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ON CERTAIN MEROMORPHIC P-VALENT FUNCTIONS

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Abstract. A certain differential operator D^n is introduced for functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in $E^* = \{z : 0 < |z| < 1\}$. The object of the present paper is to give an application of the above operator D^n to the differential inequalities.

1. Introduction

Let $\sum(p)$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \cdots\})$$

which are analytic in $E^* = \{z : 0 < |z| < 1\}$. Define

$$D^0 f(z) = f(z);$$

$$D^1 f(z) = \frac{1}{z^p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + \cdots;$$

$$D^2 f(z) = D(D^1 f(z)),$$

and for $n = 1, 2, \cdots$

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1}.$$

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Recently Uralegaddi and Somanatha [1] and Aouf and Hossen [2] have studied certain class of meromorphic multivalent functions defined by the operator $D^n f(z)$. The object of the present paper is to investigate some new properties of meromorphic p-valent functions defined by the above operator.

Definition. Let H be the set of complex valued functions $h(r,s,t):\mathbb{C}^3\to\mathbb{C}$($\mathbb{C}$ is the complex plane) such that

$$h(r,s,t,)\text{ is continuous in a domain } D\subset\mathbb{C}^3; \tag{1.1}$$

$$(1,1,1)\in D \text{ and } |h(1,1,1)|<1; \tag{1.2}$$

$$\left|h(e^{i\theta},m+e^{i\theta},\frac{m+L+3me^{i\theta}+e^{2i\theta}}{m+e^{i\theta}})\right|\geq 1, \tag{1.3}$$

whenever

$$(e^{i\theta},m+e^{i\theta},\frac{m+L+3me^{i\theta}+e^{2i\theta}}{m+e^{i\theta}})\in D$$

with $\mathrm{Re}L\geq m(m-1)$ for real θ and for real $m\geq 1$.

2. Main Result

In proving our main result, we shall need the following lemma due to Miller and Mocanu [3].

Lemma. *Let $w(z)=a+w_kz^k+\cdots$ be analytic in $E=\{z:|z|<1\}$ with $w(z)\not\equiv a$ and $k\geq 1$. If $z_0=r_0e^{i\theta}(0<r_0<1)$ and $|w(z_0)|=\max_{|z|\leq r_0}|w(z)|$. Then*

$$z_0w'(z_0)=mw(z_0) \tag{2.1}$$

and

$$\mathrm{Re}\left\{1+\frac{z_0w''(z_0)}{w'(z_0)}\right\}\geq m, \tag{2.2}$$

where m is a real number and

$$m\geq k\frac{|w(z_0)-a|^2}{|w(z_0)|^2-|a|^2}\geq k\frac{|w(z_0)|-|a|}{|w(z_0)|+|a|}.$$

Theorem. *Let $h(r,s,t)\in H$ and let $f(z)$ belonging to $\sum(p)$ satisfy*

(2.3) $$\left(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)}\right) \in D \subset \mathbb{C}^3$$

and

(2.4) $$\left| h\left(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)}\right)\right| < 1$$

for all $z \in E$ and for some $n \in N$. Then we have

$$\left|\frac{D^n f(z)}{D^{n-1} f(z)}\right| < 1 \quad (z \in E).$$

Proof. Let

$$\frac{D^n f(z)}{D^{n-1} f(z)} = w(z).$$

Then it follows that $w(z)$ is either analytic or meromorphic in $E, w(0) = 1$ and $w(z) \not\equiv 1$. With the aid of the identity (easy to verify)

$$z(D^n f(z))' = D^{n+1} f(z) - (p+1) D^n f(z),$$

we obtain

$$\frac{D^{n+1} f(z)}{D^n f(z)} = w(z) + \frac{zw'(z)}{w(z)}$$

and

$$\frac{D^{n+2} f(z)}{D^{n+1} f(z)} = w(z) + \frac{zw'(z)}{w(z)} + \frac{zw'(z) + \frac{zw'(z)}{w(z)} + \frac{z^2 w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2}{w(z) + \frac{zw'(z)}{w(z)}}.$$

We claim that $|w(z)| < 1$ for $z \in E$. Otherwise there exists a point $z_0 \in E$ such that $\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using the above lemma with $a = 1$ and $k = 1$, we see that

$$\frac{D^n f(z_0)}{D^{n-1} f(z_0)} = e^{i\theta},$$

$$\frac{D^{n+1} f(z_0)}{D^n f(z_0)} = m + e^{i\theta},$$

$$\frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)} = \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}},$$

where $L = \frac{z_0^2 w''(z_0)}{w(z_0)}$ and $m \geq 1$,

Further, an application of (2.2) in the above lemma gives

$$\mathrm{Re}L \geq m(m-1).$$

Since $h(r,s,t) \in H$, we have

$$\begin{aligned} &\left| h\left(\frac{D^n f(z_0)}{D^{n-1} f(z_0)}, \frac{D^{n+1} f(z_0)}{D^n f(z_0)}, \frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)} \right) \right| \\ &= \left| h(e^{i\theta}, m + e^{i\theta}, \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}}) \right| \\ &\geq 1, \end{aligned}$$

which contradicts the condition (2.4) of the theorem. Therefore, we conclude that

$$\left| \frac{D^n f(z)}{D^{n-1} f(z)} \right| < 1 \quad (z \in E).$$

This completes the proof of the theorem.

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