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FIXED POINT THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN PRODUCT SPACES

P. Vijayaraju

Abstract. The purpose of this paper is to prove fixed point theorems for certain type of asymptotically nonexpansive mappings in product of two locally convex spaces. These results generalize the theorems of Kirk and Yanez [7] and the author [9].

1. Introduction

Kirk and Yanez showed in [7] (see also Kirk and Sternfeld [6], Kirk [4,5]) that if K_i is a subset of a Banach space X_i for $i = 1, 2, K_1$ is weakly compact convex and has the B-G property (w.r.t. nonexpansive mappings) and if K_2 has the fixed point property for nonexpansive mappings, then $(K_1 \oplus K_2)_p$ has the fixed point property for nonexpansive mappings for $1 \leq p \leq \infty$. The author in [9] has introduced the following $\ell_p(1 \leq p < \infty)$ and ℓ_∞ direct sums and extended the above results of Kirk and Yanez to nonexpansive mappings in product of locally convex spaces.

Let X_1 and X_2 denote locally convex Hausdorff linear topological spaces with a family $(p_\alpha)_{\alpha \in J_1}$ and $(q_\beta)_{\beta \in J_2}$ of seminorms which define the topologies on X_1 and X_2 respectively, where J_1 and J_2 are index sets. Let K_i be a subset of X_i for $i = 1, 2$. Suppose that $K_1 \oplus K_2$ is the product space of K_1 and K_2 with a family of seminorms on $(K_1 \oplus K_2)_p, 1 \leq p < \infty$ and $(K_1 \oplus K_2)_\infty$ defined by

$$\gamma_{\alpha,\beta,p}((x, y)) = ([p_\alpha(x)]^p + [q_\beta(y)]^p)^{1/p}$$

and

$$\gamma_{\alpha,\beta,\infty}((x, y)) = \max[p_\alpha(x), q_\beta(y)]$$

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for each $\alpha \in J_1, \beta \in J_2$, and $(x, y) \in K_1 \oplus K_2$.

We recall the following definitions.

Definition 1.1. A self mapping T of K_1 is said to be *asymptotically nonexpansive* [8] if there is a sequence $\{k_n\}$ of real numbers with $k_n \geq 1, k_n \geq k_{n+1}$ and $k_n \to 1$ as $n \to \infty$ such that $p_\alpha(T^n x - T^n y) \leq k_n p_\alpha(x - y)$ for all x, y in K_1. If every asymptotically nonexpansive self mapping of K_1 has a fixed point in K_1, then K_1 is said to have the *fixed point property* for asymptotically nonexpansive mappings.

Definition 1.2. A self mapping T of K_1 is said to be *uniformly asymptotically regular* [8] if, for each $\alpha \in J_1$ and $\eta > 0$, there is an integer $N = N(\alpha, \eta)$ such that

$$p_\alpha(T^n x - T^{n+1} x) < \eta \text{ for all } n \geq N \text{ and for all } x \in K_1.$$

Remark. Asymptotically regular maps need not be uniformly asymptotically regular maps (Example given in [8]).

Definition 1.3. A subset K_1 of X_1 is said to have B-G *property* with *respect to asymptotically nonexpansive mappings* if for every asymptotically nonexpansive mapping T of K_1 into X_1, the mapping $I - T$ is *demiclosed* in the sense : if (x_δ) is a net in K_1 which converges weakly to x and if $((I-T)(x_\delta))$ converges to y , then $x \in K_1$ and $(I - T)x = y$.

A net (x_δ) in a set K is said to be *universal* in K [2] if, for every subset M of K, (x_δ) is eventually in M or eventually in the complement of M. The properties of universal net are given in Kelley [2].

The main aim of this paper is to extend some theorems of the author [9] and Kirk and Yanez [7] to certain type of asymptotically nonexpansive mappings in the product of two locally convex spaces. These results are new even in the case of Banach spaces. Throughout this paper, let P_i denote the coordinate projection of $K_1 \oplus K_2$ onto K_i for $i = 1, 2$.

Let T be a mapping of $(K_1 \oplus K_2)_\infty$ into itself. For $y \in K_2$, let $T_y : K_1 \to K_1$ be a mapping defined by

$$T_y(x) = P_1 \circ T(x, y) \text{ for all } x \in K_1.$$

Assume further that T satisfies the following condition:

$$(*) \qquad T_y^n(x) = P_1 \circ T^n(x, y) \text{ for all } x \in K_1, y \in K_2 \text{ and } n = 1, 2, \ldots$$

The following lemma is due to Cain and Nashed [1] which is used to prove the following Lemma 2.2.

Lemma (A) [1]. *Let K_1 be a sequentially complete subset of X_1 . If T is a contraction mapping of K_1 into itself, then T has a unique fixed point u in K_1 and $T^n x \to u$ for all $x \in K_1$.*

2. Main Results

For the proof of our theorem, we need the following lemmas.

Lemma 2.1. *Let K_i be a subset of X_i for $i = 1, 2$. Suppose that T is a uniformly asymptotically regular mapping of $(K_1 \oplus K_2)_\infty$ into itself which satisfies $(*)$. Then T_y is uniformly asymptotically regular.*

Proof. Let $x \in K_1$. Then since T is uniformly asymptotically regular and

$$\begin{aligned} p_\alpha(T_y^n x - T_y^{n+1} x) &= p_\alpha(P_1 \circ T^n(x, y) - P_1 \circ T^{n+1}(x, y)) \\ &\leq \gamma_{\alpha,\beta,\infty}(T^n(x, y) - T^{n+1}(x, y)), \end{aligned}$$

it follows that T_y is uniformly asymptotically regular.

Lemma 2.2. *Let K_1 be a complete convex subset of X_1 and $K_2 \subset X_2$. Let T be an asymptotically nonexpansive mapping of $(K_1 \oplus K_2)_\infty$ into itself which satisfies $(*)$. Then for $z \in K_1$, the map $S_{y,n} : K_1 \to K_1$ defined by $S_{y,n}(x) = (1-a_n)z + a_n T_y^n(x)$ for all $x \in K_1$, where $a_n = (1-1/n)(1/k_n), \{k_n\}$ is as in Definition* 1.1, *has a unique fixed point in K_1.*

Proof. Let $a, b \in K_1$. Then since T is asymptotically nonexpansive and

$$\begin{aligned} p_\alpha(T_y^n(a) - T_y^n(b)) &= p_\alpha(P_1 \circ T^n(a, y) - P_1 \circ T^n(b, y)) \\ &\leq \gamma_{\alpha,\beta,\infty}(T^n(a, y) - T^n(b, y)) \\ &\leq k_n \gamma_{\alpha,\beta,\infty}((a, y) - (b, y)) = k_n p_\alpha(a - b), \end{aligned}$$

it follows that T_y is asymptotically nonexpansive. Since K_1 is convex, it follows that $S_{y,n}$ maps K_1 into itself. Since $a_n = (1 - \frac{1}{n})\frac{1}{k_n}$, it follows that $S_{y,n}$ is a contraction on K_1. Using Lemma (A), there exists a point y_{a_n} in K_1 such that $S_{y,n}(y_{a_n}) = y_{a_n}$.

Using the above lemmas, we prove the following fixed point theorems.

Theorem 2.1. *Let K_1 be a weakly compact convex subset of X_1. Let K_2 be a subset of X_2 and $K = (K_1 \oplus K_2)_\infty$. Suppose that T is an asymptotically* *nonexpansive, uniformly asymptotically regular mapping of K into itself which satisfies $(*)$ such that $I - T_y$ is demiclosed. Then T_y has a fixed point in K_1.*

Proof. Let $S_{y,n}$ be defined as in Lemma 2.2. Since K_1 is weakly compact, it follows that it is complete and bounded [3, pp 155-156]. By Lemma 2.2, $S_{y,n}$ has a unique fixed point, say, y_{a_n} in K_1. i.e.,

$$y_{a_n} = S_{y,n}(y_{a_n}) = (1 - a_n)z + a_n T_y^n(y_{a_n}).$$

Therefore

$$p_\alpha(y_{a_n} - T_y^n(y_{a_n})) = (1 - a_n)p_\alpha(z - T_y^n(y_{a_n})) \to 0 \text{ as } n \to \infty, \tag{1}$$

since $a_n \to 1$ as $n \to \infty$ and K_1 is bounded.

Since T is uniformly asymptotically regular,it follows from Lemma 2.1 that T_y is uniformly asymptotically regular. Therefore

$$y_{a_n} - T_y^{n-1}(y_{a_n}) \to 0 \text{ as } n \to \infty \text{ by (1)}. \tag{2}$$

Since T_y is asymptotically nonexpansive, it follows from (1) and (2) that

$$y_{a_n} - T_y(y_{a_n}) \to 0 \text{ as } n \to \infty. \tag{3}$$

Now, suppose that (a_δ) is a universal subnet of the net $\{a_n : 0 < a_n < 1\}$ in [0,1]. Then (y_{a_δ}) is a universal net in K_1. Since K_1 is weakly compact, it follows that

$$y_{a_\delta} \overset{\delta}{\rightharpoonup} y_1 \text{ in } K_1[11, \text{p118}]. \tag{4}$$

Since $a_n \to 1$ as $n \to \infty$, it follows that $a_\delta \overset{\delta}{\to} 1$ and by(3), we obtain

$$y_{a_\delta} - T_y(y_{a_\delta}) \overset{\delta}{\to} 0. \tag{5}$$

Since $I - T_y$ is demiclosed, it follows from (4) and (5) that

$$(I - T_y)y_1 = 0, \quad \text{i.e.} \quad y_1 = T_y(y_1).$$

Theorem 2.2. *Let K_1 be a weakly compact convex subset of $X_1, K_2 \subset X_2$ and $K = (K_1 \oplus K_2)_\infty$. Suppose that K_2 has the fixed point property for asymptotically nonexpansive, uniformly asymptotically regular mappings. Suppose that T is an asymptotically nonexpansive, uniformly asymptotically regular mapping of K into itself. For fixed y in K_2, let $T_y : K_1 \to K_1$ and $S_{y,n} : K_1 \to K_1$ be mapping defined as in Lemma 2.2 such that $I - T_y$ is demiclosed,* *and suppose that T satisfies the condition $(*)$ in Lemma 2.2. Further, suppose that $G : K_2 \to K_2$ is the mapping defined by*

$$G(y) = P_2 \circ T(y_1, y) \ for\ all\ y\ in\ K_2,$$

where y_1 is a fixed point of T_y. Suppose that T satisfies the following condition:

$$(**) \qquad G^n(y) = P_2 \circ T^n(y_1, y) \ for\ all\ y \text{ in } K_2,$$

Then T has a fixed point in K.

Proof. Let u,$v \in K_2$ be arbitrary. Suppose that $S_{u,n}$ and $S_{v,n}$ are defined as in Lemma 2.2. Then u_{a_n} and v_{a_n} are unique fixed points of $S_{u,n}$ and $S_{v,n}$. Therefore

$$\begin{aligned} p_\alpha(u_{a_n} - v_{a_n}) &= a_n p_\alpha(P_1 \circ T^n(u_{a_n}, u) - P_1 \circ T^n(v_{a_n}, v)) \\ &\le a_n \gamma_{\alpha,\beta,\infty}(T^n(u_{a_n}, u) - T^n(v_{a_n}, v)) \\ &\le (1 - 1/n) \max\{p_\alpha(u_{a_n} - v_{a_n}), q_\beta(u - v)\}, \end{aligned}$$

since T is asymptotically nonexpansive. Hence

$$(6) \qquad p_\alpha(u_{a_n} - v_{a_n}) \le (1 - 1/n) q_\beta(u - v) < q_\beta(u - v).$$

Now, suppose that (a_δ) is a universal subnet of the net $\{a_n : 0 < a_n < 1\}$ in [0,1]. Then $(u_{a_\delta} - v_{a_\delta})$ is a universal net in K_1. Since K_1 is weakly compact, it follows that $u_{a\delta} - v_{a_\delta} \xrightarrow{\delta} u_1 - v_1$ in K_1. Therefore

$$(7) \qquad p_\alpha(u_1 - v_1) \le q_\beta(u - v), \text{ by } (6).$$

Now, since T is asymptotically nonexpansive,

$$\begin{aligned} q_\beta(G^n(u) - G^n(v)) &= q_\beta(P_2 \circ T^n(u_1, v) - P_2 \circ T^n(v_1, v)), \text{by}(**) \\ &\le \gamma_{\alpha,\beta,\infty}(T^n(u_1, u) - T^n(v_1, v)) \\ &\le k_n \max\{p_\alpha(u_1 - v_1), q_\beta(u - v)\} \\ &\le k_n q_\beta(u - v) \text{ by } (7). \end{aligned}$$

Therefore G is asymptotically nonexpansive. T is uniformly asymptotically regular so is G. By the hypothesis on K_2, G has a fixed point, say, w in K_2. i.e., $w = P_2 \circ T(w_1, w)$ where w_1 is such that $w_1 = P_1 \circ T(w_1, w)$. Hence $T(w_1, w) = (w_1, w)$.

The following example shows that all the conditions of Theorem 2.2 are satisfied.

Example. Let $X = \text{space}(s)$, the space of all sequences of complex numbers with a family of seminorms p_n defined by,

$$p_n(x) = \max_{1 \le i \le n} |x_i|$$

for all $x = (x_1, x_2, \ldots) \in X$ and $n = 1, 2, \ldots$. Let $K_0 = \{x \in X : |x_1| \le 1/2, |x_j| \le 1 \text{ for } j = 2, 3, \ldots\}$. Then K_0 is compact. Define a map $S : K_0 \to K_0$ by

$$Sx = (0, 2x_1, A_2 x_2, \ldots, A_k x_k, \ldots) \text{ for all } x = (x_1, x_2, \ldots, x_k, \ldots) \text{ in } K_0,$$

where $\{A_i\}$ is a sequence of real numbers with $0 < A_i < 1$ for all i and $\prod_{i=2}^{\infty} A_i = 1/2$.

Then S is an asymptotically nonexpansive, uniformly asymptotically regular mapping on K_0 [10].

It is easy to show that 0 is a unique fixed point of S in K_0. Suppose that $X_i = X$ and $K_i = K_0$ for $i = 1, 2$. Let $X = X_1 \oplus X_2$ be the locally convex space with a family of seminorms defined by

$$\gamma_{n,m,\infty}(x) = \max\{p_n(x_1), q_m(x_2)\} \text{ for all } x = (x_1, x_2) \text{ in } X,$$

where $p_n(\cdot)$ and $q_m(\cdot)$ are families of seminorms defined on X_1 and X_2 respectively. Define a map $T : K \to K$ by

$$T(x, y) = (Sx, Sy) \text{ for all } (x, y) \text{ in } K = K_1 \oplus K_2.$$

Then T is asymptotically nonexpansive, uniformly asymptotically regular.

For fixed y in K_2, we define $T_y : K_1 \to K_1$ by

$$T_y(x) = P_1 \circ T(x, y) \text{ for all } x \text{ in } K_1.$$

Then $T_y(x) = P_1 \circ (Sx, Sy) = Sx$. S is asymptotically nonexpansive so is T_y.

Now, let $x \in K_1$. Then $T_y^2(x) = S^2 x, \ldots, T_y^m(x) = S^m x = P_1 \circ T^m(x, y)$. Therefore T satisfies the condition $(*)$. To show that $I - T_y$ is demiclosed, let

$$x_\delta = (\xi_{\delta,1}, \xi_{\delta,2}, \ldots, \xi_{\delta,k}, \ldots) \rightharpoonup x = (\xi_1, \xi_2, \ldots \xi_k, \ldots)$$

and $(I - T_y)(x_\delta) \to y = (y_1, y_2, \ldots, y_k, \ldots)$. Then

$$(I - T_y)(x_\delta) - y = (\xi_{\delta,1} - y_1, \xi_{\delta,2} - 2\xi_{\delta,1} - y_2, \ldots,$$
$$\xi_{\delta,k} - A_{k-1}\xi_{\delta,k-1} - y_k, \ldots) \xrightarrow{\delta} 0.$$

Therefore

$$\xi_{\delta,1} \to y_1, \xi_{\delta,2} \to 2y_1 + y_2, \ldots,\ \xi_{\delta,k} \to \prod_{j=2}^{k-1} A_j(2y_1 + y_2) + \prod_{j=3}^{k-1} A_j y_3 + \cdots + y_k, \ldots$$

Since the weak limit is unique, it follows that

$$\xi_1 = y_1, \xi_2 = 2y_1 + y_2, \ldots, \xi_k = \prod_{j=2}^{k-1} A_j(2y_1 + y_2) + \prod_{j=3}^{k-1} A_j y_3 + \cdots + y_k, \ldots$$

Therefore $x - T_y(x) = y$. Hence $I - T_y$ is demiclosed.

Define a map $G : K_2 \to K_2$ by

$$G(y) = P_2 \circ T(0, x) \text{ for all } y \text{ in } K_2.$$

Then

$$G^2(y) = P_2(0, Sy) = Sy, \ldots, G^m(y) = S^m y = P_2 \circ T^m(0, y).$$

Therefore T satisfies the condition $(**)$. Also G is asymptotically nonexpansive. Since K_2 is compact and convex, it follows that every asymptotically nonexpansive , uniformly asymptotically regular mapping of K_2 into itself has a fixed point in K_2[8]. It is easy to show that 0 is a unique fixed point of T in K.

For the proof of the next theorem, we need the following concept.

Definition 2.1. A convex complete subset K of a locally convex space X is said to have *the effective fixed point property* for asymptotically nonexpansive mappings if there exists z in K such that for every asymptotically nonexpansive mapping T from K to X, the set of (unique) fixed points of the mappings $\{tT^n + (1-t)z : t \in (0,1) \text{ and } n = 1, 2, \ldots\}$ is precompact.

In [7], Kirk and Yanez showed that if K_i is a subset of a Banach space X_i for $i = 1, 2, K_1$ is closed convex bounded and has the effective fixed point property for nonexpansive mappings and if K_2 is closed and has the fixed point property for nonexpansive mappings, then $(K_1 \oplus K_2)_\infty$ has the fixed point property for nonexpansive mappings. This result is extended by the author in [9] to such mappings in locally convex spaces. The following theorem is an extension of the above results.

Theorem 2.3. *Let K_1 be a complete bounded convex subset of X_1 and K_1 have the effective fixed point property for asymptotically nonexpansive mappings. Suppose that K_2 is a subset of X_2 and K_2 has the fixed point property for asymptotically nonexpansive, uniformly asymptotically regular mappings. Let $T, T_y, S_{y,n}$ and G be mappings defined as in Theorem 2.1 and T satisfy the conditions $(*)$ and $(**)$. Then T has a fixed point in $(K_1 \oplus K_2)_\infty$.*

Proof. As in Theorem 2.1, we can prove that y_{a_n} is a unique fixed point of $S_{y,n}$ in K_1 and

$$(I - T_y)y_{a_n} \to 0 \text{ as } n \to \infty. \tag{8}$$

By the definition of $S_{y,n}$,

$$y_{a_n} = S_{y,n}(y_{a_n}) = (1 - a_n)z + a_n T_y^n(y_{a_n}).$$

Since K_1 has the effective fixed point property for asymptotically nonexpansive mappings, $\{y_{a_n} : n = 1, 2, \ldots\}$ is precompact.

Let $Z = c1\{y_{a_n} : n = 1, 2, \ldots\}$. Then Z is precompact [3, p. 65]. Since a closed subset of a complete space is complete, it follows that Z is complete. Therefore Z is a compact subset of K_1 [3, p. 61].

Note that if (a_δ) is a universal subnet of the net $\{a_n : 0 < a_n < 1\}$ in [0,1], then (y_{a_δ}) is a universal net in Z. Since Z is compact, it follows that $y_{a_\delta} \xrightarrow{\delta} y_1$ in Z.

Since $I - T_y$ is continuous, it follows that

$$(I - T_y)(y_{a_\delta}) \xrightarrow{\delta} (I - T_y)y_1. \tag{9}$$

From (8), we obtain

$$(I - T_y)(y_{a_\delta}) \xrightarrow{\delta} 0. \tag{10}$$

From (9) and (10) we obtain $(I - T_y)y_1 = 0$, i.e. $y_1 = T_y(y_1) = P_1 \circ T(y_1, y)$. The remaining part of the proof follows as in the proof of Theorem 2.2.

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Department of Mathematics, Anna University
Chennai (Madras) 600 025, India
E-mail: annalib@sirnetm.ernet.in