \beta e)(x') < 0 & \text{in } \omega. \end{cases}$$

Therefore, by using Taylor's formula we can find a neighborhood W of M in D such that

$$u(x) - \beta e(x) \ge 0 \quad \text{in } W.$$

This proves estimate (1.7).

(iii-b) Finally we show that the function $u = Kv$ is an interior point of P_e.

Take

$$\varepsilon = \frac{\beta}{2},$$

where β is the same constant as in estimate (1.6). Then, for all functions $w \in C_e(\overline{D})$ satisfying

$$\|w - Kv\|_e < \varepsilon,$$

we have by estimate (1.6)

$$w \leq Kv + \varepsilon e \leq (\gamma + \varepsilon) e,$$

and also

$$w \geq Kv - \varepsilon e \geq \frac{\beta}{2} e.$$

This implies that $w \in P_e$, that is, the function Kv is an interior point of P_e.

The proof of Proposition 1.6 is now complete. ■

Now we consider the resolvent K as an operator in the ordered Banach space $C_e(\overline{D})$, and prove important results concerning its eigenfunctions and corresponding eigenvalues.

First, Proposition 1.6 tells us that the operator

$$K : C_e(\overline{D}) \longrightarrow C_e(\overline{D})$$

is strongly positive and compact. This implies that K has a countable number of positive eigenvalues, μ_j, which may accumulate only at 0. Hence they may be arranged in a decreasing sequence

$$\mu_1 \geq \mu_2 \geq \cdots,$$

where each eigenvalue is repeated according to its multiplicity.

The next theorem, a sharper version of the well-known Kreĭn-Rutman theorem [10], characterizes the eigenvalues and positive eigenfunctions of the operator K (cf. [9]):

Theorem 1.7. *The resolvent K, considered as an operator $K : C_e(\overline{D}) \to C_e(\overline{D})$, has the following spectral properties:*

(i) *The largest eigenvalue μ_1 is* simple, *that is, $\mu_1 > \mu_2$, and has a* positive *eigenfunction ψ_1.*

(ii) *No other eigenvalues, μ_j, $j \geq 2$, have positive eigenfunctions.*

1.4. Proof of Theorem 0 -(2)-.

By assertions (1.4) and (1.5), it is easy to see that

$$\mathcal{U} u = \lambda u \quad \text{in } L^2(D) \longleftrightarrow Ku = \frac{1}{\lambda} u \quad \text{in } C_e(\overline{D}).$$

Therefore Theorem 0 is an immediate consequence of Theorem 1.7. ■

2. Proof of Theorem 1

The proof of Theorem 1 is based on the following bifurcation theorem from a simple eigenvalue (cf. [13, Theorem 3.2.2]; [5, Theorem 1.7]):

Theorem 2.1. *Let $f(\lambda, x)$ be a C^k map, $k \geq 3$, of a neighborhood of $(\lambda_1, 0)$ in a Banach space $\mathbb{R} \times \mathcal{X}$ into a Banach space $\mathcal{Y}$ such that*

$$f(\lambda_1, 0) = 0.$$

Assume that the following conditions are satisfied:

(i) $f_\lambda(\lambda_1, 0) = 0$.

(ii) *The null space $N(f_x(\lambda_1, 0))$ is one-dimensional, spanned by a vector x_0.*

(iii) *The range $R(f_x(\lambda_1, 0))$ has codimension one in the space $\mathcal{Y}$.*

(iv) $f_{\lambda\lambda}(\lambda_1, 0) \in R(f_x(\lambda_1, 0))$ *and* $f_{\lambda_x}(\lambda_1, 0)x_0 \notin R(f_x(\lambda_1, 0))$.

Then the point $(\lambda_1, 0)$ is a bifurcation point for the equation $f(\lambda, x) = 0$. In fact, the set of solutions of $f(\lambda, x) = 0$ near $(\lambda_1, 0)$ consists of two C^{k-2} curves Γ_1 and Γ_2 intersecting only at the point $(\lambda_1, 0)$. Furthermore the curve Γ_1 is tangent to the λ-axis at $(\lambda_1, 0)$ and may be parametrized by λ as

$$\Gamma_1 = \{(\lambda, x_1(\lambda)) : |\lambda - \lambda_1| < \varepsilon\},$$

while the curve Γ_2 may be parametrized by a variable s as

$$\Gamma_2 = \{(\lambda_2(s), sx_0 + x_2(s)) : |s| < \varepsilon\}.$$

Here

$$x_2(0) = \frac{dx_2}{ds}(0) = 0, \quad \lambda_2(0) = \lambda_1.$$

We shall apply Theorem 2.1 with

$$\begin{aligned} &\mathcal{X} = C_B^{2+\theta}(\overline{D}), \\ &\mathcal{Y} = C^\theta(\overline{D}), \\ &f(\lambda, x) := F(\lambda, u) = Au - \lambda u + G(\lambda, u). \end{aligned}$$

First, by [19, Theorems 7.3 and 7.4] it follows that $\mathcal{U}$ is a non-negative, self-adjoint operator in the Hilbert space $L^2(D)$. Hence we have, for each $\lambda > 0$, the following orthogonal decomposition:

$$L^2(D) = N(\mathcal{U} - \lambda I) \oplus R(\mathcal{U} - \lambda I). \tag{2.1}$$

However it follows from an application of regularity theorem for problem (1.1) ([21, Theorem 5.1]) that

$$N(\mathcal{U}-\lambda I)=\{u\in C_B^{2+\theta}(\overline{D}):(A-\lambda)u=0\ \text{ in }\ D\},$$

and also

$$R(\mathcal{U}-\lambda I)\cap C^{\theta}(\overline{D})=\{(A-\lambda)u:u\in C_B^{2+\theta}(\overline{D})\}.$$

Thus, restricting decomposition (2.1) to the space $C^{\theta}(\overline{D})$ and taking $\lambda=\lambda_1$ we obtain the orthogonal decomposition

$$\begin{aligned}(2.2)\qquad & C^{\theta}(\overline{D}) \\ &= \{u\in C_B^{2+\theta}(\overline{D}):(A-\lambda_1)u=0 \text{ in } D\}\oplus\{(A-\lambda_1)u:u\in C_B^{2+\theta}(\overline{D})\} \\ &= N(F_u(\lambda_1,0))\oplus R(F_u(\lambda_1,0)).\end{aligned}$$

By virtue of decomposition (2.2), it is easy to verify conditions (2) and (3) of Theorem 2.1. Indeed, Theorem 0 tells us that the null space $N(F_u(\lambda_1,0))$ is one-dimensional, spanned by the eigenfunction ψ_1.

Therefore Theorem 1 follows from an application of Theorem 2.1. ■

3. Proof of Theorems 2 and 3

First we find that problem (0.3) is equivalent to the following operator equation:

$$\lambda Ku-K(g(u))=u\quad\text{in } C(\overline{D}). \tag{3.1}$$

Indeed it suffices to recall that the operator K can be uniquely extended to an operator

$$K:C(\overline{D})\longrightarrow C^1(\overline{D}),$$

and also an operator

$$K:C^1(\overline{D})\longrightarrow C^2(\overline{D}).$$

Now let $m(x)$ be a function in $C(\overline{D})$ such that

$$m(x)>0\quad\text{on } \overline{D},$$

and consider the following eigenvalue problem for the operator K:

$$K(mu)=\mu u\quad\text{in } C(\overline{D}).$$

This problem has a countable number of positive eigenvalues, $\mu_j(m)$, which may accumulate only at 0. Hence they may be arranged in a decreasing sequence

$$\mu_1(m)\geq\mu_2(m)\geq\cdots,$$

where each eigenvalue is repeated according to its multiplicity.

In the proof of Theorem 2, we need the following generalization of Theorem 1.7:

Proposition 3.1. *The largest eigenvalue* $\mu_1(m)$ *is* simple, *i.e.,* $\mu_1(m) > \mu_2(m)$*, and has a* positive *eigenfunction. No other eigenvalues have positive eigenfunctions.*

Proof. Proposition 1.6 tells us that the operator $K : C(\overline{D}) \to C_e(\overline{D})$ is strongly positive and compact. Hence the assertions follow from an application of [1, Theorem 3.2]. ∎

Furthermore we need the following:

Proposition 3.2. *If* $m_1(x) \leq m_2(x)$ *for all* $x \in \overline{D}$*, then we have* $\mu_j(m_1) \leq u_j(m_2)$ *for all* j*. If* $m_1(x) < m_2(x)$ *for almost all* $x \in \overline{D}$*, then we have* $\mu_j(m_1) < \mu_j(m_2)$ *for all* j*.*

Proof. Proposition 3.2 is an immediate consequence of the well-known minimax property of eigenvalues. ∎

3.1. Proof of Theorem 2.

The proof of Theorem 2 is essentially the same as that of [18, Theorem 1.3]; so we only give a sketch of the proof.

(i) First, by Theorem 1 we obtain that equation (3.1) (or problem (0.3)) has precisely two branches of nontrivial solutions emanating from the point $(\lambda_1, 0)$.

(ii) Secondly, by using Propositions 3.1 and 3.2 we find that the nontrivial solutions of equation (3.1) with $\lambda_1 < \lambda \leq \lambda_2$ must necessarily be positive or negative.

(iii) In order to study *globally* the bifurcation solution curves, we need the following three lemmas:

Lemma 3.3. *If* u *is a positive (or negative) solution of equation* (3.1) *with* $\lambda_1 < \lambda < \infty$*, then* u *is a* regular point *of the mapping* $F(\lambda, u) : \mathbb{R} \times C(\overline{D}) \to C(\overline{D})$*, given by the formula*

$$F(\lambda, u) = u - \lambda K u + K(g(u)),$$

that is, the partial Fréchet derivative $F_u(\lambda, u)$ *at* u *is invertible.*

Lemma 3.4. *Equation* (3.1) *has a unique positive solution for each* $\lambda_1 < \lambda < \lambda_1 + k_+$*. No positive solutions exist for* $\lambda \geq \lambda_1 + k_+$*. The uniform norm*

$\|u_+(\lambda)\|$ of the positive solution $u_+(\lambda)$ tends to $+\infty$ as $\lambda \to \lambda_1 + k_+$. Similar assertions are valid for negative solutions $u_-(\lambda)$, with k_+ replaced by k_-.

Lemma 3.5. *There is a constant $\delta > 0$ such that equation* (3.1) *has no nontrivial solutions for $\lambda_1 - \delta \leq \lambda \leq \lambda_1$.*

Lemmas 3.3, 3.4 and 3.5 are proved just as in the proof of [18, Lemmas 2.1, 2.2 and 2.3], by using Propositions 3.1 and 3.2 and the theory of positive mappings in ordered Banach spaces.

(iii-a) By using Lemma 3.4, Lemma 3.3 and the implicit function theorem, we can prove that equation (3.1) has a unique positive solution $u_+(\lambda)$ for all $\lambda_1 < \lambda < \lambda_1 + k_+$, and that the branch Γ_+ of positive solutions emanating from $(\lambda_1, 0)$ is a C^1 curve given by the formula

$$\Gamma_+ = \{(\lambda, u) \in \mathbb{R} \times C(\overline{D}) : u = u_+(\lambda),\ \lambda_1 \leq \lambda < \lambda_1 + k_+\}.$$

The other branch Γ_- is obtained in a similar way.

(iii-b) Furthermore it follows from an application of Lemma 3.4 that no other positive or negative solutions exist for $\lambda > \lambda_1$ and also $\|u_+(\lambda)\| \to +\infty$ as $\lambda \to \lambda_1 + k_+$ and $\|u_-(\lambda)\| \to +\infty$ as $\lambda \to \lambda_1 + k_-$.

(iv) Finally, Lemma 3.5 tells us that there are no nontrivial solutions at $\lambda = \lambda_1$.

The proof of Theorem 2 is complete. ∎

3.2. Proof of Theorem 3.

The proof of Theorem 3 is carried out by using the global theory of positive mappings (cf. [6]), just as in the proof of [18, Theorems 5.1 and 5.2]. ∎

4. Proof of Theorems 4 and 5

4.1. Proof of Theorem 4.

(1) First we replace the function $c(x)$ by the function $c(x)+\omega$, where $\omega > 0$ is the same constant as in condition $(\mathrm{R})_\sigma$, and consider instead of problem (0.4) the following problem:

$$\begin{cases} (A+\omega)u = \omega u + F(u) & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases} \tag{4.1}$$

where $F(u)$ is the Nemytskii operator of $f(x, \xi)$ defined by the formula

$$Fu(x) = f(x, u(x)), \quad x \in \overline{D}.$$

It is clear that problem (0.4) is equivalent to problem (4.1). Furthermore, since $f(x,\xi) \in C^\theta(\overline{D} \times [0,\sigma])$, it is easy to verify that problem (4.1) is equivalent to an operator equation

$$u = K_\omega(\omega u + F(u)) \quad \text{in } C(\overline{D}), \tag{4.2}$$

just as in Subsection 1.2. Here $K_\omega : C(\overline{D}) \to C^1(\overline{D})$ is the compact operator introduced in Subsection 1.2 with c replaced by $c + \omega$.

(2) We let

$$H_\omega(u) = K_\omega(\omega u + F(u)), \quad u \in C(\overline{D}).$$

Then we have the following:

Lemma 4.1. *The operator $H_\omega : [\phi, \psi] \to C(\overline{D})$ is increasing. Here $[\phi, \psi]$ is the order interval defined by the formula*

$$[\phi, \psi] = \{u \in C(\overline{D}) : \phi(x) \leq u(x) \leq \psi(x) \text{ on } \overline{D}\}.$$

Proof. Let u and v be arbitrary functions in $C(\overline{D})$ satisfying $\phi \leq u \leq v \leq \psi$ on $\overline{D}$. Then we have

$$\begin{aligned} &\omega(v(x) - u(x)) + (Fv(x) - Fu(x)) \\ = &\begin{cases} 0 & \text{if } v(x) = u(x), \\ \left(\omega + \frac{Fv(x) - Fu(x)}{v(x) - u(x)}\right)(v(x) - u(x)) & \text{if } v(x) > u(x), \end{cases} \end{aligned}$$

and so by condition $(\mathrm{R})_\sigma$

$$w(v - u) + (Fu - Fv) \geq 0 \quad \text{on } \overline{D}.$$

However Lemma 4.1 tells us that $K_\omega : C(\overline{D}) \to C(\overline{D})$ is positive. Thus it follows that

$$H_\omega(v) - H_\omega(u) = K_\omega(\omega(v - u) + (F(v) - F(u))) \geq 0 \quad \text{on } \overline{D},$$

or equivalently

$$H_\omega(u) \leq H_\omega(v) \quad \text{on } \overline{D}.$$

This proves that H_ω is increasing. ■

Moreover we have the following:

Lemma 4.2. *The operator H_ω maps the order interval $[\phi, \psi]$ into itself.*

Proof. Let u be an arbitrary function in $C(\overline{D})$ satisfying $\phi \leq u \leq \psi$ on $\overline{D}$. Then it follows from an application of Lemma 4.1 that

$$H_\omega(\phi) \leq H_\omega(u) \leq H_\omega(\psi) \quad \text{on } \overline{D}.$$

Hence, in order to prove the lemma it suffices to show that

$$\phi \leq H_\omega(\phi), \quad H_\omega(\psi) \leq \psi \quad \text{on } \overline{D}.$$

If we let

$$v = H_\omega(\psi) = K_\omega(\omega\psi + F(\psi)),$$

then we have

$$\begin{cases} (A+\omega)v = \omega\psi + F(\psi) & \text{in } D, \\ Bv = 0 & \text{on } \partial D. \end{cases}$$

However, since ψ is a supersolution of problem (0.4), it follows that

$$\begin{aligned} (A+\omega)(v-\psi) &= \omega\psi + F(\psi) - (A+\omega)\psi \\ &= -(A\psi - F(\psi)) \leq 0 \ \text{ in } D, \end{aligned}$$

and

$$B(v-\psi) = -B\psi \leq 0 \ \text{ on } \quad \partial D.$$

Thus, using the maximum principle we find that

$$H_\omega(\psi) = v \leq \psi \quad \text{on } \overline{D}.$$

Indeed, if the function $v-\psi$ takes its positive maximum m at an interior point $x_0 \in D$, then we have

$$(A+\omega)(v-\psi)(x_0) \geq \omega m > 0,$$

which contradicts the condition: $(A+\omega)(v-\psi) \leq 0$ in D. On the other hand, if $v-\psi$ takes the maximum m at a boundary point $x'_0 \in \partial D$, then we have by the boundary point lemma

$$\frac{\partial}{\partial \boldsymbol{\nu}}(v-\psi)(x'_0) > 0,$$

so that by condition (H.1)

$$B(v-\psi)(x'_0) = a(x'_0)\frac{\partial}{\partial \boldsymbol{\nu}}(v-\psi)(x'_0) + b(x'_0)m > 0,$$

which contradicts the condition: $B(v-\psi) \leq 0$ on ∂D.

Similarly we can prove that

$$\phi \leq H_\omega(\phi) \quad \text{on } \overline{D}.$$

The proof of Lemma 4.2 is complete. ■

(3) Now we need an extension of Brouwer's fixed point theorem to the infinite-dimensional case, due to Schauder (see [17, Proposition 3.60]):

Theorem 4.3 (Schauder's fixed point theorem). *A compact mapping f of a closed bounded convex set K in a Banach space X into itself has a fixed point $x \in K : f(x) = x$.*

Since $K_\omega : C(\overline{D}) \to C^1(\overline{D})$ is compact, it follows from Lemma 4.2 that the mapping $H_\omega : [\phi, \psi] \to [\phi, \psi]$ is compact. Furthermore the order interval $[\phi, \psi]$ is bounded, closed and convex in the space $C(\overline{D})$. Therefore, applying Schauder's fixed point theorem we can find a function $u \in [\phi, \psi]$ such that

$$u = H_\omega(u) = K_\omega(\omega u + F(u)) \ \text{ in } \ C(\overline{D}).$$

Now the proof of Theorem 4 is complete. ■

4.2. Proof of Theorem 5.

(1) The proof of Theorem 5 is essentially based on the following (cf. [1, Theorem 24.2]):

Theorem 4.4. *Let (E, Q) be an ordered Banach space having the positive cone Q with nonempty interior. If σ is a positive number, we let*

$$\overline{Q}_\sigma = \{u \in Q : \|u\| \leq \sigma\}.$$

Assume that a mapping $f : \overline{Q}_\sigma \to E$ satisfies the following two conditions:

(A) *f is* strongly increasing, *that is, if $u, v \in \overline{Q}_\sigma$ and if $u \leq v$ and $v \neq u$, then $f(v) - f(u)$ is an interior point of Q.*

(B) *f is* strongly sublinear, *that is, $f(0) \geq 0$ and if $u \in \overline{Q}_\sigma$ and $u \neq 0$, then $f(\tau u) - \tau f(u)$ is an interior point of Q for every $0 < \tau < 1$.*

Then the mapping f has at most *one positive fixed point.*

In the proof of Theorem 5, we shall apply Theorem 4.4 with

$$E = C_e(\overline{D}),$$
$$Q = P_e = C_e(\overline{D}) \cap P = \{u \in C_e(\overline{D}) : u \geq 0 \ \text{ on } \overline{D}\},$$
$$f = H_\omega.$$

(2) If σ is a positive number, we let

$$(\overline{P_e})_\sigma = \{u \in P_e : u \le \sigma \text{ on } \overline{D}\}.$$

We have only to prove Theorem 5 in the space $(\overline{P_e})_\sigma$ for every $\sigma > 0$. Indeed, if u_1 and u_2 are two positive solutions of problem (0.4), then one can find a constant $\sigma > 0$ such that $u_1,\ u_2 \le \sigma$ on $\overline{D}$, so that $u_1,\ u_2 \in (\overline{P_e})_\sigma$.

If we take a constant $\omega = \omega(\sigma) > 0$ given in condition $(\mathrm{R})_\sigma$, then we have the following:

Lemma 4.5. *The operator H_ω maps $(\overline{P_e})_\sigma$ into P_e.*

Proof. Let u be an arbitrary function in $(\overline{P_e})_\sigma$. Then we have by condition $(\mathrm{R})_\sigma$ with $\xi = u$ and $\eta = 0$ and condition (S2)

$$F(u) \ge F(0) - \omega u \ge -\omega u \quad \text{on } \overline{D},$$

so that

$$\omega u + F(u) \ge 0 \quad \text{on } \overline{D}.$$

Hence it follows from an application of Proposition 1.6 that

$$H_\omega(u) = K_\omega(\omega u + F(u)) \in P_e.$$

■

Moreover we have the following:

Lemma 4.6. *The operator $H_\omega : (\overline{P_e})_\sigma \longrightarrow P_e$ is strongly increasing.*

Proof. Lemma 4.6 follows by combining Lemma 4.1 and Proposition 1.6. ■

Lemma 4.7. *The operator $H_\omega : (\overline{P_e})_\omega \longrightarrow P_e$ is strongly sublinear.*

Proof. Let $u(x)$ be an arbitrary function in $(\overline{P_e})_\sigma$ but $u(x) \not\equiv 0$. Then we have by condition (S)

$$\begin{cases} f(x, \tau u(x)) \ge \tau f(x, u(x)) & \text{if } u(x) > 0, \\ f(x, \tau u(x)) = f(x, 0) \ge 0 & \text{if } u(x) = 0. \end{cases}$$

This implies that

$$\omega\tau u + F(\tau u) - \tau(\omega u + F(u)) = F(\tau u) - \tau F(u) \ge 0 \text{ and } \not\equiv 0 \ \text{ on } \overline{D}.$$

Hence it follows from an application of Proposition 1.6 that the function

$$H_\omega(\tau u) - \tau H_\omega(u) = K_\omega(\omega\tau u + F(\tau u) - \tau(\omega u + F(u)))$$

is an interior point of P_e. ■

(3) Combining Lemmas 4.5, 4.6 and 4.7, we have proved that the mapping $H_\omega : (\overline{P_e})_\sigma \to P_e$ satisfies conditions (A) and (B) of Theorem 4.4 with $E = C_e(\overline{D})$ and $Q = P_e$. Therefore Theorem 5 follows from an application of the same theorem.

The proof of Theorem 5 is complete. ■

5. Proof of Theorem 6

We let

$$f(x,\xi) = \lambda\xi - h(x)\xi^p, \quad x \in \overline{D},\ \xi \geq 0.$$

(i) First it is easy to verify that the function $f(x,\xi)$ satisfies condition $(\mathrm{R})_\sigma$ for every $\sigma > 0$.

Indeed, we have, for all $x \in \overline{D}$ and $0 \leq \eta < \xi \leq \sigma$,

$$\begin{aligned} f(x,\xi) - f(x,\eta) &= \lambda(\xi - \eta) - h(x)(\xi^p - \eta^p) \\ &\geq \lambda(\xi - \eta) - \max_{x\in\overline{D}} h(x)(\xi^p - \eta^p) \\ &= \left(\lambda - \max_{x\in\overline{D}} h(x)\left(\frac{\xi^p - \eta^p}{\xi - \eta}\right)\right)(\xi - \eta) \\ &> \left(\lambda - \max_{x\in\overline{D}} h(x) \cdot p\sigma^{p-1}\right)(\xi - \eta). \end{aligned}$$

Thus, if we take a positive constant

$$\omega = \omega(\sigma,\lambda) = \max\left\{\max_{x\in\overline{D}} h(x) \cdot p\sigma^{p-1} - \lambda, 1\right\},$$

then condition $(\mathrm{R})_\sigma$ is satisfied.

(ii) Secondly we show that the function $f(x,\xi)$ satisfies condition (S).

It is clear that $f(x,0) = 0$ on $\overline{D}$, which verifies condition (S2). Furthermore, since $h(x) \geq 0$ on $\overline{D}$, we have, for all $x \in \overline{D}, \xi > 0$ and $0 < \tau < 1$,

$$\begin{aligned} f(x,\tau\xi) &= \lambda(\tau\xi) - h(x)(\tau\xi)^p \\ &= \tau(\lambda\xi - h(x)\tau^{p-1}\xi^p) \\ &\geq \tau(\lambda\xi - h(x)\xi^p) \\ &= \tau f(x,\xi). \end{aligned}$$

This verifies condition (S1).

(iii) Now we construct a positive solution $u(\lambda)$ of problem (0.5) for every $\lambda_1 < \lambda < \tilde{\lambda}_1(D_0(h))$, and further show that, for any $\lambda \geq \tilde{\lambda}_1(D_0(h))$, there exists no positive solution of problem (0.5). We remark that the uniqueness of positive solutions of problem (0.5) is an immediate consequence of Theorem 5.

(iii-a) First we begin with the following:

Lemma 5.1. *If there exists a positive solution* $u(\lambda) \in C^2(\overline{D})$ *of problem* (0.5)*, then we have*

$$\lambda > \lambda_1. \tag{5.1}$$

Proof. Let $\psi_1(x)$ be an eigenfunction corresponding to the eigenvalue λ_1:

$$\begin{cases} A\psi_1 = \lambda_1 \psi_1 & \text{in } D, \\ B\psi_1 = 0 & \text{on } \partial D. \end{cases}$$

By Theorem 0, one may assume that $\psi_1(x) > 0$ in D.

Then it follows from an application of Green's formula that

$$\begin{aligned} 0 &= \int_D (Au(\lambda) - \lambda u(\lambda) + h(x)u(\lambda)^p)\psi_1 dx \\ &= \int_D u(\lambda) A\psi_1 - \lambda \int_D u(\lambda)\psi_1 dx + \int_D h(x)u(\lambda)^p \psi_1 dx \\ &\quad - \int_{\partial D} \frac{\partial u(\lambda)}{\partial \boldsymbol{\nu}} \psi_1 d\sigma + \int_{\partial D} u(\lambda) \frac{\partial \psi_1}{\partial \boldsymbol{\nu}} d\sigma \\ &= (\lambda_1 - \lambda) \int_D u(\lambda)\psi_1 dx + \int_{\partial D} h(x)u(\lambda)^p \psi_1 dx \\ &\quad + \int_{\partial D} \left(u(\lambda) \frac{\partial \psi_1}{\partial \boldsymbol{\nu}} - \frac{\partial u(\lambda)}{\partial \boldsymbol{\nu}} \psi_1 \right) d\sigma. \end{aligned} \tag{5.2}$$

However we recall that the functions $u(\lambda)$ and ψ_1 satisfy the following boundary conditions:

$$\begin{pmatrix} \frac{\partial u(\lambda)}{\partial \nu} & u(\lambda) \\ \frac{\partial \psi_1}{\partial \nu} & \psi_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial D.$$

Thus it follows that

$$\begin{vmatrix} \frac{\partial u(\lambda)}{\partial \nu} & u(\lambda) \\ \frac{\partial \psi_1}{\partial \nu} & \psi_1 \end{vmatrix} = 0 \quad \text{on } \partial D,$$

since $(a, b) \neq (0, 0)$ on ∂D.

Therefore we obtain from formula (5.2) that

$$(\lambda_1 - \lambda) \int_D u(\lambda)\psi_1 dx + \int_D h(x)u(\lambda)^p \psi_1 dx = 0,$$

so that

$$\lambda - \lambda_1 = \frac{\int_D h(x)u(\lambda)^p \psi_1 dx}{\int_D u(\lambda)\psi_1 dx} > 0.$$

This proves inequality (5.1). ■

(iii-b) Secondly we associate with problem (0.5) a nonlinear mapping $F(\lambda, u)$ of $\mathbb{R} \times C_B^{2+\theta}(\overline{D})$ into $C^\theta(\overline{D})$ as follows:

$$F : \mathbb{R} \times C_B^{2+\theta}(\overline{D}) \longrightarrow C^\theta(\overline{D})$$
$$(\lambda, u) \longmapsto Au - \lambda u + h(x)u^p.$$

It is clear that a function $u \in C^{2+\theta}(\overline{D})$ is a solution of problem (0.5) if and only if $F(\lambda, u) = 0$.

The next lemma proves the existence of positive solutions of problem (0.5) near the point $(\lambda_1, 0)$:

Lemma 5.2. *There exists a positive bifurcation solution curve $(\lambda, u(\lambda))$ of the equation $F(\lambda, u) = 0$ starting at $(\lambda_1, 0)$.*

Lemma 5.2 follows from an application of Theorem 1. Indeed it is easy to see that the mapping $G(\lambda, u) = h(x)u^p$ satisfies conditions (i) through (iv) of the same theorem, since $p > 1$.

(iii-c) Thirdly, by using the implicit function theorem we show that there exists a critical value $\overline{\lambda}(h) \in (\lambda_1, \infty)$ such that one can extend the above bifurcation solution curve $(\lambda, u(\lambda))$ to all $\lambda_1 < \lambda < \overline{\lambda}(h)$:

Lemma 5.3. *There exists a constant $\overline{\lambda}(h) \in (\lambda_1, \infty)$ such that we have a positive solution $(\lambda, u(\lambda))$ of the equation $F(\lambda, u) = 0$ for all $\lambda_1 < \lambda < \overline{\lambda}(h)$.*

Proof. By applying Theorem 1.1 to our situation, we obtain that the Fréchet derivative

$$F_u(\lambda, u(\lambda)) : C_B^{2+\theta}(\overline{D}) \longrightarrow C^\theta(\overline{D})$$
$$v \longmapsto Av - \lambda v + ph(x)u(\lambda)^{p-1}v$$

is a Fredholm operator with index zero. Hence, in order to prove the lemma it suffices to show that $F_u(\lambda, u(\lambda))$ is injective. Indeed, by using the implicit function theorem one can find a constant $\overline{\lambda}(h) \in (\lambda_1, \infty)$ such that $F(\lambda, u(\lambda)) = 0$

and the derivative $F_u(\lambda, u(\lambda))$ is an algebraic and topological isomorphism for all $\lambda_1 < \lambda < \overline{\lambda}(h)$.

The next claim proves the injectivity and hence surjectivity of $F_u(\lambda, u(\lambda))$:

Claim 1. *We define a densely defined, closed linear operator* $\mathcal{U}(\lambda) : L^2(D) \to L^2(D)$ *as follows.*

(a) *The domain of definition* $D(\mathcal{U}(\lambda))$ *is the space*

$$D(\mathcal{U}(\lambda)) = \{v \in H^{2,2}(D) : Bv = 0 \text{ on } \partial D\}.$$

(b) $\mathcal{U}(\lambda)v = Av + ph(x)u(\lambda)^{p-1}v, v \in D(\mathcal{U}(\lambda))$.

Then the first eigenvalue $\mu_1(\lambda)$ *of* $\mathcal{U}(\lambda) - \lambda I$ *is* positive *for all* $\lambda_1 < \lambda < \overline{\lambda}(h)$.

Proof. Let $\mu_1(\lambda)$ and $v_1(\lambda)$ be the first eigenvalue and associated eigenfunction of $\mathcal{U}(\lambda) - \lambda I$, respectively:

$$(\mathcal{U}(\lambda) - \lambda I)v_1(\lambda) = \mu_1(\lambda)v_1(\lambda),$$

or equivalently

$$\begin{cases} (A - \lambda + ph(x)u(\lambda)^{p-1})v_1(\lambda) = \mu_1(\lambda)v_1(\lambda) & \text{in } D, \\ Bv_1(\lambda) = 0 & \text{on } \partial D. \end{cases}$$

By Theorem 0, one may assume that $v_1(\lambda) > 0$ in D. Then we have by Green's formula

$$\begin{aligned} \mu_1(\lambda) \int_D u(\lambda)v_1(\lambda)dx &= \int_D \Big(Av_1(\lambda) - \lambda v_1(\lambda) + ph(x)u(\lambda)^{p-1}v_1(\lambda)\Big)u(\lambda)dx \\ &= \int_D v_1(\lambda)(A-\lambda)u(\lambda)dx + p\int_D h(x)v_1(\lambda)u(\lambda)^p dx \\ &\quad - \int_{\partial D} \frac{\partial v_1(\lambda)}{\partial \boldsymbol{\nu}} u(\lambda)d\sigma + \int_{\partial D} v_1(\lambda)\frac{\partial u(\lambda)}{\partial \boldsymbol{\nu}} d\sigma \\ &= -\int_D h(x)u(\lambda)^p v_1(\lambda)dx + p\int_D h(x)v_1(\lambda)u(\lambda)^p dx \\ &\quad + \int_{\partial D} \left(v_1(\lambda)\frac{\partial u(\lambda)}{\partial \boldsymbol{\nu}} - \frac{\partial v_1(\lambda)}{\partial \boldsymbol{\nu}} u(\lambda)\right) d\sigma \\ &= (p-1)\int_D h(x)u(\lambda)^p v_1(\lambda)dx. \end{aligned}$$

Indeed, it suffices to note that the functions $u(\lambda)$ and $v_1(\lambda)$ satisfy the following boundary conditions:

$$\begin{pmatrix} \frac{\partial u(\lambda)}{\partial \nu} & u(\lambda) \\ \frac{\partial v_1(\lambda)}{\partial \nu} & v_1(\lambda) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial D.$$

Therefore we obtain that

$$\mu_1(\lambda) = \frac{(p-1)\int_D h(x)u(\lambda)^p v_1(\lambda)dx}{\int_D u(\lambda)v_1(\lambda)dx} > 0,$$

since $p > 1$ and $h(x) \geq 0$ in D. ■

The proof of Lemma 5.3 is complete. ■

By Lemma 5.3, we have a positive bifurcation solution curve $(\lambda, u(\lambda))$ of the equation $F(\lambda, u) = 0$ for all $\lambda_1 < \lambda < \overline{\lambda}(h)$. Then it is easy to see that the solution curve $u(\lambda)$ is of class C^1 with respect to λ, and further that it is increasing in λ and also blows up as $\lambda \to \overline{\lambda}(h)$, just as in [1, Theorem 25.4].

(iii-d) Finally we prove that there exists no positive solution of problem (0.5) for any $\lambda \geq \tilde{\lambda_1}(D_0(h))$, and further that:

$$\overline{\lambda}(h) = \tilde{\lambda_1}(D_0(h)).$$

First we begin with the following:

Lemma 5.4. *If $u(\lambda) \in C^2(\overline{D})$ is a positive solution of problem (0.5) for $\lambda > \lambda_1$, then we have*

$$\lambda < \lambda_1(D_i(h)), \quad 1 \leq i \leq l. \tag{5.3}$$

In particular, we have

$$\overline{\lambda}(h) \leq \tilde{\lambda_1}(D_0(h)). \tag{5.4}$$

Proof. Let φ_1 be an eigenfunction corresponding to the first eigenvalue $\lambda_1(D_i(h))$ of the Dirichlet problem

$$\begin{cases} A\varphi_1 = \lambda_1(D_i(h))\varphi_1 & \text{in } D_i(h), \\ \varphi_1 = 0 & \text{on } \partial D_i(h). \end{cases} \tag{0.6}$$

One may assume (cf. [26, Section 24.6, Theorem]) that $\varphi_1(x) > 0$ in $D_i(h)$.

On the other hand, it follows that

$$Au(\lambda) = \lambda u(\lambda) + h(x)u(\lambda)^p = \lambda u(\lambda) \quad \text{in } D_i(h),$$

since $h(x) = 0$ in $D_i(h)$.

Then we have, by a direct calculation,

$$\begin{aligned} \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(u(\lambda)^2 \sum_{j=1}^N a^{ij}\frac{\partial}{\partial x_j}\left(\frac{\varphi_1}{u(\lambda)}\right)\right) &= -u(\lambda)\cdot A\varphi_1 + \varphi_1 \cdot Au(\lambda) \\ &= -u(\lambda)\cdot \lambda_1(D_i(h))\varphi_1 + \varphi_1 \cdot \lambda u(\lambda) \\ &= (\lambda - \lambda_1(D_i(h)))u(\lambda)\cdot\varphi_1 \quad \text{in } D_i(h), \end{aligned}$$

so that

$$(\lambda - \lambda_1(D_i(h)))\varphi_1 = \frac{1}{u(\lambda)} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(u(\lambda)^2 \sum_{j=1}^{N} a^{ij} \frac{\partial}{\partial x_j} \left(\frac{\varphi_1}{u(\lambda)} \right) \right) \quad \text{in} \quad D_i(h).$$

Therefore, by integration by parts it follows that

$$\begin{aligned}
&(\lambda - \lambda_1(D_i(h))) \int_{D_i(h)} \varphi_1^2 dx \\
&= \int_{D_i(h)} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(u(\lambda)^2 \sum_{j=1}^{N} a^{ij} \frac{\partial}{\partial x_j} \left(\frac{\varphi_1}{u(\lambda)} \right) \right) \cdot \frac{\varphi_1}{u(\lambda)} dx \\
&= - \int_{D_i(h)} u(\lambda)^2 \sum_{i,j=1}^{N} a^{ij} \frac{\partial}{\partial x_i} \left(\frac{\varphi_1}{u(\lambda)} \right) \frac{\partial}{\partial x_j} \left(\frac{\varphi_1}{u(\lambda)} \right) dx \\
&< 0.
\end{aligned}$$

This proves inequality (5.3).

The proof of Lemma 5.4 is complete. ■

The next lemma proves the reverse inequality of inequality (5.4):

Lemma 5.5. *We have*

$$\tilde{\lambda}_1(D_0(h)) \leq \overline{\lambda}(h). \tag{5.5}$$

Proof. (1) First it follows from an application of [14, Theorems 2 and 3] that, for every $\lambda_1(D) < \lambda < \tilde{\lambda}_1(D_0(h))$, one can find a positive solution $\phi(\lambda)$ of the semilinear Dirichlet problem

$$\begin{cases} A\phi(\lambda) - \lambda\phi(\lambda) + h(x)\phi(\lambda)^p = 0 & \text{in } D, \\ \phi(\lambda) = 0 & \text{on } \partial D, \end{cases}$$

and also a positive solution $\psi(\lambda)$ of the semilinear Neumann problem

$$\begin{cases} A\psi(\lambda) - \lambda\psi(\lambda) + h(x)\psi(\lambda)^p = 0 & \text{in } D, \\ \frac{\partial\psi(\lambda)}{\partial\nu} = 0 & \text{on } \partial D. \end{cases}$$

Then we obtain that the function $\psi(\lambda)$ is a supersolution of problem (0.5), since we have

$$\begin{cases} A\psi(\lambda) - \lambda\psi(\lambda) + h(x)\psi(\lambda)^p = 0 & \text{in } D, \\ B\psi(\lambda) = b\psi(\lambda) \geq 0 & \text{on } \partial D. \end{cases}$$

Further it follows that the function $\phi_\varepsilon(\lambda) = \varepsilon\phi(\lambda)$ is a subsolution of problem (0.5) for each $0 < \varepsilon < 1$. Indeed, we have

$$\begin{aligned} & A\phi_\varepsilon(\lambda) - \lambda\phi_\varepsilon(\lambda) + h(x)\phi_\varepsilon(\lambda)^p \\ = & \ \varepsilon(A\phi(\lambda) - \lambda\phi(\lambda) + h(x)\varepsilon^{p-1}\phi(\lambda)^p) \\ = & \ \varepsilon h(x)(\varepsilon^{p-1} - 1)\phi(\lambda)^p \\ \leq & \ 0 \quad \text{in } D, \end{aligned}$$

and also by the boundary point lemma

$$B\phi_\varepsilon(\lambda) = a\varepsilon\frac{\partial\phi(\lambda)}{\partial\boldsymbol{\nu}} \leq 0 \quad \text{on } \partial D.$$

Here we may choose a constant $0 < \varepsilon(\lambda) < 1$ so small that

$$0 < \varepsilon(\lambda)\phi(\lambda) \leq \psi(\lambda) \quad \text{in } D.$$

Therefore, applying Theorem 4 to our situation we can find a solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ of problem (0.5) such that

$$0 < \varepsilon(\lambda)\phi(\lambda) \leq u(\lambda) \leq \psi(\lambda) \quad \text{in } D.$$

This proves that problem (0.5) has a positive solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ for every $\lambda_1(D) < \lambda < \tilde{\lambda}_1(D_0(h))$.

(2) The next claim asserts that a positive solution curve $(\lambda, u(\lambda))$ of the equation $F(\lambda, u) = 0$ may bifurcate only at the point $(\lambda_1, 0)$ (cf. [1, Proposition 18.1]):

Claim 2. *Assume that* $\{(\lambda_j, u_j)\}$ *is a sequence in the space* $\mathbb{R} \times C^2(\overline{D})$ *such that*

(a) $\lambda_j \geq 0, u_j > 0$ *in* D;

(b) $F(\lambda_j, u_j) = 0$;

(c) $(\lambda_j, u_j) \to (\lambda, 0)$ *in* $\mathbb{R} \times C^2(\overline{D})$.

Then we have $\lambda = \lambda_1$.

Proof. First we remark that the equation $F(\lambda_j, u_j) = 0$ is equivalent to the operator equation

$$u_j = K(\lambda_j u_j - hu_j^p) \quad \text{in } C(\overline{D}). \tag{5.6}$$

If we let

$$v_j = \frac{u_j}{\|u_j\|},$$

then we have

$$\begin{cases} v_j \geq 0 \quad \text{on } \overline{D}, \\ \|v_j\| = 1, \end{cases}$$

and also by equation (5.6)

$$\begin{aligned} v_j - \lambda K v_j &= \frac{u_j}{\|u_j\|} - \lambda \frac{K u_j}{\|u_j\|} \\ &= \frac{1}{\|u_j\|}(u_j - \lambda_j K u_j) + \frac{1}{\|u_j\|}(\lambda_j - \lambda) K u_j \\ &= -\frac{1}{\|u_j\|} K(h u_j^p) + \frac{1}{\|u_j\|}(\lambda_j - \lambda) K u_j. \end{aligned}$$

Hence it follows from condition (c) that

$$\|v_j - \lambda K v_j\| \leq \max_{x \in \overline{D}} h(x) \|K\| \; \|u_j\|^{p-1} + |\lambda_j - \lambda| \; \|K\| \to 0.$$

This implies that

$$0 \in \overline{(I - \lambda K) S^+}, \tag{5.7}$$

where S^+ is the closed unit semi-sphere in $C(\overline{D})$ defined by the formula

$$S^+ = \{u \in C(\overline{D}) : u \geq 0 \quad \text{on } \overline{D}, \; \|u\| = 1\}.$$

However it is easy to see that the set $(I - \lambda K) S^+$ is closed in the space $C(\overline{D})$. Indeed, if $\{u_j\}$ is a sequence in S^+ such that $(I - \lambda K) u_j \to v$ in $C(\overline{D})$ as $j \to \infty$, then, by the compactness of K, one may assume that the sequence $\lambda K u_j$ converges to some function ω in $C(\overline{D})$. Hence we have

$$\begin{cases} u_j \in S^+, \\ u_j = (I - \lambda K) u_j + \lambda K u_j \longrightarrow v + \omega \quad \text{in } C(\overline{D}). \end{cases}$$

If we let

$$u = v + \omega \in S^+,$$

then it follows from the continuity of K that

$$\omega = \lim_{j \to \infty} \lambda K u_j = \lambda K u,$$

so that

$$v = u - \omega = (I - \lambda K) u \in (I - \lambda K) S^+.$$

This proves the closedness of $(I - \lambda K) S^+$.

Thus we obtain from assertion (5.7) that there exists a function $v_1 \in S^+$ such that $v_1 = \lambda K v_1$, that is,

$$\begin{cases} v_1 \geq 0 \quad \text{on } \overline{D}, \\ \|v_1\| = 1, \\ K v_1 = \frac{1}{\lambda} v_1. \end{cases}$$

In view of Theorem 1.7, this implies that

$$\frac{1}{\lambda} = \mu_1 = \frac{1}{\lambda_1}. \qquad \blacksquare$$

By virtue of inequality (5.3) and Claim 2, it is easy to see that

$$\lambda_1 < \lambda_1(D) < \overline{\lambda}(h).$$

(3) Summing up, we have proved that problem (0.5) has a positive solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ for every $\lambda_1 < \lambda < \tilde{\lambda}_1(D_0(h))$. This proves the desired inequality (5.5). $\blacksquare$

The proof of Theorem 6 is now complete. $\blacksquare$

Appendix: The Maximum Principle

Let D be a bounded domain of Euclidean space $\mathbb{R}^N$, with boundary ∂D, and let A be a second-order *elliptic* differential operator with real coefficients such that

$$Au(x) = -\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x),$$

where:

(1) $a^{ij} \in C(\mathbb{R}^N), a^{ij}(x) = a^{ji}(x)$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \mathbb{R}^N, \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N.$$

(2) $b^i \in C(\mathbb{R}^N), 1 \leq i \leq N$.

(3) $c \in C(\mathbb{R}^N)$ and $c(x) \geq 0$ in D.

First we have the following result:

Theorem A.1 (The weak maximum principle). *Assume that a function $u \in C(\overline{D}) \cap C^2(D)$ satisfies one of the conditions*

(a) $$Au \geq 0 \ \textit{ and } \ c > 0 \quad \textit{in } D;$$

(b) $$Au > 0 \ \textit{ and } \ c \geq 0 \quad \textit{in } D.$$

Then the function u may take its negative *minimum only on the boundary ∂D.*

Secondly we have the following (cf. [15, Chapter 2, Section 3, Theorem 6]; [20, Theorem 7.2.1]):

Theorem A.2 (The strong maximum principle). *Assume that a function $u \in C(\overline{D}) \cap C^2(D)$ satisfies the condition*

$$Au \geq 0 \quad \textit{in } D.$$

Then, if the function u attains its non-positive minimum at an interior *point of D, then it is constant.*

Now assume that D is a domain of class C^2, that is, each point of the boundary ∂D has a neighborhood in which ∂D is the graph of a C^2 function of $N-1$ of the variables $x_1, x_2, \ldots, x_N$. We consider a function $u \in C(\overline{D}) \cap C^2(D)$ which satisfies the condition

$$Au \geq 0 \ \text{ in } \ D,$$

and study the conormal derivative $\partial u/\partial \boldsymbol{\nu}$ at a point where the function u takes its non-positive minimum.

The boundaly point lemma reads as follows:

Lemma A.3 (The boundary point lemma). *Let D be a domain of class C^2. Assume that a function $u \in C(\overline{D}) \cap C^2(D)$ satisfies the condition*

$$Au \geq 0 \ \textit{ in } \ D,$$

and that there exists a point x_0' of the boundary ∂D such that

$$\begin{cases} u(x_0') = \min_{x \in \overline{D}} u(x) \leq 0, \\ u(x) > u(x_0'), \ x \in D. \end{cases}$$

Then the conormal derivative $\partial u/\partial \boldsymbol{\nu}(x_0')$ of u at x_0', if it exists, satisfies the condition

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') < 0.$$

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