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BOUNDEDNESS STABILITY PROPERTIES OF LINEAR AND AFFINE OPERATORS

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Abstract. Let E be a vector space in which some notion of boundedness is defined. Then $T : E \to E$ is said to have the boundedness stability property (BSP) if for each $x \in E$, the sequence $(T^n_x)_{n=1}^{\infty}$ is bounded whenever a subsequence $(T^{n_i}x)_{i=1}^{\infty}$ is bounded. It is shown that (1) every affine operator on a finite-dimensional Banach space has the (BSP); (2) every affine operator on an infinite-dimensional vector space has the functional (BSP); (3) when E is an infinite-dimensional Banach space, an affine operator T on E has the (BSP) if its linear part $A_T = T - T(0)$ is a compact perturbation of a bounded linear operator with spectral radius less than one and (4) when E is a Hilbert space, every normal or subnormal bounded linear operator has the (BSP). Some results on affine operators on a Hilbert space whose linear parts are normal or subnormal are also obtained. Finally, some problems are posed.

1. Introduction

Let X be a vector space over the field Φ which is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Then a map $T : X \to X$ is an *affine operator* if the operator $A_T : X \to X$, defined by $A_T x = Tx - T0$ for all $x \in X$, is linear. (A_T is called the linear part of T.) Let $T : X \to X$ and $x_0 \in X$. Then x_0 is a *fixed point* of T if $T(x_0) = x_0$. Suppose some notion of boundedness is defined in the vector space X. Then $T : X \to X$ is said to have the *boundedness stability property* (BSP) if for each $x \in X$, the sequence $(T^n(x))_{n=1}^{\infty}$ is bounded whenever a

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subsequence $(T^{n_i}(x))_{i=1}^{\infty}$ is bounded. It is clear from the definition that if X is a Banach space, $T, S : X \to X$ are bounded linear operators such that T has the (BSP) (where bounded means bounded in norm) and S is similar to T, then S also has the (BSP). Also, it is easy to see that if X is a Banach space, $T : X \to X$ is an affine operator such that $\|A_T\| \leq 1$ with $1 \notin \sigma(A_T)$ (the spectrum of A_T), then T has the (BSP) (where bounded means bounded in norm).

In this paper, we first show that if X is a finite-dimensional Banach space, then every (linear or) affine operator on X has the (BSP) (where bounded means bounded in norm). Next, as an application, when X is any (infinite-dimensional) vector space, it is shown that every (linear or) affine operator on X has the (BSP) (where bounded means functionally bounded as defined below). When X is an infinite-dimensional Banach space, it is also shown that an affine operator T on X has the (BSP) if its linear part A_T is either compact, a compact perturbation of a strict contraction, a quasi-nilpotent operator, a Riesz operator or a compact perturbation of operators with Spectral radii less than 1. Moreover, it is proved that every normal or subnormal bounded linear operator on a Hilbert space also has the (BSP). Some results on affine operators whose linear parts are normal or subnormal bounded linear operators on a Hilbert space are also given. Finally, some remarks are made and some problems are posed.

2. The Finite-Dimensional Case

We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $N = [a_{ij}]$ be an $m \times m$ matrix. Then N is a *Jordan cell* if $a_{ij} = 1$ whenever $j = i + 1$ for $i = 1, \cdots, m-1$ and $a_{ij} = 0$ otherwise. If $x = [x_1, \cdots, x_m]$ is a (row) vector, we shall denote by x^t the *transpose* of x; i.e. the (column) vector

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

Lemma 1. *Let I be the $m \times m$ identity matrix, N be the $m \times m$ Jordan cell, $\lambda \in \mathbb{C}$ and $A = \lambda I + N$. If for some $m \times 1$ vector x_0, a subsequence $(A^{n_i}x_0)_{i=1}^{\infty}$ of $(A^n x_0)_{n=1}^{\infty}$ is bounded, then $(A^n x_0)_{n=1}^{\infty}$ is itself bounded.*

Proof. The assertion is clearly true if $x_0 = 0$. Now assume $x_0 \neq 0$. Let $x_0 = [x_1, \cdots, x_m]^t$ and let $m_0 = \max\{i \in \{1, \cdots, m\} : x_i \neq 0\}$. Note that for each $p = 0, 1, 2, \cdots$,

$$\begin{aligned} A^{m+p} &= (\lambda I + N)^{m+p} \\ &= \sum_{j=0}^{m-1} \binom{m+p}{j} \lambda^{m+p-j} N^j . \end{aligned}$$

It follows that for each $p = 0, 1, 2, \cdots$,

$$A^{m+p} x_0 = \sum_{j=0}^{m-1} \binom{m+p}{j} \lambda^{m+p-j} N^j x_0$$

$$= \begin{bmatrix} \sum_{j=0}^{m_0-1} \binom{m+p}{j} \lambda^{m+p-j} x_{j+1} \\ \vdots \\ \sum_{j=0}^{m_0-k} \binom{m+p}{j} \lambda^{m+p-j} x_{j+k} \\ \vdots \\ \lambda^{m+p} x_{m_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a\\a\\a\\a\\a\\a \end{matrix}}\right\} (m_0 \ rows) \\ \left.\vphantom{\begin{matrix} a\\a\\a \end{matrix}}\right\} (m - m_0 \ rows) \end{matrix} .$$

Let $n_i = m + p_i$ for all $i \leq i_0$. Since $(A^{n_i} x_0)_{i=1}^{\infty}$ is bounded, $(\lambda^{m+p_i} x_{m_0})_{i=1}^{\infty}$ is bounded; as $x_{m_0} \neq 0$, we must have $|\lambda| \leq 1$.

Case 1. Suppose $m_0 = 1$. Then for each $p = 0, 1, 2, \cdots, A^{m=p} x_0 = [\lambda^{m+p} x_1, 0, \cdots, 0]^t$ so that $(A^n x_0)_{n=1}^{\infty}$ is bounded as $|\lambda| \leq 1$.

Case 2. Suppose $m_0 \geq 2$. Since $\left(\lambda^{m+p_i} x_{m_0-1} + \lambda^{m+p_i-1} \binom{m+p_i}{1} x_{m_0}\right)_{i=1}^{\infty}$ is bounded and $x_{m_0} \neq 0$, we must have $|\lambda| < 1$. Let $\alpha = \max_{1 \leq i \leq m_0} |x_i|$. Then for each $k = 1, \cdots, m_0$,

$$\begin{aligned} &\left| \sum_{j=0}^{m_0-k} \binom{m+p}{j} \lambda^{m+p-j} x_{j+k} \right| \\ &\leq \sum_{j=0}^{m_o-k} \binom{m+p}{j} |\lambda|^{m+p-j} \alpha \to 0 \ as \ \ p \to \infty, \end{aligned}$$

which shows that $A^n x_0 \to 0$ as $n \to \infty$ so that $(A^n x_0)_{n=1}^{\infty}$ is bounded. $\square$

Let $(X, \|\cdot\|)$ be a real Banach space. Let $X_C = X \times X$. If (x_1, y_1), $(x_2, y_2) \in X_C$, define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and if $z = (x, y) \in X_C$ and $\alpha = a + ib$ where $a, b, \in \mathbb{R}$ define $az = (ax - by, bx + ay)$. Then X_C is a complex vector space. Define

$$|(x,y)| = \|x\| + \|y\|,$$
$$|||(x,y)||| = \frac{1}{\sqrt{2}} \sup\{|e^{i\theta}(x,y)| : \theta \in [0, 2\pi]\}$$

for each $(x, y) \in X_C$, then $(X_C, |||\cdot|||)$ is a complex Banach space. Clearly $r \longmapsto (x, 0)$ is an isometry from X into X_C. The space $(X_C, |||\cdot|||)$ is called the complexification of $(X, \|\cdot\|)$ (see, e.g. [8]). Now if $A : X \to X$ is a bounded (real) linear operator, define $A_C : X_C \to X_C$. by $A_C(x, y) = (A_x, A_y)$ for all $x, y \in X$. Then A_C is a complex linear operator on X_C such that $|||A_C||| = \|A\|$, $A_C^n(x, 0) = (A^n x, 0)$ and $|||A_C^n(x, 0)||| = \|A^n x\|$ for all $x \in X$ and for all $n \in \mathbb{N}$. The operator A_C is called the complexification of A (see, e.g. [8]). Note that

(a) $|||A_C^n||| = \|A^n\|$ for all $n \in \mathbb{N}$;

(b) A is compact if and only if A_C is compact;

(c) A is finite-rank if and only if A_C is finite-rank;

(d) A is a strict contraction (i.e., $\|A\| < 1$) if and only if A_C is a strict contraction.

Theorem 1. *Let X be a, finite-dimensional Banach space and $A : X \to X$ be linear. Then A has the (BSP).*

Proof. Let dim $X = m$. *Case* 1. If X is a complex Banach space, without loss of generality, we may assume that $X = \mathbb{C}^m$. Furthermore, we may, by the Jordan canonical form, assume that A is of the form $A_1 \oplus \cdots \oplus A_r$, where each A_k is the sum of a scalar matrix and a Jordan cell. Then the assertion of the (BSP) of A follows easily from Lemma 1.

Case 2. Suppose X is a real Banach space. Let $X_C = X \times X$ be the complexification of X and A_C be the complexification of A. Then $x \longmapsto (x, 0)$ is an isometry from X into X_C. Since $(A^{n_i} x_0)_{i=1}^{\infty}$ is bounded in X, the subsequence $(A_C^{n_i}(x_0, 0))_{i=1}^{\infty}$ of $(A_C^n(x_0, 0))_{n=1}^{\infty}$ is also bounded in X_C. Hence by Case 1, the sequence $(A_C^n(x_0, 0))_{n=1}^{\infty}$ is bounded in X_C. Therefore $(A^n x_0)_{n=1}^{\infty}$ is bounded in X. □

The following simple fact can be easily proved by induction; its proof is thus omitted:

Lemma 2. *Let E be a vector space, $A : E \to E$ be a linear operator, $a \in E$ and $T : E \to E$ be the affine operator defined by $T(x) = Ax + a$ for all $x \in E$. If $\eta \in E$ is a fixed point of T, then for each $x \in E$,*

$$T^n(x) - \eta = A^n(x - \eta) \quad for\ all\ n \in \mathbb{N}.$$

Let E be a vector space over the field $\Phi(= \mathbb{C}$ or $\mathbb{R})$ and S be a non-empty subset of E. Then S is said to be functionally bounded in E [4] if for each linear functional f on E, $f(S)$ is bounded in Φ. The following simple result is Lemma 2.4 in [4]:

Lemma 3. *Let E be a vector space and S be a non-empty subset of E. If S is functionally bounded in E, then the linear span of S is finite-dimensional.*

The following result is Theorem 2.2 in [4]:

Lemma 4. *Let E be a vector space and $T : E \to E$ be an affine operator. If T has no fixed point in E, then there exists a linear functional f on E such that for each $x \in E$, $f(T^n(x)) \to \infty$ as $n \to \infty$.*

As an application of Theorem 1, we have:

Theorem 2. *Let E be a vector space and $T : E \to E$ be affine. Then T has the functional (BSP); i.e., for each $x \in E$, if a subsequence $(T^{n_i}(x))_{i=1}^{\infty}$ of $(T^n(x))_{n=1}^{\infty}$ is functionally bounded in E, then $(T^n(x))_{n=1}^{\infty}$ is itself functionally bounded in E.*

Proof. Let $A : E \to E$ be linear and $a \in E$ be such that $T(x) = Ax + a$ for all $x \in E$.

Let $x \in E$ be such that $(T^{n_i}(x))_{i=1}^{\infty}$ is functionally bounded in E. By Lemma 4, T has a fixed point $\eta \in E$. If $x = \eta$, then clearly $(T^n(x))_{n=1}^{\infty}$ is functionally bounded. Thus we may assume that $x \neq \eta$. It follows that $\{T^{n_i}(x) - \eta : i \in \mathbb{N}\}$ is also functionally bounded in E. By Lemma 3, $\{T^{n_i}(x) - \eta : i \in \mathbb{N}\}$ spans a finite-dimensional subspace of E so that $\{T^{n_i}(x) - \eta : i \in \mathbb{N}\}$ is linearly dependent. By Lemma 2, $\{A^{n_i}(x - \eta) : i \in \mathbb{N}\}$ is linearly dependent and hence $\{A^n(x - \eta) : n \in \mathbb{N}\}$ is linearly dependent.

If $z = x - \eta$, then $z \neq 0$. Let p be the smallest positive integer such that $A^p z$ is a linear combination of $\{z, Az, \cdots, A^{p-1}z\}$. Then it is easy to verify that the whole sequence $(A^n z)_{n=1}^{\infty}$ lies in the subspace F of E which is spanned by $\{z, Az, \cdots, A^{p-1}z\}$. Now equip F with the Euclidean topology; then F is isometrically isomorphic to Φ^p (for $\Phi = \mathbb{C}$ or $\mathbb{R}$). Note that $(A^{n_i}z)_{i=1}^{\infty}$ is functionally bounded in E and is hence functionally bounded in F. As every linear functional on F is continuous on F and functional boundedness in F is equivalent to boundedness in F. $(A^{n_i}z)_{i=1}^{\infty}$ is bounded in F. By Theorem 1, $(A^n z)_{n=1}^{\infty}$ is bounded in F and is therefore functionally bounded in E. By Lemma 2 again, $(T^n(x)-\eta)_{n=1}^{\infty}$ is functionally bounded in E. It follows that $(T^n(x))_{n=1}^{\infty}$ is functionally bounded in E. □

Theorem 2 answers the Conjecture in [4] in the affirmative.

Again, since in a finite-dimensional Banach space, functional boundedness is equivalent to boundedness (in norm), we have the following immediate consequence of Theorem 2 extending Theorem 1 from linear operators to affine operators:

Theorem 3. *Let X be a finite-dimensional Banach space and $T : X \to X$ be an affine operator. Then T has the (BSP).*

3. The Infinite-Dimensional Case

Let X be the complex Hilbert space ℓ_2 of all sequences $x = (x_n)_{n=1}^{\infty}$ of complex numbers with $\|x\| = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2} < \infty$. Let $\{e_1, e_2, \cdots\}$ be the standard orthonormal basis for X and S be the forward shift operator on X, i.e., the bounded linear operator defined by $Se_i = e_{i+1}$ for all $i \in \mathbb{N}$. Let S^* be the (Hilbert space) adjoint of S (the backward shift). In [9], it is shown that for each $\alpha > 1$, the operator $A = \alpha S^*$ has a dense orbit, i.e., there exists a vector $x_0 \in X$ such that $\{A^n x_0 : n \in \mathbb{N}\}$ is dense in X. It follows that there are bounded subsequences of $(A^n x_0)_{n=1}^{\infty}$ while the whole sequence $(A^n x_0)_{n=1}^{\infty}$ is not bounded. This shows that Theorem 1 cannot be extended to infinite-dimensional Banach spaces without additional assumptions on A. Also, in [3, Theorem 2.1], it is shown that there exists a continuous affine operator $T : X \to X$ such that the sequence $(T^n 0)_{n=1}^{\infty}$ is unbounded while its subsequence $(T^{n!} 0)_{n=1}^{\infty}$ converges to 0. This shows that Theorem 3 cannot be extended to infinite-dimensional Banach spaces without additional assumptions on T (or on its linear part A_T)

Let $(X, \|\cdot\|)$ be a Banach space and $A : X \to X$ be a bounded linear operator. Then the spectral radius of A, denoted by $r(A)$, is defined as $\lim_{n\to\infty} \|A^n\|^{1/n}$. Note that if $(X, \|\cdot\|)$ is a complex Banach space, then by Gel'fand's theorem, $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A.

Lemma 5. *Let X be a complex Banach space and $A : X \to X$ be a bounded linear operator. Suppose $\sigma(A) = \sigma_1 \cup \sigma_2 \cup \sigma_3$ where σ_1, σ_2 and σ_3 are all closed, σ_1 is in the interior of the unit circle, σ_2 is on the unit circle* *and σ_3 is in the exterior of the unit circle. Let $X = X_1 \oplus X_2 \oplus X_3$ be the Riesz decomposition of X [7] with corresponding $A = A_1 \oplus A_2 \oplus A_3$ such that $\sigma_i = \sigma(A_i), i = 1, 2, 3$. If $x_0 = x_1 \oplus x_2 \oplus x_3$ is any vector in X with $x_i \in X_i$ for $i = 1, 2, 3$, then $(A^{n_i} x_0)_{i=1}^{\infty}$ is bounded in X if and only if $x_3 = 0$ and $(A_2^{n_i} x_2)_{i=1}^{\infty}$ is bounded in X_2.*

Proof. Since $r(A_1)$, the spectral radius of A_1, is strictly less than 1, $A_1^n \to 0$ as $n \to \infty$. Since $r(A_3) > 1$ (so that $r(A_3^{-1}) < 1$), $(A_3^{n_i} x_3)_{i=1}^{\infty}$ is bounded if and only if $x_3 = 0$. Hence $(A^{n_i} x_0)_{i=1}^{\infty}$ is bounded if and only if $x_3 = 0$ and $(A_2^{n_i} x_2)_{i=1}^{\infty}$ is bounded. □

Lemma 6. *Let X be a complex Banach space and $T : X \to X$ be affine. Suppose $A = A_T$ satisfies the same hypotheses as in Lemma 5. Assume X_2 is finite-dimensional. Then T has the (BSP).*

Proof. Let $a = T(0) = a_1 \oplus a_2 \oplus a_3$ and let $x_0 = x_1 \oplus x_2 \oplus x_3$, where $x_i, a_i \in X_i$ for $i = 1, 2, 3$, be such that a subsequence $(T^{n_i}(x_0))_{i=1}^{\infty}$ of $(T^n(x_0))_{n=1}^{\infty}$ is bounded. Let $T = T_1 \oplus T_2 \oplus T_3$ corresponding to $A = A_1 \oplus A_2 \oplus A_3$; then $T_j(x_j) = A_j x_j + a_j$ for $j = 1, 2, 3$. Since $1 \notin \sigma(A_1) \cup \sigma(A_3)$, we can define $\eta_j = (I_j - A_j)^{-1} a_j$ for $j = 1, 3$. Then $T_j(\eta_j) = \eta_j$ so that η_j is a fixed point of T_j, $j = 1, 3$. Since $(T_j^{n_i}(x_j))_{i=1}^{\infty}$ is bounded, by Lemma 2, $(A_j^{n_i}(x_j - \eta_j))_{i=1}^{\infty}$ is bounded for $j = 1, 3$. By Lemma 5, $x_3 = \eta_3$ so that $(A_3^n(x_3 - \eta_3))_{n=1}^{\infty}$ is trivially bounded. Since $r(A_1) < 1$, $(A_1^n(x_1 - \eta_1))_{n=1}^{\infty}$ is clearly bounded. By Lemma 2 again, $(T_j^n(x_j) - \eta_j)_{n=1}^{\infty}$ is also bounded for $j = 1, 3$. Since X_2, is finite-dimensional, $(T_2^n(x_2))_{n=1}^{\infty}$ is bounded by Theorem 3. Therefore $(T^n(x_0))_{n=1}^{\infty}$ is bounded. □

Recall that a bounded linear operator A on a Banach space X is quasi-nilpotent if $r(A) = 0$. In the case when the Banach space X is complex, this is equivalent to the condition $\sigma(A) = \{0\}$. If $(X, \|\cdot\|)$ is a real Banach space and A_C is the complexification of A, then $r(A_C) = r(A)$ so that A is quasi-nilpotent if and only if A_C is quasi-nilpotent. A Riesz operator A on a complex Banach space is a bounded linear operator for which the non-zero elements in $\sigma(A)$ behave like those for compact operators. More precisely, a bounded linear operator A is a Riesz operator if and only if for each $\lambda \neq 0$, $N(A - \lambda I)$ (the null space of $A - \lambda I$, where I is the identity operator) and $R(A - \lambda I)$ (the range space of $A - \lambda I$) have finite dimension and finite codimension respectively (see [2]). It follows that if $\epsilon > 0$ is given, then $A = S + K$ where S and K are bounded linear operators such that $r(S) < \epsilon$ and K is of finite rank.

Theorem 4. *Let X be a Banach space and $T : X \to X$ be an affine operator. If $A = A_T$ is a compact perturbation of an operator with spectral radius less than 1, then T has the (BSP).*

Proof. Case 1. Suppose X is a complex Banach space. Suppose $A = S + K$, where $S, K : X \to X$ are bounded linear operators such that $r(S) < 1$ and K is compact. Let $\eta(S)$ be the polynomially convex hull of $\sigma(S)$; i.e., the smallest set containing $\sigma(S)$ with a connected complement). Then by Theorem 5.7.4 in [1], $\sigma(A)\backslash\eta(S)$ is contained in the set of isolated eigenvalues λ with finite-rank associated riesz operator (see Theorem 3.3.4 in [1]) and the accumulation points of $\sigma(A)\backslash\eta(S)$ are in $\eta(S)$. (Here, I is again the identity operator on X.) Since $\sigma(S)$ is in the interior of the unit circle, the intersection of the unit circle with $\sigma(A)$ has the properties of σ_2) in Lemma 6. The conclusion follows from Lemma 6.

Case 2. Suppose X is a real Banach space. Let $A = S + K$, where $S, K : X \to X$ are bounded linear operators such that $r(S) < 1$ and K is compact. Let $X_C = X \times X$ be the complexification of X and A_C, S_C and K_C be the complexifications of A, S and K, respectively. Then X_C is a complex Banach space, $r(S_C) < 1$, K_C is compact and $A_C = S_C + K_C$. Define $T_C : X_C \to X_C$ by $T_C(x, y) = A_C(x, y) + (a, 0)$ for all $(x, y) \in X_C$. Then by Case 1, T_C has the (BSP) so that T also has the (BSP). $\square$

By the remark just preceding Theorem 4, a Riesz operator is a special case of a compact perturbation of an operator with spectral radius less than 1. Also strict contractions and quasinilpotent operators have spectral radii less than 1. Thus Theorem 4 implies the following

Corollary 1. *Let X be a Banach space and $T : X \to X$ be an affine operator. Assume that $A = A_T$ is a bounded linear operator in any of the following classes:*

(1) *compact operators,*

(2) *compact perturbations of strict contractions,*

(3) *quasi-nilpotent operators,*

(4) *Riesz operators.*

Then T has the (BSP).

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Let $H_C = H \times H$. If $(x_1, y_2), (x_2, y_2) \in H_C$, define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and if $z = (x, y) \in H_C$ and $\alpha = a + ib$ where $a, b \in \mathbb{R}$, define $\alpha z = (ax - by, bx + ay)$. Then H_C is a complex vector space. Define $[\cdot, \cdot] : H_C \times H_C \to \mathbb{C}$ by

$$[(x_1, y_1), (x_2, y_2)] = \langle x_1, x_2 \rangle + i\langle y_1, x_2 \rangle - i\langle x_1, y_2 \rangle + \langle y_1, y_2 \rangle,$$

for each $(x_1, y_1), (x_2, y_2) \in H_C$, then $[\cdot,\cdot]$ is an inner product on H_C. If $|||(x,y)||| = ([(x,y),(x,y)])^{1/2}$, then $|||(x,y)||| = (\|x\|^2 + \|y\|^2)^{1/2}$ for all $x, y \in H$. Hence H_C is a Hilbert space. Clearly $x \longmapsto (x, 0)$ is an isometry frorm H into H_C. The space H_C is called the complexification of H (see e.g., [6]). Now if $A : H \to H$ is a bounded (real) linear operator, define $A_C : H_C \to H_C$ by $A_C(x,y) = (Ax, Ay)$ for all $x, y \in H$. Then A_C is a complex linear operator on H_C such that $|||A_C||| = \|A\|$, $A_C^n(x,0) = (A^n x, 0)$ and $|||A_C^n(x,0)||| = \|A^n x\|$ for all $x \in H$ and for all $n \in \mathbb{N}$. The operator A_C is called the complexification of A. Note that

(a) A is normal if and only if A_C is normal.
(b) $\sigma(A_C) \cap \mathbb{R} = \sigma(A)$.
(c) A is subnorma1 if and only if A_C is subnormal.

When A is a bounded normal or subnormal linear operator on a Hilbert space, Lemma 6 can be improved as follows:

Theorem 5. *Let H be a Hilbert space and $A : H \to H$ be a bounded normal linear operator. Then A has the (BSP).*

Proof. Case 1. Suppose H is a complex Hilbert space. Since A is normal, by the spectral theorem for normal operators (see, e.g. [7]), there exists a finite Borel measure space (X, μ) and a, bounded measurable function ϕ on X such that A is unitarily equivalent to M_ϕ on $\mathcal{L}^2(X,\mu)$ defined by $M_\phi f = \phi f$ for all $f \in \mathcal{L}^2(X,\mu)$. It is sufficient to show that for $f_0 \in \mathcal{L}^2(X,\mu)$, if $(\int_X |\phi^{n_i} f_0|^2 d\mu)_{i=1}^\infty$ is bounded, then $(\int_X |\phi^n f_0|^2 d\mu)_{n=1}^\infty$ is bounded. Let $M = \sup\limits_{i\geq 1} \displaystyle\int_X |\phi^{n_i} f_0|^2 d\mu$,

$$X_1 = \phi^{-1}\{z \in \mathbb{C} : |z| \leq 1\} \;\; \textit{and} \;\; X_2 = \phi^{-1}\{z \in \mathbb{C} : |z| > 1\}.$$

For a given $n \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $n_i \leq n \leq n_{i+1}$. It follows that

$$\begin{aligned}
\int_X |\phi^n f_0|^2 d\mu &= \int_{X_1} |\phi^n f_0|^2 d\mu + \int_{X_2} |\phi^n f_0|^2 d\mu \\
&\leq \int_{X_1} |\phi^{n_i} f_0|^2 d\mu + \int_{X_2} |\phi^{n_{i+1}} f_0|^2 d\mu \\
&\leq \int_X |\phi^{n_i} f_0|^2 d\mu + \int_X |\phi^{n_{i+1}} f_0|^2 d\mu \\
&\leq 2M.
\end{aligned}$$

Therefore $(\int_X |\phi^n f_0|^2 d\mu)_{n=1}^\infty$ is bounded.

Case 2. Suppose H is a real Hilbert space. Let $H_C = H = H \times H$ be the complexification of H and A_C be the complexification of A. Then H_C is a complex Hilbert space and A_C is a bounded norma1 linear operator on H_C such that $A_C(x, y) = (Ax, Ay)$ and $|||(x, y)|||^2 = \|x\|^2 + \|y\|^2$ for all $x, y \in H$. Since $(A^{n_i}x_0)_{i=1}^{\infty}$ is bounded in H, $(A_C^{n_i}(x_0, 0))_{i=1}^{\infty}$ is bounded in H_C. Hence by Case 1, $(A_C^{n}(x_0, 0))_{n=1}^{\infty}$ is bounded in H_C. Therefore $(A^{n}x_0)_{n=1}^{\infty}$ is bounded in H. □

Corollay 2. *Let H be a Hilbert space and $A : H \to H$ be a bounded subnormal linear operator. Then A has the (BSP).*

Proof. By definition, there exist a Hilbert space $\widehat{H} \supset H$ and a normal operator $\widehat{A}$ on $\widehat{H}$ such that $\widehat{A}H \subset H$ and $A = \widehat{A}|H$. If $x_0 \in H$, then $A^m x_0 = \widehat{A}^m x_0$ for all $m = 1, 2, \cdots$. Thus the conclusion follows from Theorem 5. □

Theorem 6. *Let H be a Hilbert space and $T : H \to H$ be affine. If $A = A_T$ is a bounded normal linear operator such that $1 \notin \sigma(A)$, then T has the (BSP).*

Proof. Since $1 \notin \sigma(A)$, we may take $\eta = (I - A)^{-1}a$, where I is the identity operator on H and $a = T(0)$; then η is a fixed point of T. Suppose $x \in H$ such that $(T^{n_i}(x))_{i=1}^{\infty}$ is bounded; by Lemn1a 2, $(A^{n_i}(x - \eta))_{i=1}^{\infty}$ is also bounded. By Theorem 5, $(A^{n}(x - \eta))_{n=1}^{\infty}$ is bounded . By Lemma 2 again, $(T^{n}(x))_{n=1}^{\infty}$ is therefore bounded. □

Corollary 3. *Let H be a Hilbert space and $T : H \to H$ be affine. If $A = A_T$ is a bounded subnormal linear operator such that $1 \notin \sigma(A)$, then T has the (BSP).*

Proof. Case 1. Suppose H is a complex Hilbert space. Let $\widehat{A}$ be the minimal normal extension of A, then $\sigma(\widehat{A}) \subset \sigma(A)$ (see, e.g. [5, Problem 200]). Thus $1 \notin \sigma(A)$ and the conclusion now follows from Theorem 6.

Case 2. Suppose H is a real Hilbert space. Let H_C be the complexification of H and A_C be the complexification of A. Then A_C is a bounded subnormal linear operator on H_C. Since $1 \notin \sigma(A) = \sigma(A_C) \cap \mathbb{R}$, we have $1 \notin \sigma(A_C)$. Define $T_C : H_C \to H_C$ by $T_C(x, y) = A_C(x, y) + (a, 0)$ for all $(x, y) \in H_C$. Suppose $x \in H$ such that $(T^{n_i}(x))_{i=1}^{\infty}$ is bounded; then $(T_C^{n_i}(x, 0))_{i=1}^{\infty}$ is also bounded so that $(T_C^{n}(x, 0))_{n=1}^{\infty}$ is bounded by Case 1. Therefore $(T^{n}(x))_{n=1}^{\infty}$ is bounded. □

Theorem 7. *Let H be a complex Hilbert space and $T : H \to H$ be an affine operator. If $A = A_T$ is a bounded normal linear operator such that 1 is an isolated point in $\sigma(A)$, then T has the (BSP).*

Proof. Since 1 is an isolated point in $\sigma(A)$, it is an eigenvalue of A (this fact can be proved by applying the spectral theorem for normal operators which is used in proving Theorem 5; see e.g. [7]). Let $H_1 = N(A - I)$ and $H_2 = H_1^{\perp}$, where I is the identity operator on H. Then $A = I \oplus A_2$ and $T = T_1 \oplus T_2$ corresponding to the decomposition $H = H_1 \oplus H_2$, where $1 \notin \sigma(A_2)$.

Let $a = T(0) = a_1 \oplus a_2$ and let $x = x_1 \oplus x_2$, where $a_i, x_i \in H_i$ for $i = 1, 2$, be such that $(T^{n_i}(x))_{i=1}^{\infty}$ is bounded. The assumption that $(T^{n_i}(x))_{i=1}^{\infty}$ is bounded implies that $a_1 = 0$ and $(T_2^{n_i}(x_2))_{i=1}^{\infty}$ is bounded. By Theorem 6, $(T_2^{n}(x_2))_{n=1}^{\infty}$ is bounded. Therefore $(T^n(x))_{n=1}^{\infty}$ is also bounded. □

Corollary 4. *Let H be a complex Hilbert space and $T : H \to H$ be an affine operator. If $A = A_T$ is a bounded subnormal linear operator such that 1 is an isolated point in $\sigma(A)$, then T has the (BSP).*

Proof. Let $\widehat{A}$ be the minimal normal extension of A. Since $\sigma(\widehat{A}) \subset \sigma(A)$, 1 has to be an isolated point of $\sigma(A)$ if it belongs to the set. (It actually does, because $\sigma(A)\backslash\sigma(\widehat{A})$ has the property that each bounded component of the complement of $\sigma(\widehat{A})$ is either entirely contained in it or is disjoint from it, see e.g. [5, Problem 201]). The conclusion now follows from Theorem 7. □

We remark that if H is a real Hilbert space, the conclusions of Theorem 7 and Corollary 4 remain valid if the condition "1 is an isolated point in $\sigma(A)$" is replaced by "1 is an isolated point in $\sigma(A_C)$, where A_C is the complexification of A".

Finally, we shall provide an example showing that the conclusion of Theorem 7 may be false if the given Hilbert space is real instead of complex.

Example 1. Let X be the complex Hilbert space ℓ_2 of all sequences $x = (x_n)_{n=1}\infty$ of complex numbers with $\|x\| = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2} < \infty$. Define $A : X \to X$ by

$$Ax = (e^{2\pi i/n!} x_n)_{n=1}^{\infty} \quad for\ each \quad x = (x_n)_{n=1}^{\infty} \in X.$$

Let $a = (a_n)_{n=1}^{\infty}$, where $a_n = 1 - e^{2\pi i/n!}$ for each $n \geq 1$; then $a \in X$. Define $T : X \to X$ by $T(x) = Ax + a$ for all $x \in X$. Then as shown in [3]:

(a) A is linear and unitary;

(b) the sequence $(T^k(0))_{k=1}^{\infty}$ is unbounded but contains a subsequence $(T^{k!}(0))_{k=1}^{\infty}$ which converges to 0;

(c) $\sigma(A) = \{e^{2\pi i/n!} : n \in \mathbb{N}\} \cup \{1\}$.

Let A^* be the adjoint of A and $\overline{a} = (\overline{a}_n)_{n=1}^{\infty}$, where $\overline{a}_n$ denotes the complex conjugate of a_n. Let $H = X \times X$ and

$$B = \begin{bmatrix} A & 0 \\ 0 & A^* \end{bmatrix} \quad \textit{and} \quad b = \begin{bmatrix} a \\ \overline{a} \end{bmatrix}.$$

Define $S : H \to H$ by $S(x) = Bx + b$ for all $x \in H$. Then

(a)' B is linear and unitary;

(b)' the sequence $(S^k(0))_{k=1}^{\infty}$ is unbounded but contains a subsequence $(S^{k!}(0))_{k=1}^{\infty}$ which converges to 0 (by observing that A is diagonal relative to the standard basis and so is A^*).

(c)' $\sigma(B) = \sigma(A) \cup \sigma(A^*)$.

Let I be the identity operator on X and define

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}.$$

Then M is unitary and

$$M^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -iI \\ I & iI \end{bmatrix}.$$

Define $B_0, T_0 : H \to H$ and $b_0 \in H$ by $B_0 = MBM^{-1}$, $T_0(x) = B_0 x + b_0$ for all $x \in H$ and $b_0 = Mb$. Then

(a)" B_0 is linear and unitary;

(b)" the sequence $(T_0^k(0))_{k=1}^{i} nfty$ is unbounded but contains a subsequence $(T_0^{k!}(0))_{k=1}^{\infty}$ which converges to 0.

(c)" $\sigma(B_0) \cap \mathbb{R} = \{1\}$.

However,

$$B_0 = \frac{1}{2} \begin{bmatrix} A + A^* & -iA + iA^* \\ iA - iA^* & A + A^* \end{bmatrix} \quad \textit{and} \quad B_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} a + \overline{a} \\ i(a - \overline{a}) \end{bmatrix},$$

which show that both B_0 and b_0 are real. We can now consider H as a real Hilbert space. Then $T_0 : H \to H$ is affine such that

(a)" ' its linear part is B_0 which is unitary and hence normal;

(b)" ' 1 is isolated in $\sigma(B_0)$;

(c)" ' T_0 does not have the (BSP) by (b)".

This shows the hypothesis that the Hilbert space H is complex in Theorem 7 is essential.

4. Remarks

In this section, we shall make some observations and pose some problems suggesting that further work may be fruitful.

First we note the following:

(1) Suppose $(X, \|\cdot\|)$ is a Banach space, $T, S : X \to X$ are affine and $R : X \to X$ is bounded linear and invertible. If T has the (BSP), $A_S = RA_TR^{-1}$ and $S0 = RT0$, then S also has the (BSP).

(2) Suppose H is a Hilbert space and $T : H \to H$ is bounded linear with the (BSP). Then its adjoint T^* may not have the (BSP). For example, if S is the forward shift operator as defined in the first paragraph of the preceding section, then $2S$ is subnormal (in fact, S is an isometry, so it can be extended to a unitary operator). Thus $2S$ has the (BSP), but, as remarked before, $2S^*$ fails to have the (BSP).

Let E be a vector space over the field $\Phi(= \mathbb{C}$ or $\mathbb{R})$ and B be a non-empty subset of E. Denote by $ac(B)$ the absolutely convex hull of B; i.e., $ac(B) = \{\sum_{j=1}^m \lambda_j x_j : \lambda_j \in \Phi$ and $x_j \in B$ for all $j = 1, \cdots, m$ with $\sum_{j=1}^m |\lambda_j| \leq 1,\ m \in \mathbb{N}\}$. The set B is said to be *linearly bounded* in E if for each $x \in E$ with $x \neq 0$, the set $\{\lambda \in \Phi : \lambda x \in ac(B)\}$ is bounded in Φ. It is easy to see that

(a) B is functionally bounded in E if and only if $ac(B)$ is functionally bounded in E;

(b) If B is functionally bounded in E, then B is linearly bounded in E;

(c) If B is linearly bounded in E, B need not be functionally bounded in E;

(d) If E is finite-dimensional, then B is functionally bounded in E if and only if B is linearly bounded in E.

Problem 1. *Let E be a vector space and $T : E \to E$ be affine.* (1) *Does T have the linear (BSP); i.e., for any $x \in E$, if a subsequence $(T^{n_i}(x))_{i=1}^\infty$ of $(T^n(x))_{n=1}^\infty$ is linearly bounded in E, then is $(T^n(x))_{n=1}^\infty$ itself linearly bounded in E?* (2) *If for some $x_0 \in E$, a subsequence $(T^{n_i}(x_0))_{i=1}^\infty$ of $(T^n(x_0))_{n=1}^\infty$ is linearly bounded in E, does T have a fixed point in E?*

Let E be a vector space over the field $\Phi(= \mathbb{C}$ or $\mathbb{R})$ and B be a non-empty subset of E. The set B is said to be *radially bounded* in E if for each $x \in E$ with $x \neq 0$, the set $\lambda \in \Phi : \lambda x \in B\}$ is bounded in Φ. It is easy to see that

(a) If B is functionally bounded in E, then B is radially bounded in E,.

(b) If B is radially bounded in E, B need not be functionally bounded in E.

Problem 2. *Let E be a vector space and $T : E \to E$ be affine.* (1) *Does T have the radial (BSP); i.e., for any $x \in E$, if a subsequence $(T^{n_i}(x))_{i=1}^{\infty}$ of $(T^n(x))_{n=1}^{\infty}$ is radially bounded in E, then is $(T^n(x))_{n=1}^{\infty}$ itself radially bounded in E?* (2) *If for some $x_0 \in E$, a subsequence $(T^{n_i}(x_0))_{i=1}^{\infty}$ of $(T^n(x_0))_{n=1}\infty$ is radially bounded in E, does T have a fixed point in E?*

As noted earlier, if $(X, \|\cdot\|)$ is a Banach space, $T, S : X \to X$ are bounded linear operators such that T has the (BSP) and S is similar to T, then S also has the (BSP). However, we pose the following

Problem 3. *Let $(X, \|\cdot\|)$ be a non-reflexive Banach space and $T, S : X^* \to X^*$ bounded linear operators. Suppose T has the w^* (BSP); i.e., for each $x_0 \in X^*$, if a subsequence $(T^{n_i}(x_0))_{i=1}^{\infty}$ of $(T^n(x_0))_{n=1}^{\infty}$ is w^* bounded, then $(T^n(x_0))_{n=1}^{\infty}$ is itself w^* bounded. Suppose S similar to T. Does S also have the w^* (BSP)?*

Last but not least, we pose the following:

Problem 4. *Let $(X, \|\cdot\|)$ be a non-reflexive Banach space and $T : X^* \to X^*$ be a bounded linear operator belonging to any one of the classes* $(1) - (4)$ *in Corollary* 1. *Does T have the w^* (BSP)?*

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