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ON A CONJECTURE ON THE UNIFORM CONVERGENCE OF A SEQUENCE OF WEIGHTED BOUNDED POSITIVE DEFINITE FUNCTIONS ON FOUNDATION SEMIGROUPS

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Abstract. In the present paper, we shall establish one of our earlier conjectures by proving that on compact subsets of a $*$-foundation semigroup S with identity and with a locally bounded Borel measurable weight function w, the pointwise convergence and the uniform convergence of a sequence of w-bounded positive definite functions on S which are also continuous at the identity are equivalent.

1. Introduction

In our earlier paper [7], we proved that if S is a foundation topological $*$-semigroup with an identity e and with a Borel measurable weight function w such that $0 < w \leq 1$ and $1/w$ is locally bounded (i.e. bounded on compact subsets of S), then a sequence (φ_n) in $\mathcal{P}_e(S, w)$, the set of w-bounded Borel measurable positive definite functions on S which are continuous at e, converges pointwise on S to a function $\varphi \in \mathcal{P}_e(S, w)$ if and only if (φ_n) converges to φ uniformly on compact subsets of S. In that paper we also conjectured that this result remains true for any Borel measurable weight function w such that w and $1/w$ are locally bounded.

In the present paper we shall first prove that on a foundation $*$-semigroup S with an identity and with a locally bounded Borel measurable weight function w, the set of w-bounded Borel measurable positive definite functions which are continuous at the identity is identical with the set of w-bounded continuous positive definite functions. We then introduce two new topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{F}}$

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on $\mathcal{P}(S,w)$ (the set of w-bounded continuous positive definite functions on S) and we show that these topologies coincide on $\mathcal{P}(S,w)$ (see Theorem 5). An application of this result has enabled us to establish our earlier conjecture in [7]. We conclude the paper with an example of a weighted foundation semigroup with a uniformly convergent sequence of w-bounded continuous positive definite functions on compact subsets of S but not uniformly convergent on the whole of S. It should be noted that the class of foundation semigroups is extensive, and includes all discrete semigroups, all locally compact non-locally-null subsemigroups of locally compact groups. For many other examples, see [8; Appendix B].

2. Preliminaries

Throughout this paper, except in Lemma 3, S will denote a locally compact, Hausdorff topological semigroup with an identity. A topological semigroup S is called a $*$-semigroup if there is a continuous mapping $*: S \to S$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. A locally bounded (i.e. bounded on compact subsets of S) mapping $w : S \to \mathcal{R}^+$($\mathcal{R}^+$ denotes the set of positive real numbers) is called a weight function on S if $w(xy^*) \leq w(x)w(y)$ for all $x, y \in S$. A function $f : S \to \mathcal{C}$ ($\mathcal{C}$ denotes the set of complex numbers) is called w-bounded if there is a positive number k such that $|f(x)| \leq kw(x)\,(x \in S)$. A complex-valued function φ on S is called positive definite whenever

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{c}_j\varphi(x_ix_j^*) \geq 0$$

for all choices $\{x_1, \dots, x_n\}$ from S and $\{c_1, \dots, c_n\}$ from $\mathcal{C}$. We denote by $\mathcal{P}_e(S,w)\,(\mathcal{P}(S,w)$, respectively) the set of w-bounded, Borel measurable, continuous at e and positive definite functions on S (the set of w-bounded continuous positive definite functions on S, respectively). A $*$-representation of S by bounded operators on a Hilbert space $\mathcal{H}$ is a homomorphism: $x \to \pi(x)$ of S into $\mathcal{L}(\mathcal{H})$, the space of all bounded linear operators on $\mathcal{H}$, such that $\pi(x^*) = (\pi(x))^*$ for all $x \in S$ and $\pi(e)$ is the identity operator on $\mathcal{H}$. A representation π is called cyclic if there is a (cyclic) vector $\xi \in \mathcal{H}$ such that the set $\{\pi(x)\xi : x \in S\}$ is dense in $\mathcal{H}$, and π is called w-bounded if there is a positive number k such that $\|\pi(x)\| \leq kw(x)$ $(x \in S)$. Note that a $*$-representation π is w-bounded if and only if $\|\pi(x)\| \leq w(x)$ $(x \in S)$. For further information on the representations on topological $*$-semigroups we refer the reader to [2], [5], [6].

Recall that (see for example, [1] or [4]) $\tilde{L}(S)$ or $M_a(S)$ denotes the set of all measures $\mu \in M(S)$ (the convolution measure algebra of bounded complex measures on S with the total variation norm $\|\cdot\|$) for which the mapping $x \in \delta_x * |\mu|$ and $x \to |\mu| * \delta_x$ (where δ_x denotes the point mass at x for $x \in S$) from S into $M(S)$ are weakly continuous. If w is a locally bounded Borel measurable weight function on S, then we denote by $M_a^{\mathcal{K}}(S, w)$ the set of all complex regular measures μ on S such that $w\mu \in M_a^{\mathcal{K}}(S)$, where $M_a^{\mathcal{K}}(S)$ denotes the set of all measures in $M_a(S)$ with compact support. We observe that $M_a^{\mathcal{K}}(S, w)$ with the convolution

$$(\mu * \nu)(f) = \int f(xy) d\mu(x) d\nu(y) \quad (f \in C_c(S)),$$

where $C_c(S)$ denotes the space of all continuous complex-valued functions on S with compact support, defines a normed algebra. Moreover, $\delta_x * \mu \in M_a^{\mathcal{K}}(S, w)$ for every $x \in S$ and $\mu \in M_a^{\mathcal{K}}(S, w)$. A semigroup S is called foundation if $\cup\{\mathrm{supp}(\mu) : \mu \in M_a(S)\}$ is dense in S. It is well-known that $M_a(S)$ is a two-sided closed L-ideal of $M(S)$ and if S is also a foundation semigroup with identity, then both mappings $x \to \delta_x * \mu$ and $x \to \mu * \delta_x (\mu \in M_a(S))$ from S into $M_a(S)$ are norm continuous (c.f. [8; Theorem 5.8]). We observe that if S is a foundation semigroup with identity and with a locally bounded Borel measurable weight function w, then both the mappings $x \to \delta_x * \mu$ and $x \to \mu * \delta_x$ ($\mu \in M_a^{\mathcal{K}}(S, w)$) from S into $M_a^{\mathcal{K}}(S, w)$ are $\|\cdot\|_w$-norm continuous, where $\|\mu\|_w = \|w\mu\|$ for every $\mu \in M_a^{\mathcal{K}}(S, w)$.

3. The τ_u-Topology and the $\tau_{\mathcal{F}}$-Topology of $\mathcal{P}(S, w)$

The following two definitions are needed for the proof of the main result.

Definition 1. For each compact subset F of S, positive numbers α, β, and $\varphi_0 \in \mathcal{P}(S, w)$ of a foundation $*$-semigroup S with an identity e and with a locally bounded Borel measurable weight function w we define

$$\mathcal{U}_{F;\alpha,\beta}(\varphi_0) = \{\varphi \in \mathcal{P}(S, w) : |\varphi(x) - \varphi_0(x)| < \alpha \text{ and } |\varphi(xx^*) - \varphi_0(xx^*)| < \beta \text{ for all } x \in F\}. \tag{1}$$

The family of the sets of the form (1) defines a base for a topology on $\mathcal{P}(S, w)$ which is denoted by $\tau_{\mathcal{U}}$.

Definition 2. For $\mu_1, \ldots, \mu_m \in M_a^{\mathcal{K}}(S, w)$, positive real numbers α, β, γ, and $\varphi_0 \in \mathcal{P}(S, w)$ let

$$\mathcal{F}_{\mu_1,\ldots,\mu_m;\alpha,\beta,\gamma}(\varphi_0) = \Big\{ \varphi \in \mathcal{P}(s,w) : \Big|\int_S [\varphi(y) - \varphi_0(y)] d\mu_j(y)\Big| < \alpha,$$

$$\Big|\int_S [\varphi(yy^*) - \varphi_0(yy^*)] d\mu_j(y)\Big| < \beta, \tag{2}$$

$$\text{for } j = 1, \ldots, m, \quad \text{and} \quad |\varphi(e) - \varphi_0(e)| < \gamma \Big\}$$

The family of the sets of the form (2) defines a base for a topology $\tau_{\mathcal{F}}$ on $\mathcal{P}(S,w)$.

The following result is the key lemma to this paper.

Lemma 3. *Let S be a $*$-semigroup (not necessarily topological) with an identity and with a weight w. Then every w-bounded positive definite function φ on S satisfies the following inequality*

$$|\varphi(x) - \varphi(xy)|^2 \le \varphi(e) w^2(x) [\varphi(e) - 2\operatorname{Re}\varphi(y) + \varphi(yy^*)] \ (x, y \in S). \tag{3}$$

Proof. Since φ is w-bounded, from Proposition 4.1.3 of [3] and the proof of 4.1.14 of [3] it follows that there exists a w-bounded cyclic $*$-representation π of S by bounded operators on a Hilbert space $\mathcal{H}$ with a cyclic vector ξ such that $\|\xi\|^2 = \varphi(e)$ and

$$\varphi(x) = \langle \pi(x)\xi, \xi \rangle \quad (x \in S).$$

For every $x, y \in S$ we have

$$\begin{aligned}
|\varphi(x) - \varphi(y^*x)|^2 &= |\langle \pi(x)\xi, \xi\rangle - \langle \pi(y^*x)\xi, \xi\rangle|^2 \\
&= |\langle \pi(x)\xi, \xi - \pi(y)\xi\rangle|^2 \\
&\le \|\pi(x)\xi\|^2 \|\xi - \pi(y)\xi\|^2 \\
&= \langle \pi(x)\xi, \pi(x)\xi\rangle [\|\xi\|^2 - 2 \ \operatorname{Re} \ \varphi(y) + \varphi(yy^*)] \\
&= \varphi(xx^*)[\varphi(e) - 2 \ \operatorname{Re} \ \varphi(y) + \varphi(yy^*)] \\
&\le \varphi(e)(w(x))^2 [\varphi(e) - 2 \ \operatorname{Re} \ \varphi(y) + \varphi(yy^*)].
\end{aligned}$$

By replacing y by y^*, we obtain the desired inequality.

Lemma 4. *Let S be a foundation $*$-semigroup with identity and with a locally bounded Borel measurable weight function w. Then $\mathcal{P}_e(S,w) = \mathcal{P}(S,w)$.*

Proof. Let $\varphi \in \mathcal{P}_e(S,w)$. Take a fixed $x_0 \in S$ and let W be a fixed compact neighbourhood of x_0. Since w is locally bounded, there exists a positive real number M such that $w(x) \leq M$ for all $x \in W$. Given $\varepsilon > 0$, by the continuity of φ at e there exists a neighbourhood U of e such that

$$[\varphi(e) - 2 \ \text{Re} \ \varphi(u) - \varphi(uu^*)]^{1/2} < \frac{\varepsilon}{2M[(\varphi(e))^{1/2} + 1]} \ (u \in U).$$

By Theorem 3.1.2 of [4] $W_1 = [U^{-1}(Ux) \cap (xU)U^{-1}] \cap W$ defines a neighbourhood of e. Let $z \in W_1$, then $uz = vx$ for some $u, v \in U$. So by (3)

$$\begin{aligned} |\varphi(z) - \varphi(x)| &\leq |\varphi(z) - \varphi(uz)| + |\varphi(vx) - \varphi(x)| \\ &\leq (\varphi(e))^{1/2} w(z)([\varphi(e) - 2 \ \text{Re} \ \varphi(u) + \varphi(uu^*)])^{1/2} \\ &\quad +(\varphi(e))^{1/2} w(x)([\varphi(e) - 2 \ \text{Re} \ \varphi(v) + \varphi(vv^*)])^{1/2} \\ &< 2M(\varphi(e))^{1/2} \frac{\varepsilon}{2M[(\varphi(e))^{1/2} + 1]} \\ &< \varepsilon. \end{aligned}$$

So $\varphi \in \mathcal{P}(S, w)$ and the proof is complete.

The following theorem is the main result of this paper and it generalizes Theorem 2.4 of [7]. Note that $\mathcal{P}_e(S, w) = \mathcal{P}(S, w)$, by Lemma 4.

Theorem 5. *Let S be a foundation $*$-semigroup with identity and with a locally bounded Borel measurable weight function w. Then the $\tau_{\mathcal{U}}$-topology and the $\tau_{\mathcal{F}}$-topology are identical on $\mathcal{P}(S, w)$.*

Proof. Take φ_0 fixed in $\mathcal{P}(S, w)$. Let $\mathcal{F}_{\mu_1,\ldots,\mu_m;\beta,\gamma,\lambda}(\varphi_0)$ be an arbitrary basic $\tau_{\mathcal{F}}$-neighbourhood of φ_0. Choose a positive number η such that $\eta \leq \lambda$ and $2\eta + \eta \max \{\|\mu_1\|, \ldots, \|\mu_n\|\} < \min(\beta, \gamma)$. Choose a compact set F_0 such that $e \in F_0$ with

$$\int_{S \setminus F_0} (w(y))^2 d|\mu_j|(y) < \eta, \text{ and } \int_{S \setminus F_0} w(y) d|\mu_j|(y) < \eta \quad (j = 1, \ldots, m).$$

Then it is clear that

$$\mathcal{U}_{F_0;\eta,\eta}(\varphi_0) \subseteq \mathcal{F}_{\mu_1,\ldots,\mu_m;\beta,\gamma,\lambda}(\varphi_0).$$

Conversely, suppose that $\mathcal{U}_{F;\alpha_0,\beta_0}(\varphi_0)$ is an arbitrary $\tau_{\mathcal{U}}$-neighbourhood of φ_0. Let $\beta = \min\{\alpha_0, \beta_0\}$ and M be a positive number such that $w(x) \leq M$ for all $x \in F$. Put

$$\begin{aligned} \gamma &= \min \left\{ \frac{\beta^2}{81M^4(1 + \varphi_0(e))}, \frac{\beta^2}{81M^2(1 + (\varphi_0(e))} \right\}, \\ \delta &= \min \left\{ \frac{\beta}{6(\varphi_0(e) + 1)}, 1 \right\}. \end{aligned}$$

By the continuity of φ_0 at e there exists a compact neighbourhood U of e such that for all $y \in U$

$$|\varphi_0(y) - \varphi_0(e)| < \gamma \text{ and } |\varphi_0(yy^*) - \varphi_0(e)| < \gamma. \tag{4}$$

Now choose a positive measure $\mu \in M_a^{\mathcal{K}}(S, w)$ such that $\mu(S) = 1$ and $e \in$ supp $(\mu) \subseteq U$. By the $\|\cdot\|_w$-norm continuity of the mapping $x \to \delta_x * \mu$ from S into $M_a^{\mathcal{K}}(S, w)$ and the compactness of F we can find a finite subset $\{x_1, \ldots, x_n\}$ of F such that the set $\{\delta_x * \mu : x \in F\}$ can be covered by $\{\mathcal{N}_{x_1}, \ldots, \mathcal{N}_{x_n}\}$, where $\mathcal{N}_{x_i} = \{\lambda \in M_a^{\mathcal{K}}(S, w) : \|\lambda - \delta_{x_i} * \mu\|_w < \delta\}$ for $i = 1, \ldots, n$. Again by the $\|\cdot\|_w$-norm continuity of the mapping $x \to \delta_{xx^*} * \mu$ from S into $M_a^{\mathcal{K}}(S, w)$, we can find $s_1, s_2, \ldots, s_\ell \in S$ such that the set $\{\delta_{xx^*} * \mu : x \in F\}$ can be covered by $\{\mathcal{N}'_{s_1}, \ldots, \mathcal{N}'_{s_\ell}\}$, where $\mathcal{N}'_{s_j} = \{\lambda \in M_a^{\mathcal{K}}(S, w) : \|\lambda - \delta_{s_j s_j^*} * \mu\|_w < \delta\}$ $(j = 1, \ldots, \ell)$. Put $z_i = x_i,\ i = 1, \ldots, n, z_{n+j} = s_j s_j^*$ for $1 \le j \le \ell$. Put $p = n + \ell$. and let $\mu_k = \delta_{z_k} * \mu(k = 1, 2, \ldots, p)$. We shall prove that

$$\mathcal{F}_{\mu_1,\mu_2,\ldots,\mu_p;\delta,\delta,\delta}(\varphi_0) \cap \mathcal{F}_{\mu;\gamma,\gamma,\gamma}(\varphi_0) \subseteq \mathcal{U}_{F;\beta,\gamma}(\varphi_0).$$

To prove this we choose $\varphi \in \mathcal{F}_{\mu_1,\ldots,\mu_p;\delta,\delta,\delta}(\varphi_0)$. Let x be a fixed but arbitrary element in F. Then

$$\|\delta_x * \mu - \delta_{x_j} * \mu\|_w < \delta \quad \text{and} \quad \|\delta_{x * x^*} - \delta_{x_q * x_q^*} * \mu\|_w < \delta$$

for some j and $q \in \{1, 2, \ldots, p\}$. Therefore

$$\begin{aligned}
&|\delta_x * \mu(\varphi) - \delta_x * \mu(\varphi_0)| \\
&= \left| \int [\varphi(y) - \varphi_0(y)] d\delta_x * \mu(y) \right| \\
&\le \left| \int \varphi(y) d(\delta_x * \mu - \delta_{x_j} * \mu)(u) \right| + \left| \int [\varphi(y) - \varphi_0(y)] d\mu_j(y) \right| \\
&\quad + \left| \int \varphi_0(y) d(\delta_{x_j} * \mu - \delta_x * \mu)(y) \right| \\
&\le \varphi(e) \|\delta_x * \mu - \delta_{x_j} * \mu\|_w + \delta + \varphi_0(e) \|\delta_{x_j} * \delta_x * \mu\|_w \\
&< \delta(\varphi(e) + \varphi_0(e) + 1) < \beta/3.
\end{aligned} \tag{5}$$

(In the above we have used Proposition 4.1.12 of [3].) Similarly by using the inequality $\|\delta_{xx^*} * \mu - \delta_{x_q x_q^*} * \mu\|_w < \delta$, we can prove that

$$|\delta_{xx^*} * \mu(\varphi) - \delta_{xx^*}(\varphi_0)| < \beta/3. \tag{6}$$

Suppose now that

$$\varphi \in \mathcal{F}_{\mu;\gamma,\gamma,\gamma}(\varphi_0).$$

Then for every $x \in F$ with the aid of (3) and the Hölder inequality we have

$$\begin{aligned} |\delta_x * \mu(\varphi) &- \varphi(x)| \\ &\leq \Big|\int_S \varphi(xy)d\mu(y) - \int_S \varphi(x)d\mu(y)\Big| \\ &\leq \int_S |\varphi(xy) - \varphi(x)|d\mu(y) \\ &\leq w(x)(\varphi(e))^{1/2}\Big(\int_U [\varphi(e) - 2 \ \text{Re} \ \varphi(y) + \varphi(yy^*)]d\mu(y)\Big)^{1/2} \\ &\leq M\varphi(e)^{1/2}\Big(\int_U [\varphi(e) - 2 \ \text{Re} \ \varphi(y) + \varphi(yy^*)]d\mu(y)\Big)^{1/2}. \end{aligned}$$

Now if we apply (4), then we obtain

$$\begin{aligned} \int_S [\varphi(e) &- 2 \ \text{Re} \ \varphi(y) + \varphi(yy^*)]d\mu(y) \\ &\leq 2\Big|\int_U [\varphi(e) - \text{Re} \ \varphi(y)]d\mu(y)\Big| + \Big|\int_U [\varphi(yy^*) - \varphi(e)]d\mu(y)\Big| \\ &\leq 2\Big|\int_U [\varphi(e) - \varphi(y)]d\mu(y)\Big| + \Big|\int_U [\varphi(yy^*) - \varphi(e)]d\mu(y)\Big| \\ &\leq 2\Big[\int_U [\varphi(e) - \varphi_0(e)|d\mu(y) + \int_U |\varphi_0(e) - \varphi_0(y)|d\mu(y) \\ &\quad + \int_U |\varphi_0(y) - \varphi(y)|d\mu(y)\Big] + \int_U |\varphi(yy^*) - \varphi_0(yy^*)|d\mu(y) \\ &\quad + \int_U |\varphi_0(yy^*) - \varphi_0(e)|d\mu(y) + \int_U |\varphi_0(e) - \varphi(e)|d\mu(y) \\ &< 9\gamma. \end{aligned}$$

So for every $x \in F$

$$|\delta_x * \mu(\varphi) - \varphi(x)| \leq 3M(\varphi(e)\gamma)^{1/2} < \beta/3. \tag{7}$$

Similarly for every $x \in F$

$$\begin{aligned} |\delta_{xx^*} * \mu(\varphi) &- \varphi(xx^*)| \\ &\leq w(xx^*)\varphi(e)^{1/2}\Big(\int_U [\varphi(e) - 2 \ \text{Re} \ \varphi(y) + \varphi(yy^*)]d\mu(y))^{1/2} \\ &< 3M^2(\varphi(e)\gamma)^{1/2} < \beta/3. \end{aligned} \tag{8}$$

Finally, for every $\varphi \in \mathcal{F}_{\mu_1,\ldots,\mu_p;\delta,\delta,\delta}(\varphi_0) \cap \mathcal{F}_{\mu;\gamma,\gamma,\gamma}(\varphi_0)$ and every $x \in F$ from (7) and (5) we have

$$\begin{aligned} |\varphi(x) - \varphi_0(x)| &\leq |\varphi(x) - \delta_x * \mu(\varphi)| + |\delta_x * \mu(\varphi) - \delta_x * \mu(\varphi_0)| \\ &\quad + |\delta_x * \mu(\varphi_0) - \delta_x * \mu(\varphi)| \\ &< \frac{\beta}{3} + \frac{\beta}{3} + \frac{\beta}{3} = \beta. \end{aligned}$$

Similarly for every $x \in F$ from (8) and (6) we conclude that

$$|\varphi(xx^*) - \varphi_0(xx^*)| < \beta.$$

That is $\varphi \in \mathcal{U}_{F,\beta,\beta}(\varphi_0)$. The proof is now complete, since $\mathcal{U}_{F;\beta,\beta} \subseteq \mathcal{U}_{F;\alpha_0,\beta_0}$.

We are now in a position to establish our earlier conjecture in [7].

Theorem 6. *Let S be a foundation topological $*$ -semigroup with an identity and with a locally bounded Borel measurable w. Then a sequence (φ_n) of w-bounded continuous positive definite functions on S converges pointwise to a continuous function φ if and only if (φ_n) converges to φ in the topology of uniform convergence on compact subsets of S.*

Proof. Suppose that (φ_n) converges to φ pointwise on S and φ is also continuous. Then it is clear that $\varphi \in \mathcal{P}(S,w)$. From the Lebesgue dominated convergence theorem it follows that $\varphi_n \to \varphi$ in $\tau_{\mathcal{F}}$-topology. So by Theorem 5, $\varphi_n \to \varphi$ in $\tau_{\mathcal{U}}$-topology. The converse is obvious.

The following example shows that the uniform convergence of the sequence (φ_n) in the above theorem on compact subsets of S does not imply the convergence is uniform on the whole of S.

Example 7. Let S denotes the set of positive real numbers, and let S be endowed with the usual topology. Then S with the usual multiplication on the real line and the involution $x^* = x (x \in S)$ defines a foundation $*$ -semigroup with identity. If we define $w(x) = 1/x \, (x \in S)$, then w defines a continuous weight function on S. For every positive integer n define $\varphi_n(x) = 1/nx \, (x \in S)$. Then (φ_n) defines a sequence of w-bounded continuous positive definite functions on S which converges uniformly to 0 on each compact subset of S and it is clear that this convergence is not uniform on the whole space S.

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