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FIXED POINTS, MINIMAX INEQUALITIES AND EQUILIBRIA OF NONCOMPACT ABSTRACT ECONOMIES*

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Abstract. Several new fixed point theorems in H-space are first proved. Next, by applying the fixed point theorems, some minimax inequalities and existence theorems of maximal elements for $\mathcal{L}_F$ correspondences and $\mathcal{L}_F$-majorized correspondences in H-spaces are obtained. Finally, using the existence theorems of maximal elements, some equilibrium existence theorems for one-person games, qualitative games and noncompact abstract economies with $\mathcal{L}_F$-majorized correspondences in H-spaces are obtained. Our theorems improve and generalize most known results due to Border, Borglin-Keiding, Ding-Kim-Tan, Ding-Tan, Ding-Tarafdar, Mehta-Tarafdar, Shafer-Sonnenschein, Tan-Yuan, Tarafdar, Toussaint, Tulcea, Yannelis and Yannelis-Prabhakar etc.

1. Introduction

Recently, Ding-Kim-Tan [7, 8], Ding-Tan [10, 11, 12], Ding-Zhuang [14], Tan-Yuan [24, 25], Tian [32, 33] have proved some very general equilibrium existence theorems for noncompact abstract economies (= generalized games) with an infinite number of agents, with infinitely dimensional strategy spaces and with majorized type preference correspondences defined on noncompact strategy sets of agents. These theorems improve and generalize most known results due to Borglin-Keiding [4], Shafer-sonnenschein [21], Tarafdar [30], Toussaint [34], Tuclea [35, 36], Yannelis-Prabhakar [38] and others. To my best knowledge, all the above existence theorems are proved by assuming that the strategy sets are nonempty convex or nonempty compact convex subsets

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of topological vector spaces. The assumptions are very restrictive since the strategy sets of agents generally are not compact and convex in any topology of commodity spaces; the commodity spaces may not have the linear structure and various kinds of preference and constraint correspondences will be encountered in general economic situations. Thus, it is important and of interest to establish some equilibrium existence theorems for an abstract economy with noncompact and nonconvex strategy sets of agents.

Tarafdar [31] and Ding [6] have established some equilibrium existence theorems for qualitative games and abstract economies under H-space setting without linear structure which generalize some known results under the setting of topological vector spaces.

In this paper, we shall first introduce the notions of correspondence of class $\mathcal{L}_F$, $\mathcal{L}_F$-majorant of ϕ at x and $\mathcal{L}_F$-majorized correspondence in H-space. These notions generalize the corresponding definitions of Borglin-Keiding [4], Toussaint [34], Tulcea [35, 36], Ding-Kim-Tan [7], Ding-Tan [10, 11, 12], Tan-Yuan [25] and Ding-Tarafdar [13]. Next some new fixed point theorems for set-valued mappings with noncompact domain in H-space are proved under very weak coercive condition and some equivalent forms are also given. These theorems unify and generalize many known results in the literature. As applications. an existence theorem of maximal elements for $\mathcal{L}_F$-majorized correspondence is obtained which generalizes the corresponding results of Borglin-Keiding [4], Yannelis [37], Yannelis-Prabhaka [38], Toussaint [34], Tulcea [35, 36], Ding-Kim-Tan [7], Ding-Tan [10, 11, 12], Tan-Yuan [25] and Ding-Tarafdar [13]. By applying earlier results, several equilibrium existence theorems of one-person games and qualitative games with an infinite number of agents and with $\mathcal{L}_F$-majorized preference correspondences are proved under H-space setting. Finally, some equilibrium existence theorems of an abstract economy with an infinite number of agents and with $\mathcal{L}_F$-majorized correspondences defined on noncompact and nonconvex strategy sets of agents are obtained under H-space setting. Our results are new and generalize most known results in the economies literature mentioned above.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A, by $int_X(A)$ the interior of A in X, and by $cl_X(A)$ the closure of A in X. A subset A of X is said to be compactly open (resp., closed) in X if for each nonempty compact subset K of X, $A \cap K$ is open (resp., closed) in K. If X and Y are topological spaces and $T, S : X \to 2^Y$ are two correspondences, then $T \cap S : X \to 2^Y$ is a corresopndence defined by $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in X$. The graph of T is the set

$Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ and the correspondence $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in X : (x, y) \in cl_{X\times Y}(Gr(T))\}$ (the set $cl_{X\times Y}(Gr(T))$ is called the adherence of the graph of T) and the correspondence $cl\ T : X \to 2^Y$ is defined by $(cl\ T)(x) = cl_Y(T(x))$ for each $x \in X$. it is easy to see that $(cl\ T)(x) \subset \overline{T}(x)$ for each $x \in X$.

The following notions, which were introduced by Bardaro-Ceppitelli [1, 2], were motivated by the earlier works of Horvath in [18, 19].

A pair $(E, \{\Gamma_A\})$ is said to be an H-space if E is a topological space and $\{\Gamma_A\}$ is a family of contractible subsets of E indexed by $A \in \mathcal{F}(E)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$ where $\mathcal{F}(E)$ denotes the family of all nonempty finite subsets of E. Clearly, each topological vector space and its convex subset are all H-spaces with $\Gamma_A = co(A)$ for each $A \in \mathcal{F}(E)$ where $co(A)$ is the convex hull of A. A subset D of an H-space $(E, \{\Gamma_A\})$ is said to be (i) H-convex if $\Gamma_A \subset D$ for each $A \in \mathcal{F}(D)$ and (ii) weakly H-convex if $\Gamma_A \cap D$ is contractible for each $A \in \mathcal{F}(D)$.

Following Tarafdar [29], for a nonempty subset D of an H-space $(E, \{\Gamma_A\})$, we define the H-convex hull of D, denoted by $H\text{-}co(D)$, as

$$H\text{-}co(D) = \cap\{B \subset X : D \subset B \text{ and } B \text{ is } H\text{-convex}\}.$$

By Lemma 1 of Tarafdar [29], we have

$$H\text{-}co(D) = \cup\{H\text{-}co(A) : A \in \mathcal{F}(\mathcal{D})\}.$$

Let D be a nonempty set, $(X, \{\Gamma_A\})$ be an H-space and $G : D \to 2^X$ be a correspondence. The correspondence $H\text{-}co\,G : D \to 2^X$ is defined by $(H\text{-}co\,G)(x) = H\text{-}co(G(x))$ for each $x \in D$.

The following notions are more general than the corresponding notions due to Ding-Tan [10, 12], Tan-Yuan [25] and Ding-Tarafdar [13]. Let X be a topological space, Y be a nonempty subset of an H-space $(E, \{\Gamma_A\})$. $\theta : X \to E$ be a map and $\phi : X \to 2^Y$ be a correspondence. Then (1) ϕ is said to be of class $\mathcal{L}_{\theta,F}$ if (a) for each $x \in X, H\text{-}co(\phi(x)) \subset Y$ and $\theta(x) \notin H\text{-}co(\phi(x))$ for each $x \in X$ and (b) there exists a correspondence $\psi : X \to 2^Y$ such that $\psi(x) \subset \phi(x)$ for each $x \in X$ and for each $y \in Y, \psi^{-1}(y) = \{x \in X : y \in \psi(x)\}$ is compactly open in X and $\{x \in X : \phi(x) \neq \emptyset\} = \{x \in X : \psi(x) \neq \emptyset\}$; (2) (ϕ_x, ψ_x, N_x) is called a $\mathcal{L}_{\theta,\mathcal{F}}$-majorant of ϕ at x if $\phi_x, \psi_x : X \to 2^Y$ and N_x is an open neighborhood of x in X such that (a) for each $z \in N_x, \phi(z) \subset \phi_x(z)$ and $\theta(z) \notin H\text{-}co(\phi_x(z))$, (b) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $H\text{-}co(\phi_x(z)) \subset Y$ and (c) for each $y \in Y$, $\psi_x^{-1}(y)$ is compactly open in X; (3) ϕ is called $\mathcal{L}_{\theta,F}$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $\mathcal{L}_{\theta,F}$-majorant (ϕ_x, ψ_x, N_x) of ϕ at x such that for any nonempty finite subset A of the set

$\{x \in X : \phi(x) \neq \emptyset\}$, we have

$$\{z \in \cap_{x\in A} N_x : \cap_{x\in A} H\text{-}co(\phi_x(z)) \neq \emptyset\}$$
$$= \{z \in \cap_{x\in A} N_x : \cap_{x\in A} H\text{-}co(\psi_x(z)) \neq \emptyset\}.$$

It is clear that every correspondence of class $\mathcal{L}_{\theta,F}$ is $\mathcal{L}_{\theta,F}$-majorized. We note that our notions of the correspondence ϕ being of class $\mathcal{L}_{\theta,F}$ and $\mathcal{L}_{\theta,F}$-majorized correspondence generalize the corresponding notions of Ding-Tan [10, 12], Ding-Kim-Tan [7], Tan-Yuan [25], Ding-Tarafdar [13] and Tulcea [35]. In this paper, we shall deal mainly with either the case (I) $X = Y$ and is a nonempty H-space and $\theta = I_X$, the identity mapping on X, or the case (II) $X = \Pi_{i\in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and X_j is an H-space. In both cases (I) and (II), we shall write $\mathcal{L}_F$ in place of $\mathcal{L}_{\theta,F}$.

3. Fixed Point Theorems

Lemma 3.1. *Let D be a topological space, $(X, \{\Gamma_A\})$ be an H-space and $F : D \to 2^X$ be such that $F^{-1}(y)$ is compactly open in D for each $y \in X$. Then the correspondence $H\text{-}co\, F : D \to 2^X$ satisfies that $(H\text{-}co\ F)^{-1}(y)$ is also compactly open in D for each $y \in X$.*

Proof. For any fixed $y \in X$ and for any nonempty compact subset C of D, let $x \in (H\text{-}co\, F)^{-1}(y) \cap C$, then $x \in C$ and $y \in H\text{-}co(F(x))$. By Lemma 1 of Tarafdar [29], there exists a finite set A of $F(x)$ such that $y \in H - co(A)$. Hence we have that $x \in F^{-1}(y) \cap C$ for each $y \in A$ and each $F^{-1}(y) \cap C$ is open in C since each $F^{-1}(y)$ is compactly open in D. Let $U = \cap_{y\in A}(F^{-1}(y) \cap C)$, then U is an open neighborhood of x in C. If $z \in U$, then $z \in C$ and $y \in F(z)$ for all $y \in A$ and hence $A \subset F(z)$ and $y \in H\text{-}co(A) \subset H\text{-}co(F(z))$. It follows that

$$z \in (H\text{-}co\, F)^{-1}(y) \cap C \text{ for all } z \in U.$$

Therefore $(H\text{-}co\, F)^{-1}(y) \cap C$ is open in C and so $(H\text{-}co\, F)^{-1}(y)$ is compactly open in D for each $y \in X$.

The following result is Corollary 1 of Ding-Tan [9].

Lamma 3.2. *Let $(X, \{\Gamma_A\})$ be an H-space and $G : X \to 2^X$ be such that*

(1) *G is an H-KKM mapping,*
(2) *for each $x \in X$, $G(x)$ is closed in X and for some $x_0 \in X$, $G(x_0)$ is compact.*

Then $\cap_{x\in X} G(x) \neq \emptyset$.

We shall first show the following main fixed point theorem.

Theorem 3.1. *Let* $(X, \{\Gamma_A\})$ *be an H-space,* $F, G : X \to 2^X$ *and* K *be a nonempty compact subset of* X *such that*

(1) *for each* $x \in X,\ F(x) \subset G(x)$,

(2) *for each* $y \in X,\ F^{-1}(y)$ *is compactly open in* X,

(3) *for each* $N \in \mathcal{F}(X)$*, there exists a compact weakly H-convex subset* L_N *of* X *with* $N \subset L_N$ *such that for each nonempty compact subset* C *of* X,

$$L_N \cap \cap_{x\in L_N} cl_C((X\backslash(H\text{-}co\, G)^{-1}(x)) \cap C) \subset K,$$

(4) *for each* $x \in K,\ F(x) \neq \emptyset$.

Then there exists $\hat{y} \in X$ *such that* $\hat{y} \in H\text{-}co(G(\hat{y}))$.

Proof. Suppose that the conclusion does not hold. Then for each $x \in X,\ x \notin H\text{-}co(G(x))$. By assumption (2) and Lemma 3.1, for each nonempty compact subset C of X and for each $y \in X$, $(H\text{-}co\, F)^{-1}(y) \cap C$ is open in C and $(H\text{-}co\, F)^{-1}(y)$ is compactly open in X for each $y \in X$.

For each $x \in X$, let

$$\begin{aligned} T(x) &= cl_X(X\backslash(H\text{-}co\, G)^{-1}(x)) \cap K, \text{ and} \\ S(x) &= (X\backslash(H\text{-}co\, F)^{-1}(x)) \cap K. \end{aligned}$$

We shall prove that the family $\{T(x) : x \in X\}$ has the finite intersection property. Let $N \in \mathcal{F}(X)$. By (3) there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ and hence $(L_N, \{\Gamma_A \cap L_N\})$ is a compact H-space. Define two mappings $T_0, S_0 : L_N \to 2^{L_N}$ by

$$\begin{aligned} T_0(x) &= cl_{L_N}((X\backslash(H\text{-}co\, G)^{-1}(x)) \cap L_N), \text{ and} \\ S_0(x) &= (X\backslash(H\text{-}co\, F)^{-1}(x)) \cap L_N \end{aligned}$$

for each $x \in L_N$. Then we have

(a) for each $x \in L_N,\ S_0(x)$ is closed in L_N since $(H\text{-}co\, F)^{-1}(x)$ is compactly open and L_N is compact,

(b) for each $x \in L_N,\ T_0(x) \subset S_0(x)$ by (1) and (a),

(c) for each $x \in L_N,\ T_0(x)$ is compact,

(d) T_0 is an H-KKM mapping. Indeed, it is enough to show that the mapping $T^* : L_N \to 2^{L_N}$ defined by

$$T^*(x) = (X \backslash (H\text{-}co\, G)^{-1}(x)) \cap L_N \text{ for each } x \in L_N,$$

is an H-KKM mapping. If this were false, then there exists $A \in \mathcal{F}(L_N)$ and a point $z \in H - co(A)$ such that

$$\begin{aligned} z &\notin \cup_{x\in A} T^*(x) \\ &= \cup_{x\in A}[(X\backslash(H\text{-}co\, G)^{-1}(x)) \cap L_N] \\ &= (X \backslash \cap_{x\in A} (H\text{-}co\, G)^{-1}(x)) \cap L_N. \end{aligned}$$

It follows that $z \in \cap_{x\in A}(H\text{-}co\, G)^{-1}(x)$ and hence $A \subset H\text{-}co(G(z))$. Therefore we have

$$z \in H\text{-}co(A) \subset H\text{-}co(G(z)),$$

which contradicts the fact that for each $x \in X,\ x \notin H\text{-}co((G(x))$. Hence T^* is an H-KKM mapping and so T_0 is also an H-KKM mapping. By applying Lemma 3.2, we have

$$\begin{aligned} \emptyset &\neq \cap_{x\in L_N} T_0(x) \\ &= \cap_{x\in L_N} cl_{L_N}((X\backslash(H\text{-}co\, G)^{-1}(x))\cap_{L_N}) \\ &\subset \cap_{x\in L_N} cl_X(x\backslash(H\text{-}co\, G)^{-1}(x)) \cap L_N. \end{aligned}$$

Take $\hat{y} \in \cap_{x\in L_N} cl_{L_N}((X\backslash(H\text{-}co\, G)^{-1}(x)) \cap L_N)$. By assumption (3), we must have $\hat{y} \in K$ and hence

$$\begin{aligned} \hat{y} &\in \cap_{x\in L_N} cl_X(X\backslash(H\text{-}co\, G)^{-1}(x)) \cap K \\ &= \cap_{x\in L_N} T(x) \\ &\subset \cap_{x\in N} T(x). \end{aligned}$$

That is, the family $\{T(x) : x \in X\}$ has the finite intersection property. By the compactness of $K, \cap_{x\in X} T(x) \neq \emptyset$. By (1) and (2), we have $T(x) \subset S(x)$ for each $x \in X$ and hence

$$\begin{aligned} \emptyset &\neq \cap_{x\in X} S(x) \\ &= \cap_{x\in X}(X\backslash(H\text{-}co\, F)^{-1}(x)) \cap K \\ &= K \cap (X \backslash \cup_{x\in X} (H\text{-}co\, F)^{-1}(x)) \\ &= K \backslash \cup_{x\in X} (H\text{-}co\, F)^{-1}(x). \end{aligned}$$

But, by the assumption (4), for each $x \in K,\ F(x) \neq \emptyset$ and hence

$$K \subset \cup_{x\in X}(H\text{-}co\, F)^{-1}(x))$$

which is a contradiction. Therefore the conclusion must hold.

Corollary 3.1. *Let X be a nonempty convex subset of a topological vector space and $F, G : X \to 2^X$ be such that*

(1) *for each $x \in X$, $F(x) \subset G(x)$,*
(2) *for each $y \in X$, $F^{-1}(y)$ is compactly open in X,*
(3) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$, there is an $x \in co(X_0 \cup \{y\})$ with $y \notin cl_C((X \backslash (co\, G)^{-1}(x)) \cap C)$ for any nonempty compact subset C of X,*
(4) *for each $x \in K$, $F(x) \neq \emptyset$.*

Then there exists a point $\hat{y} \in X$ such that $\hat{y} \in co(G(\hat{y}))$.

Proof. For each $A \in \mathcal{F}(X)$, let $\Gamma_A = co(A)$, then $(X, \{\Gamma_A\})$ is an H-space. Since X_0 is compact and convex, for each $N \in \mathcal{F}(X)$, $L_N = co(X_0 \cup N)$ is also a nonempty compact and convex subset of X with $N \subset L_N$. Now we claim that condition (3) implies condition (3) of Theorem 3.1. Indeed, if condition (3) of Theorem 3.1 does not hold, then there exist a finite set $N \in \mathcal{F}(X)$ and a nonempty compact subset C of X such that

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (co\, G)^{-1}(x)) \cap C) \not\subset K.$$

Thus there exists a point $y \in L_N \backslash K$ such that $y \in cl_C((X \backslash (co\, G)^{-1}(x)) \cap C)$ for all $x \in L_N$ which contradicts assumption (3) since $y \in L_N$ implies $co(X_0 \cup \{y\}) \subset L_N$. The conclusion of Corollary 3.1 follows from Theorem 3.1.

Remark 3.1. Corollary 3.1 is Theorem 3.1 of Ding-Tarafdar [13] which improves and generalizes Theorem 3" of Ding-Tan [11] (also see Lemma 1 of Ding-Tan [12]). Corollary 3.1 also generalizes Theorems 2.4', 2.4" and 2.4"' of Tan-Yuan [25], Theorem 1 of Ding-Tan [10], Theorem 1 of Tarafdar [26] (also Theorem 1 in [20]), Theorem 2 of Tarafdar [28] and Theorem 1.2 of Tarafdar [27].

Corollary 3.2. *Let $(x, \{\Gamma_A\})$ be an H-space, $G : X \to 2^X$ and K be a nonempty compact subset of X such that*

(1) *for each $y \in X$, $G^{-1}(y)$ is compactly open in X,*
(2) *for each $N \in \mathcal{F}(X)$, there exists a nonempty compact weakly H-convex subset L_N of X with $N \subset L_N$ such that for each $y \in L_N \backslash K$, there is an $x \in L_N$ with $x \in H - co(G(y))$*
(3) *for each $x \in K$, $G(x) \neq \emptyset$.*

Then there exists a point $\hat{y} \in X$ such that $\hat{y} \in H\text{-}co(G(\hat{y}))$.

Proof. Condition (2) implies that for each $N \in \mathcal{F}(X)$,

$$L_N \backslash K \subset \cup_{x \in L_N} (H\text{-}co\, G)^{-1}(x)$$

and hence

$$\begin{aligned} L_N \cap \cap_{x \in L_N} (X \backslash (H\text{-}co\, G)^{-1}(x)) &= L_N \cap (X \backslash \cup_{x \in L_N} (H\text{-}co\, G)^{-1}(x)) \\ &= L_N \backslash \cup_{x \in L_N} (H\text{-}co\, G)^{-1}(x) \subset K. \end{aligned}$$

Thus for any nonempty compact subset C of X, we have

$$L_N \cap \cap_{x \in L_N} ((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) \subset K \cap C \subset K.$$

By assumption (1) and Lemma 3.1, $(H\text{-}co\, G)^{-1}(x)$ is compactly open in X for each $x \in X$. It follows that each $(X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C$ is closed in C and so

$$cl_C((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) = X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C.$$

Hence we have

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) \subset K.$$

By applying Theorem 3.1 with $F = G$, there exists a point $\hat{y} \in X$ such that $\hat{y} \in H\text{-}co(G(\hat{y}))$.

Remark 3.2. Corollary 3.2 improves and generalizes Theorem 3" of Ding-Tan [11] (also see, Lemma 1 of Ding-Tan [12]).

The following are some equivalent versions of Theorem 3.1.

Theorem 3.2. *Let $(X, \{\Gamma_A\})$ be an H-space, $F, G : X \to 2^X$ and K be a nonempty compact subset such that*

(1) *for each $x \in X$, $F(x) \subset G(x)$ and $G(x)$ is H-convex,*
(2) *for each $y \in X$, $F^{-1}(y)$ is compactly open in X,*
(3) *for each $N \in \mathcal{F}(X)$ there exists a compact weakly H-convex subset L_N with $N \subset L_N$ such that for any nonempty compact subset C of X,*

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash G^{-1}(x)) \cap C) \subset K,$$

(4) *for each $y \in K$, $F(y) \neq \emptyset$.*

Then there exists a point $\hat{y} \in X$ such that $\hat{y} \in G(\hat{y})$.

Theorem 3.3. *Let $(X, \{\Gamma_A\})$ be an H-space, $G : X \to 2^X$ and K be a nonempty compact subset of X such that*

(1) *for each $y \in X$, $G^{-1}(y)$ contains a compactly open subset O_y (which may be empty) of X,*

(2) *for each $N \in \mathcal{F}(X)$, there exists a compact weakly H-convex subset L_N with $N \subset L_N$ such that for any nonempty compact subset C of X,*

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co)^{-1}(x)) \cap C) \subset K$$

and $K \subset \cup_{x \in X} O_y$.

Then there exists a point $\hat{y} \in X$ such that $\hat{y} \in H\text{-}co(G(\hat{y}))$.

Theorem 3.4. *Let $(X, \{\Gamma_A\})$ be an H-space, $G : X \to 2^X$ and K be a nonempty compact subset of X such that*

(1) *for each $x \in X$, $G(x)$ is H-convex,*

(2) *for each $y \in X$, $G^{-1}(y)$ contains a compactly open subset O_y (which may be empty) of X,*

(3) *condition (3) of Theorem 3.2 holds and $K \subset \cup_{x \in X} O_y$.*

Then there exists a point $\hat{y} \in X$ such that $\hat{y} \in G(\hat{y})$.

Sketch of Proofs:

(1) Theorem 3.1 $\Longleftrightarrow$ Theorem 3.2 and Theorem 3.3 $\Longleftrightarrow$ Theorem 3.4 are obvious.

(2) Theorem 3.1 $\Longrightarrow$ Theorem 3.3: Define a mapping $F : X \to 2^X$ by $F(x) = \{y \in X : x \in O_y\}$ for each $x \in X$.

(3) Theorem 3.3 $\Longrightarrow$ Theorem 3.1: For each $y \in X$. Iet $O_y = F^{-1}(y)$ where F is given in Theorem 3.1.

Remark 3.3. Theorem 3.2 generalizes Theorem 2.4' of Tan-Yuan [25] to H-space and the coercive condition (3) in Theorem 3.2 is weaker than the condition (d) in Theorem 2.4' of Tan-Yuan [25]. Theorem 3.3 improves and extends Theorem 3.3 of Ding-Tarafdar [13] and Theorem 2.4"" of Tan-Yuan [25] to H-space and in turn generalizes Theorem 1 of Tarafdar [26], Theorem 2 of Tarafdar [28] and Theorem 1.2 of Tarafdar [27] and the corresponding results of Ding-Tan [7], Metha-Tarafdar [20], Border [3], Browder [5] and Yannalis [37].

4. Minimax Inequalities

In this section, we shall show some minimax inequalities in H-space which generalize some recent results in the literature.

Let $(X, \{\Gamma_A\})$ be an H-space and $\psi : X \times X \to \mathbf{R} \cup \{\pm\infty\}$. $\psi(x, y)$ is said to be γ-H-diagonally quasiconcave (in short, γ-HDQCV) in x for some $\gamma \in \mathbf{R} \cup \{\pm\infty\}$, if for any $A \in \mathcal{F}(X)$ and for any $x_0 \in H-co(A)$, $\min_{x\in A} \psi(x, x_0) \le \gamma$. Clearly, the notion "$\psi(x, y)$ is γ-HDQCV in x" is a generalization of the notion "$\psi(x, y)$ is γ-DQCV in x" introduced by Zhou-Chen [40].

Lemma 4.1. *Let $(X, \{\Gamma_A\})$ be an H-space and $\psi : X \times X \to \mathbf{R} \cup \{\pm\infty\}$. Then the following conditions are equivalent:*

(1) *$\psi(x, y)$ is 0-HDQCV in x,*

(2) *the map $x \longmapsto F(x) = \{y \in X : \psi(x, y) \le 0\}$ is H-KKM,*

(3) *for each $x \in X$, $x \notin H\text{-}co(G(x))$ where $G(y) = \{x \in X : \psi(x, y) > 0\}$ for each $y \in X$.*

Proof. (1) $\Longrightarrow$ (2). Suppose (1) holds. If (2) does not hold, then there exists $A \in \mathcal{F}(X)$ such that $H-co(A) \not\subset \cup_{x\in A} F(x)$. Choose any $x_0 \in H\text{-}co(A)$ such that $x_0 \notin \cup_{x\in A} F(x)$. It follows that $\psi(x, x_0) > 0$ for all $x \in A$ and hence $\min_{x\in A} \psi(x, x_0) > 0$ which contradicts (1).

(2) $\Longrightarrow$ (1). Suppose (2) holds. If (1) is not true, then there exists $A \in \mathcal{F}(X)$ and $x_0 \in H\text{-}co(A)$ such that $\min_{x\in A} \psi(x, x_0) > 0$. Hence we have $x_0 \notin F(x)$ for all $x \in A$ so that $x_0 \notin \cup_{x\in A} F(x)$ which contradicts (2).

(1) $\Longrightarrow$ (3). Suppose (1) holds. If there exists a point $x_0 \in X$ such that $x_0 \in H\text{-}co(G(x_0))$. By Lemma 1 of Tarafdar [29], there exists $A \in \mathcal{F}(G(x_0))$ such that $x_0 \in H\text{-}co(A)$ and hence $\psi(x, x_0) > 0$ for all $x \in A$ which contradicts (1).

(3) $\Longrightarrow$ (1). Suppose (3) holds. If $\psi(x, y)$ is not 0-HDQCV in x, then there exist $A \in \mathcal{F}(X)$ and $x_0 \in H\text{-}co(A)$ such that $\psi(x, x_0) > 0$ for all $x \in A$. Hence we have $A \subset G(x_0)$ and so $x_0 \in H\text{-}co(A) \subset H\text{-}co(G(x_0))$. This contradicts (3).

Theorem 4.1. *Let $(X, \{\Gamma_A\})$ be an H-space, $\phi, \psi : X \times X \to \mathbf{R} \cup \{\pm\infty\}$ and K be a nonempty compact subset of X such that*

(a) *for each $(x, y) \in X \times X$, $\phi(x, y) \le \psi(x, y)$,*

(b) *for each $x \in X$, $y \longmapsto \phi(x, y)$ is lower semicontinuous (in short, l.s.c.) on each nonempty compact subset C of X,*

(c) *$\psi(x, y)$ is 0-HDQCV in x,*

(d) *for each* $N \in \mathcal{F}(X)$*, there exists a compact weakly H-convex subset* L_N *of* X *with* $N \subset L_N$ *such that for any nonempty compact subset* C *of* X*,*

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) \subset K,$$

where $G(y) = \{x \in X : \psi(x,y) > 0\}$ *and* $(H\text{-}co\, G)^{-1}(x) = \{y \in X : x \in H\text{-}co(G(y))\}$*.*

Then there exists $\hat{y} \in X$ *such that* $\phi(x,\hat{y}) \leq 0$ *for all* $x \in X$*.*

Proof. Define the maps $F, G : X \to 2^X$ by

$$F(y) = \{x \in X : \phi(x,y) > 0\},\ G(y) = \{x \in X : \psi(x,y) > 0\}$$

for each $y \in X$, respectively. Then conditions (1), (2) and (3) of Theorem 3.1 are satisfied by asumptions (a), (b) and (d). By assumption (c) and Lemma 4.1, the conclusion of Theorem 3.1 does not hold. Hence condition (4) of Theorem 3.1 is not true and so there exists a point $\hat{y} \in K$ such that $F(\hat{y}) = \emptyset$. This shows that $\phi(x,\hat{y}) \leq 0$ for all $x \in X$.

In fact, Theorem 3.1 and Theorem 4.1 are equivalent.

Proof of Theorem 3.1 *using Theorem* 4.1. Define $\phi, \psi : X \times X \to \mathbf{R}$ by

$$\phi(x,y) = \begin{cases} 1 & \text{if } x \in F(y), \\ 0 & \text{if } x \notin F(y); \end{cases}$$

$$\psi(x,y) = \begin{cases} 1 & \text{if } x \in G(y), \\ 0 & \text{if } x \notin G(y). \end{cases}$$

By assumptions (1), (2) and (3) of Theorem 3.1, conditions (a), (b) and (d) of Theorem 4.1 are satisfied. If the conclusion of Theorem 3.1 does not hold, by Lemma 4.1, $\psi(x,y)$ is 0-HDQCV in x. By applying Theorem 4.1, there exists $\hat{y} \in K$ such that $\psi(x,\hat{y}) \leq 0$ for all $x \in X$ and hence $x \notin G(\hat{y})$ for all $x \in X$. Therefore $G(\hat{y}) = \emptyset$ which contradicts assumption (4) of Theorem 3.1. Hence the conclusion of Theorem 3.1 must hold.

The following results are direct consequences of Theorem 4.1.

Corollary 4.1. *Let* X *be a nonempty convex subset of a topological vector space and* $\phi, \psi : X \times X \to \mathbf{R} \cup \{\pm\infty\}$ *be such that*

(a) *for each* $(x,y) \in X \times X$, $\phi(x,y) \leq \psi(x,y)$*,*

(b) *for each* $x \in X$, $y \longmapsto \phi(x,y)$ *is l.s.c. on each nonempty compact subset* C *of* X*,*

(c) *$\psi(x, y)$ is 0-DQCV in x, (see,* [40]*)*

(d) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$, there is an $x \in co(X_0 \cup \{y\})$ satisfying $y \notin cl_C((X \backslash (co\, G)^{-1}(x)) \cap C)$ for any nonempty compact subset C of X, where $G(y) = \{x \in X : \psi(x, y) > 0\}$ for each $y \in X$ and $(co\, G)^{-1}(x) = \{y \in X : x \in co(G(y))\}$.*

Then there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$.

Proof. For each $A \in \mathcal{F}(X)$, let $\Gamma_A = co(A)$. Then $(X, \{\Gamma_A\})$ is an H-space. Assumptions (a), (b) and (c) imply that conditions (a), (b) and (c) of Theorem 4.1 are satisfied. For each $N \in \mathcal{F}(X)$, let $L_N = co(X_0 \cup N)$, then L_N is a nonempty compact convex subset of X with $N \subset L_N$. If condition (d) of Theorem 4.1 does not hold, then for some $N \in \mathcal{F}(X)$ and some nonempty compact subset C of X,

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (co\, G)^{-1}(x)) \cap C) \not\subset K.$$

Therefore there is a $y \in L_N \backslash K$ such that $y \in cl_C((X \backslash (co\, G)^{-1}(x)) \cap C)$ for each $x \in L_N$. This contradicts assumption (d) of the Corollary since $y \in L_N$ implies $co(X_0 \cup \{y\}) \subset L_N$. Hence condition (d) of Theorem 4.1 holds. The conclusion follows from Theorem 4.1.

Corollary 4.2. *Let X be a nonempty convex subset of a topological vector space and $\phi, \psi : X \times X \to \mathbf{R} \cup \{\pm\infty\}$ be such that*

(1) *conditions (a), (b) and (c) of Corollary 4.1 hold,*

(2) *there exist a nonempty closed and compact subset K of X and an $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \backslash K$.*

Then there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$.

Proof. If $X_0 = \{x_0\}$, then X_0 is a nonempty compact convex subset of X and $x_0 \in co(X_0 \cup \{y\})$ for any $y \in X \backslash K$. Let $G(y) = \{x \in X : \psi(x, y) > 0\}$ for each $y \in X$. Assumption (2) implies

$$X \backslash K \subset \{y \in X : \psi(x_0, y) > 0\} = G^{-1}(x_0) \subset (co\, G)^{-1}(x_0)$$

and hence $X \backslash (co\, G)^{-1}(x_0) \subset K$. It follows that

$$cl_C((X \backslash (co\, G)^{-1}(x)) \cap C) \subset K$$

for any nonempty compact subset C of X. Therefore for each $y \in X \backslash K$, there exists $x_0 \in co(X_0 \cup \{y\})$ such that $y \notin cl_C((X \backslash (co\, G)^{-1}(x_0) \cap C)$. The conclusion of the Corollary follows from Corollary 4.1.

Remark 4.1. Corollary 4.2 is Theorem 2.2 of Tan-Yuan [25]. Hence Theorem 4.1 improves and generalizes Theorem 2.2 of Tan-Yuan [25] to H-spaces.

The following results are equivalent formulations of Theorem 4.1.

Theorem 4.2. (First Geometric Form) *Let* $(X, \{\Gamma_A\})$ *be an H-space,* $B \subset D \subset X \times X$ *and* K *be a nonempty compact subset of* X *such that*

(1) *for each* $x \in X$, *the set* $\{y \in X : (x, y) \in B\}$ *is compactly open in* X,

(2) *for each* $A \in \mathcal{F}(X)$ *and for each* $y \in H\text{-}co(A)$, *there excists* $x \in A$ *such that* $(x, y) \notin D$,

(3) *for each* $N \in \mathcal{F}(X)$, *there exists a nonempty compact weakly H-convex subset* L_N *of* X *with* $N \subset L_N$ *such that for any nonempty compact subset* C *of* X,

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) \subset K,$$

where $G(y) = \{x \in X : (x, y) \in D\}$ *and* $(H\text{-}co\, G)^{-1}(x) = \{y \in X : x \in H\text{-}co(G(y))\}$.

Then there exists $\hat{y} \in K$ *such that* $\{x \in X : (x, \hat{y}) \in B\} = \emptyset$.

Theorem 4.3. (Second Geometric Form) *Let* $(X, \{\Gamma_A\})$ *be an H-space,* $M \subset L \subset X \times X$ *and* K *be a nonempty compact subset of* X *such that*

(1) *for each* $x \in X$, *the set* $\{y \in X : (x, y) \in L\}$ *is compactly closed in* X,

(2) *for each* $A \in \mathcal{F}(X)$ *and for each* $y \in H\text{-}co(A)$, *there exists* $x \in A$ *such that* $(x, y) \in M$,

(3) *for each* $N \in \mathcal{F}(X)$, *there exists a nonempty compact weakly H-convex subset* L_N *of* X *with* $N \in L_N$ *such that for any nonempty compact subset* C *of* X,

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) \subset K,$$

where $G(y) = \{x \in X : (x, y) \notin M\}$ *and* $(H\text{-}co\, G)^{-1}(x) = \{y \in X : x \in H\text{-}co(G(y))\}$.

Then there exists $\hat{y} \in K$ *such that* $X \times \{\hat{y}\} \subset L$.

Sketch of Proofs:

(1) Theorem 4.1 $\Longrightarrow$ Theorem 4.2. Let $\phi, \psi : X \times X \to \mathbf{R}$ be the characteristic functions of B and D, respectively.

(2) Theorem 4.2 $\Longrightarrow$ Theorem 4.1. Define $B = \{(x, y) \in X \times X : \phi(x, y) > 0\}$ and $D = \{(x, y) \in X \times X : \psi(x, y) > 0\}$.

(3) Theorem 4.2 $\Longrightarrow$ Theorem 4.3. Let $B = X \times X \backslash L$ and $D = X \times X \backslash M$.

(4) Theorem 4.3 $\Longrightarrow$ Theorem 4.2. Let $L = X \times X \backslash B$ and $M = X \times X \backslash D$.

Remark 4.2. Theorems 4.2 and 4.3 improve and generalize Theorems 2.2' and 2.2" of Tan-Yuan [25] and the corresponding results of Shih-Tan [22, 23], Yen [39] and Ky Fan [16].

5. Existence of Maximal Elements

Let X be a topological space and $T : X \to 2^X$ be a correspondence. A point $x_0 \in X$ is said to be a maximal element of T if $T(x_0) = \emptyset$.

The following result is an equivalent formulation of Theorem 3.1.

Theorem 5.1. *Let $(X, \{\Gamma_A\})$ be an H-space, $F, G : X \to 2^X$ and K be a nonempty compact subset of X such that*

(1) *for each $x \in X$, $F(x) \subset G(x)$ and for each $x \in K$, $x \notin H\text{-}co(G(x))$,*

(2) *for each $y \in X$, $F^{-1}(y)$ is compactly open in X,*

(3) *for each $N \in \mathcal{F}(X)$, there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ such that for any nonempty compact subset C of X,*
$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) \subset K.$$

Then there exists $\hat{y} \in K$ such that $F(\hat{y}) = \emptyset$, i.e. $\hat{y}$ is a maximal element of F.

Remark 5.1. Theorem 5.1 improves and generalizes Theorem 2.2"' of Tan-Yuan [25] to H-space.

Theorem 5.2. *Let $(X, \{\Gamma_A\})$ be an H-space, $G : X \to 2^X$ be of class $\mathcal{L}_F$ and K be a nonempty compact subset of X. Suppose that for each $N \in \mathcal{F}(X)$, there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ such that for any nonempty compact subset C* of *X,*

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\, G)^{-1}(x)) \cap C) \subset K.$$

Then there exists $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$.

Proof. Since G is of class $\mathcal{L}_F$, we have

(a) for each $x \in X$, $x \notin H\text{-}co(G(x))$,

(b) there exists a correspondence $F : X \to 2^X$ such that (1) for each $x \in X, F(x) \subset G(x)$; (2) for each $y \in X$, $F^{-1}(y)$ is compactly open in X; (3) $\{x \in X : F(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}$.

Suppose $G(x) \neq \emptyset$ for all $x \in K$, then by (3), $F(x) \neq \emptyset$ for each $x \in K$. By applying Theorem 3.1, there exists a point $\hat{y} \in K$ such that $\hat{y} \in H - co(G(\hat{y}))$ which contradicts (a). Hence there must exist an $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$.

Remark 5.2. Theorem 5.2 generalizes Theorem 4.1 of Ding-Tarafdar [13], Theorems 3 and 4 of Ding-Tan [10] and Theorem 3.2 of Tan-Yuan [25] to H-spaces.

Lemma 5.1. *Let X be a regular topological space, Y be a nonempty subset of an H-space $(E, \{\Gamma_A\})$. Let $\theta : X \to E$ and $P : X \to 2^Y$ be $\mathcal{L}_{\theta,F}$-majorized. If each open subset of X containing the set $B = \{x \in X : P(x) \neq \emptyset\}$ is paracompact, then there exists a correspondence $\phi : X \to 2^Y$ of class $\mathcal{L}_{\theta,F}$ such that $P(x) \subset \phi(x)$ for all $x \in X$.*

Proof. Since P is $\mathcal{L}_{\theta,F}$-majorized, for each $x \in B$, let N_x be an open neighborhood of x in X and $\psi_x, \phi_x : X \to 2^Y$ be such that

(1) for each $z \in N_x, P(z) \subset \phi_x(z)$ and $\theta \notin H\text{-}co(\phi_x(z))$,

(2) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $H\text{-}co(\phi_x(z)) \subset Y$,

(3) for each $y \in Y$, $\psi_x^{-1}(y)$ is compactly open in X,

(4) for each $A \in \mathcal{F}(B)$,

$$\{z \in \cap_{x\in A} N_x : \cap_{x\in A} H\text{-}co(\phi_x(z)) \neq \emptyset\}$$
$$= \{z \in \cap_{x\in A} N_x : \cap_{x\in A} H\text{-}co(\psi_x(z)) \neq \emptyset\}.$$

Since X is regular, for each $x \in B$, there exists an open neighborhood G_x of x in X such that $cl_X G_x \subset N_x$. Let $G = \cup_{x\in B} G_x$, then G is open in X which contains B so that G is paracompact by the assumption. By Theorem VIII.1.4 of Dugundji [15, p. 162], the open covering $\{G_x\}$ of G has an open precise neighborhood-finite refinement $\{G'_x\}$. For each $x \in B$, we define $\psi'_x, \phi'_x : G \to 2^Y$ by

$$\psi'_x(z) = \begin{cases} H\text{-}co(\psi_x(z)) & \text{if } z \in G \cap cl_X G'_x, \\ Y & \text{if } z \notin G \cap cl_X G'_x, \end{cases}$$

$$\phi'_x(z) = \begin{cases} H\text{-}co(\phi_x(z)) & \text{if } z \in G \cap cl_X G'_x, \\ Y & \text{if } z \notin G \cap cl_X G'_x. \end{cases}$$

Then we have

(i) for each $z \in G$, $\psi'_x(z) \subset \phi'_x(z)$ by (2),

(ii) $\{z \in G : \psi'_x(z) \neq \emptyset\} = \{z \in G : \phi'_x(z) \neq \emptyset\}$ by (4), and

(iii) for each $y \in Y$,

$$\begin{aligned}(\psi'_x)^{-1}(y) &= \{z \in G : y \in \psi'_x(z)\}\\ &= \{z \in G \cap cl_X G'_x : y \in \psi'_x(z)\} \cup \{z \in G \backslash cl_X G'_x : y \in \psi'_x(z)\}\\ &= \{z \in G \cap cl_X G'_x : y \in H\text{-}co(\psi_x(z))\} \cup \{z \in G \backslash cl_X G'_x : y \in Y\}\\ &= [(G \cap cl_X G'_x) \cap (H\text{-}co\,\psi_x)^{-1}(y)] \cup (G \backslash cl_X G'_x)\\ &= (G \cap (H\text{-}co\,\psi_x)^{-1}(y)) \cup (G \backslash cl_X G'_x).\end{aligned}$$

It follows from (3) and Lemma 3.1 that for each nonempty compact subset C of X, $(\psi'_x)^{-1}(y) \cap C = (G \cap (H\text{-}co\,\psi_x)^{-1}(y) \cap C) \cup ((G \backslash cl_X G'_x) \cap C)$ is open in C.

Now define $\psi, \phi : X \to 2^Y$ by

$$\psi(z) = \begin{cases} \cap_{x \in B} \psi'_x(z) & \text{if } z \in G,\\ \emptyset & \text{if } z \in X \backslash G;\end{cases}$$

$$\phi(z) = \begin{cases} \cap_{x \in B} \phi'_x(z) & \text{if } z \in G,\\ \emptyset & \text{if } z \in X \backslash G.\end{cases}$$

Let $z \in X$ be given. Clearly, (2) implies $\psi(z) \subset \phi(z)$ and $H\text{-}co(\phi(z)) \subset Y$. If $z \in X \backslash G$, then $\phi(z) = \emptyset$ so that $\theta(z) \notin H\text{-}co(\phi(z))$; if $z \in G$, then $z \in G \cap cl_X G'_x$ for some $x \in B$ so that $\phi'_x(z) = H\text{-}co(\phi_x(z))$ and hence $\phi(z) \subset H$-$co(\phi_x(z))$. As $\theta(z) \notin H\text{-}co(\phi'_x(z))$ by (1) we must also have $\theta(z) \notin H\text{-}co(\phi(z))$. Therefore $\theta(z) \notin H\text{-}co(\phi(z))$ for all $z \in X$. Now we show that for each $y \in Y$, $\psi^{-1}(y)$ is compactly open in X. Indeed, let $y \in Y$ be such that $\psi^{-1}(y) \neq \emptyset$ and C be a nonempty compact subset of X; fix an arbitrary $u \in \psi^{-1}(y) \cap C = \{z \in X : y \in \psi(z)\} \cap C = \{z \in G : y \in \psi(z)\} \cap C$. Since $\{G'_x\}$ is a neighborhood- finite refinement, there exists an open neighborhood M_u, of u in G such that $\{x \in B : M_u \cap cl_X G'_x \neq \emptyset\} = \{x_1, \cdots, x_n\}$. Note that for each $x \in B$ with $x \notin \{x_1, \cdots, x_n\}, \emptyset = M_u \cap G'_x = M_u \cap cl_X G'_x$ so that $\psi'_x(z) = Y$ for all $z \in M_u$. Thus we have

$$\psi(z) = \cap_{x \in B} \psi'_x(z) = \cap_{i=1}^n \psi'_{x_i}(z)$$

for all $z \in M_u$. It follows that

$$\begin{aligned}\psi^{-1}(y) &= \{z \in X : y \in \psi(z)\} = \{z \in G : y \in \cap_{x \in B} \psi'_x(z)\}\\ &\subset \{z \in M_u : y \in \cap_{x \in B} \psi'_x(z)\}\\ &= \{z \in M_u : y \in \cap_{i=1}^n \psi'_{x_i}(z)\}\\ &= M_u \cap \{z \in G : y \in \cap_{i=1}^n \psi'_{x_i}(z)\}\\ &= M_u \cap [\cap_{i=1}^n (\psi'_{x_i})^{-1}(y)].\end{aligned}$$

But then $M'_u = M_u \cap [\cap_{i=1}^n (\psi'_{x_i})^{-1}(y)] \cap C$ is an open neighborhood of u in C such that $M'_u \subset \psi^{-1}(y) \cap C$ since $(\psi'_{x_i})^{-1}(y)$ is compactly open in X. This shows that for each $y \in Y$, $\psi^{-1}(y)$ is compactly open in X. Next we claim $\{z \in X : \phi(z) \neq \emptyset\} = \{z \in X : \psi(z) \neq \emptyset\}$. Indeed, for each $w \in X$ with $\phi(w) \neq \emptyset$, we must have $w \in G$. Since $\{G'_x\}$ is neighborhood- finite, the set $\{x \in B : w \in cl_X G'_x\} = \{x'_1, \cdots, x'_m\}$ is finite so that if $x \notin \{x'_1, \cdots, x'_m\}$, then $w \notin cl_X G'_x$ and $\phi'_x(w) = \psi'_x(w) = Y$. Thus we have

$$\phi(w) = \cap_{x \in B} \phi'_x(w) = \cap_{i=1}^m H\text{-}co(\phi'_{x'_i}(w)),$$
$$\psi(w) = \cap_{x \in B} \psi'_x(w) = \cap_{i=1}^m H\text{-}co(\psi'_{x'_i}(w)).$$

Since $w \in \cap_{i=1}^m cl_X G'_{x'_i} \subset \cap_{i=1}^m N_{x'_i}$, it follows form (4) that $\psi(w) \neq \emptyset$. Hence $\{z \in X : \phi(z) \neq \emptyset\} \subset \{z \in X : \psi(z) \neq \emptyset\}$. Conversely, (2) implies that $\{z \in X : \psi(z) \neq \emptyset\} \subset \{z \in X : \phi(z) \neq \emptyset\}$. Therefore

$$\{z \in X : \psi(z) \neq \emptyset\} = \{z \in X : \phi(z) \neq \emptyset\}.$$

This shows that ϕ is of class $\mathcal{L}_{\theta,F}$. To complete the proof, it remains to show that $P(z) \subset \phi(z)$ for each $z \in X$. Indeed, let $z \in X$ with $P(z) \neq \emptyset$. Note then $z \in G$. For each $x \in B$, if $z \in G \backslash cl_X G'_x$, then $\phi'_x(z) = Y \subset P(z)$ and if $z \in G \cap cl_X G'_x$, we have $z \in cl_X G'_x \subset cl_X G_x \subset N_x$ so that by (1), $P(z) \subset \phi_x(z) \subset \phi'_x(z)$. It follows that $P(z) \subset \phi'_x(z)$ for each $x \in B$ so that $P(z) \subset \cap_{x \in B} \phi'_x(z) = \phi(z)$.

Remark 5.2. Lemma 5.1 generalizes Lemma 2 of Ding-Tan [10, 12], Lemma 2 of Ding-Kim-Tan [7], Lemma 3.1 of Tan-Yuan [25] and Proposition 1 of Tulcea [35] to H-spaces.

As an application of Theorem 3.1 and Lemma 5.1, we shall now present the following result concerning the existence of a maximal element.

Theorem 5.3. *Let $(X, \{\Gamma_A\})$ be a paracompact H-space, $P : X \to 2^X$ be an $\mathcal{L}_F$- majorized correspondence and K be a nonempty compact subset of X. Suppose that for each $N \in \mathcal{F}(X)$, there exists a nonempty compactly weakly H-convex subset L_N of X with $N \subset L_N$ such that for any nonempty compact subset C of X,*

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\, P)^{-1}(x)) \cap C) \subset K.$$

Then there exists a $\hat{y} \in K$ such that $P(\hat{y}) = \emptyset$.

Proof. We first prove that for each $x \in X \backslash K$, $P(x) \neq \emptyset$. Indeed, for $x \in X \backslash K$, $L_{\{x\}}$ is a nonempty compact subset of X with $\{x\} \subset L_{\{x\}}$ and by the assumption, we must have

$$x \notin \cap_{x \in L_{\{x\}}} cl_{L_{\{x\}}}((X \backslash (H\text{-}co\,P)^{-1}(x)) \cap L_{\{x\}}).$$

It follows that there exists a $y \in L_{\{x\}}$ such that $x \in (H\text{-}co\,P)^{-1}(y)$ and hence $y \in H\text{-}co(P(x))$ and $P(x) \neq \emptyset$. Now suppose that the conclusion of the theorem does not hold. Then $P(x) \neq \emptyset$ for all $x \in X$ and hence the set $\{x \in X : P(x) \neq \emptyset\} = X$ is paracompact. By Lemma 5.1, there exists a correspondence $\phi : X \to 2^X$ of class $\mathcal{L}_F$ such that $P(x) \subset \phi(x)$ for each $x \in X$. It follows that for each $N \in \mathcal{F}(X)$ and for any nonempty compact subset C of X,

$$\begin{aligned} &L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\,\phi)^{-1}(x)) \cap C) \\ &\quad \subset L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\,P)^{-1}(x)) \cap C) \subset K. \end{aligned}$$

By Theorem 5.2, there exists an $\hat{y} \in K$ such that $\phi(\hat{y}) = \emptyset$ so that $P(\hat{y}) = \emptyset$. which is a contradiction. Therefore there exists a point $\hat{y} \in K$ such that $P(\hat{y}) = \emptyset$.

Remark 5.3. Theorem 5.3 generalizes Theorem 1 of Ding-Tan [12] and Theorem 3.3 of Tan-Yuan [25] to H-spaces and in turn generalizes Theorem 5 of Ding-Tan [10], Corollary 1 of Borglin-Keiding [4]. Theorem 2.2 of Toussaint [34], Theorem 2 of Tulcea [35], Theorem 5.1 and Corollary 5.1 of Yannelis-Prabhakar [38] and Theorem 2 of Yannelis [37] to H-spaces.

6. Equilibrium Existence Theorems

Let I be a (possibly infinite) set of agents. For each $i \in I$, let its choice or strategy set X_i be a nonempty subset of a topological space. Let $X = \Pi_{i \in I} X_i$. For each $i \in I$, let $P_i : X \to 2^{X_i}$ be a preference correspondence. Following the notion of Gale and Mas-Colell [17], the collection $\Gamma = (X_i, P_i)_{i \in I}$ will be called a qualitative game. A point $x \in X$ is said to be an equilibrium of the game Γ if $P_i(x) = \emptyset$ for all $i \in I$. For each $i \in I$, let A_i be a subset of X_i. Then for each fixed $k \in I$, we define

$$\prod_{j \in I, j \neq k} A_j \otimes A_k = \{x = (x_i)_{i \in I} : x_i \in A_i \text{ for all } i \in I\}.$$

An abstract economy (=generalized game) is a family of quadruples $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ where I is a (finite or infinite) set of agents (players) such that for each $i \in I, X_i$ is a nonempty subset of a topological space and $A_i, B_i : X = \Pi_{i \in I} X_i \to 2^{X_i}$ are constraint correspondences and $P_i : X \to 2^{X_i}$

is a preference correspondence. When $I = \{1, \cdots, N\}$ where N is a positive integer, $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ is also called an N-person game. An equilibrium point of Γ is a point $x \in X$ such that for each $i \in I$, $x_i \in \overline{B}_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$. we remark that when each X_i is a nonempty subset of a topological vector space and $\overline{B}_i(x) = cl_{X_i} B_i(x)$ (which is the case when B_i has a closed graph; in particular, when $cl\ B_i$ is u.s.c. with closed values), the definitions of an abstract economy and an equilibrium point coincide with that of Ding-Kim-Tan [7] and Ding-Tan [10, 12]; and if in addition, $A_i = B_i$ for each $i \in I$, the definitions of an abstract economy and an equilibrium point coincide with the standard definitions; e.g. in Borglin Keiding [4], Tulcea [35, 36] and Yannelis-Prabhakar [38].

As an application of Theorem 5.2, we shall prove the following equilibrium existence theorem for a one-person game.

Theorem 6.1. *Let $(X, \Gamma_A\})$ be an H-space, $A, B, P : X \to 2^X$ and K be a nonempty compact subset of X such that*

(1) *for each $x \in X$, $H\text{-}co(A(x)) \subset \overline{B}(x)$,*

(2) *for each $y \in X$, $A^{-1}(y)$ is compactly open in X,*

(3) *$A \cap P$ is of class $\mathcal{L}_F$,*

(4) *for each $N \in \mathcal{F}(X)$, there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ such that for any nonempty compact subset C of X,*

$$L_N \cap \cap_{x\in L_N} cl_C((X \backslash (H\text{-}co\, A \cap P)^{-1}(x)) \cap C) \subset K,$$

(5) *for each $x \in K$, $A(x) \neq \emptyset$.*

Then there exists a point $\hat{x} \in K$ such that $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. If $M = \{x \in X : x \notin \overline{B}(x)\}$, then M is open in X. Define $\phi : X \to 2^X$ by

$$\phi(x) = \begin{cases} A(x) \cap P(x) & \text{if } x \notin M, \\ A(x) & \text{if } x \in M. \end{cases}$$

Since $A \cap P$ is of class $\mathcal{L}_F$, for each $x \in X$, $x \notin H\text{-}co(A(x) \cap P(x))$ and there exists a correspondence $\beta : X \to 2^X$ such that

(a) for each $x \in X$, $\beta(x) \subset A(x) \cap P(x)$,

(b) for each $y \in X$, $\beta^{-1}(y)$ is compactly open in X, and

(c) $\{x \in X : \beta(x) \neq \emptyset\} = \{x \in X : A(x) \cap P(x) \neq \emptyset\}$.

Now define a correspondence $\psi : X \to 2^X$ by

$$\psi(x) = \begin{cases} \beta(x) & \text{if } x \notin M, \\ A(x) & \text{if } x \in M. \end{cases}$$

Clearly for each $x \in X$, $\psi(x) \subset \phi(x)$ and $\{x \in X : \psi(x) \neq \emptyset\} = \{x \in X : \phi(x) \neq \emptyset\}$ by (c). For each $y \in X$, it is easy to see that $\psi^{-1}(y) = (M \cup \beta^{-1}(y)) \cap A^{-1}(y)$ and is compactly open in X by (2) and (b). For each $x \in X$, if $x \in M$, then $x \notin \overline{B}(x)$, it follows from (1) that $x \notin H\text{-}co(\phi(x))$; if $x \notin M$, then $x \notin H\text{-}co(A(x) \cap P(x)) = H\text{-}co(\phi(x))$ since $x \notin H\text{-}co(A(x) \cap P(x))$ for all $x \in X$. This shows that ϕ is of class $\mathcal{L}_F$. By (4) and the definition of ϕ, for each $N \in \mathcal{F}(X)$ and for any nonempty compact subset C of X, we have

$$L_N \cap \cap_{x \in L_N} cl_C((X \backslash (H\text{-}co\,\phi)^{-1}(x)) \cap C) \subset K.$$

By applying Theorem 5.2, there exists a point $\hat{x} \in K$ such that $\phi(\hat{x}) = \emptyset$. Since for each $x \in K$, $A(x) \neq \emptyset$ and the assumption (4) implies that $A(x) \neq \emptyset$ for all $x \in X \backslash K$. Hence $A(x) \neq \emptyset$ for all $x \in X$ so that we must have $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Remark 6.1. Theorem 6.1 improves and generalizes Theorem 5.1 of Ding-Tarafdar [13], Theorem 4 of Ding-Tan [11], Theorem 4.1 of Tan-Yuan [25] and Theorem 6 of Ding-Tan [10] to H-spaces.

As an application of Theorem 5.3, we shall show the following equilibrium existence theorem for a noncompact qualitative game in H-spaces.

Theorem 6.2. *Let $\Gamma = (X_i, P_i)_{i \in I}$, be a qualitative game such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

(1) *for each $i \in I$, $(X_i, \{\Gamma_{A_i}\})$ is an H-space,*
(2) *for each $i \in I$, $P_i : X \to 2^{X_i}$ is an $\mathcal{L}_F$-majorized correspondence,*
(3) $\cup_{i \in I}\{x \in X : P_i(x) \neq \emptyset\} = \cup_{i \in I} int_X\{x \in X : P_i(x) \neq \emptyset\}$,
(4) *For each $N \in \mathcal{F}(X)$ there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ and there exists a nonempty compact subset K of X such that for each $y \in X \backslash K$, there is an $x \in L_N$ with $y \notin cl_C((X \backslash (H\text{-}co\,P_i)^{-1}(x_)) \cap C)$ for each $i \in I$ and for any nonempty compact subset C of X.*

Then Γ has an equilibrium point in K.

Proof. For each $x \in X$, let $I(x) = \{i \in I : P_i(x) \neq \emptyset\}$. For each $i \in I$, define a correspondnce $P'_i : X \to 2^X$ by $P'_i(x) = \Pi_{j \in I, j \neq i} X_j \otimes P_i(x)$.

Furthermore, define the correspondence $P : X \to 2^X$ by

$$P(x) = \begin{cases} \cap_{i\in I(x)} H\text{-}co(P_i'(x)) & \text{if } I(x) \neq \emptyset, \\ \emptyset & \text{if } I(x) = \emptyset. \end{cases}$$

Then for each $x \in X$, $P(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. Now we prove that P is an $\mathcal{L}_F$-majorized correspondence. For each $x \in X$ with $P(x) \neq \emptyset$, by (3), let $i(x) \in I$ be such that $x \in int_X\{z \in X : P_i(x)(z) \neq \emptyset\}$ and by (2), $N(x)$ be an open neighborhood of x in X and $\phi_i(x), \psi_i(x) : X \to 2^{X_i}$ be correspondences such that

(i) for each $z \in N(x)$, $P_{i(x)}(z) \subset \phi_{i(x)}(z)$ and $z_{i(x)} \notin H\text{-}co(\phi_{i(x)}(z))$,

(ii) for each $z \in X$, $\psi_{i(x)}(z) \subset \phi_{i(x)}(z)$,

(iii) for each $y \in X_{i(x)}$, $\psi^{-1}_{i(x)}(y)$ is compactly open in X,

(iv) for each finite subset $\{x_1, \cdots, x_n\}$ of the set $\{x \in X : P(x) \neq \emptyset\}$ with $i(x_1) = \cdots = i(x_n)$,

$$\begin{aligned} &\{z \in \cap_{j=1}^n N(x_j) : \cap_{j=1}^n H\text{-}co(\psi_{i(x_j)}(z)) \neq \emptyset\} \\ &\quad = \{z \in \cap_{j=1}^n N(x_j) : \cap_{j=1}^n H\text{-}co(\phi_{i(x_j)}(z)) \neq \emptyset\}. \end{aligned}$$

Without loss of generality we may assume that $N(x) \subset int_X\{z \in X : P_{i(x)}(z) \neq \emptyset\}$ so that $P_{i(x)}(z) \neq \emptyset$ and hence $i(x) \in I(z)$ for all $z \in N(x)$. Let $x \in X$ be such that $P(x) \neq \emptyset$; define $\phi'_{i(x)}, \psi'_{i(x)} : X \to 2^X$ by

$$\begin{aligned} \psi'_{i(x)}(z) &= \prod_{j\in I, j\neq i(x)} X_j \otimes H\text{-}co(\psi_{i(x)}(z)), \\ \phi'_{i(x)}(z) &= \prod_{j\in I, j\neq i(x)} X_j \otimes H\text{-}co(\phi_{i(x)}(z)) \end{aligned}$$

for each $x \in X$. Then we have

(a) for each $z \in N(x)$, by (i),

$$\begin{aligned} P(z) &= \cap_{i\in I(z)} H\text{-}co(P_i'(z)) \subset H\text{-}co(P'_{i\in i(x)}(z)) \\ &= \prod_{j\in I, j\neq i(x)} X_j \otimes H\text{-}co(P_{i(x)}(z)) \\ &\subset \prod_{j\in I, j\neq i(x)} X_j \otimes H\text{-}co(\phi_{i(x)}(z)) \\ &= \phi'_{i(x)}(z) \end{aligned}$$

and $z_{i(x)} \notin H\text{-}co(\phi'_{i(x)}(z))$;

(b) for each $z \in X$, by (ii), $\psi'_{i(x)}(z) \subset \phi'_{i(x)}(z)$;

(c) for each $y \in X, (\psi'_{i(x)})^{-1}(y) = (H\text{-}co\,\psi^{-1}_{i(x)}(y_{i(x)})$ is compactly open in X by (iii) and Lemma 3.1;

(d) for any finite set A of $\{x \in X : P(x) \neq \emptyset\}$, let $\cup\{I(x) : x \in A\} = \{i_1, \cdots, i_k\}$ where $i_1, \cdots, i_k$ are all distinct and for each $t = 1, \cdots, k$ let $A_t = \{x \in A : i(x) = i_t\}$. Note that for each $z \in X$,

$$\begin{aligned} \cap_{x\in A} H\text{-}co(\psi'_{i(x)}(z)) &= \cap_{x\in A} \prod_{j\in I, j\neq i(x)} X_j \otimes H\text{-}co(\psi_{i(x)}(z)) \\ &= \cap_{t=1}^{k} \prod_{j\in I, j\neq i_t} X_j \otimes (\cap_{x\in A_t} H\text{-}co(\psi_{i(x)}(z)) \end{aligned}$$

so that for each $z \in \cap_{x\in A} N(x)$, if $\cap_{x\in A} H\text{-}co(\psi'_{i(x)}(z)) = \emptyset$, then there exists $m \in \{1, \cdots, k\}$ such that $\cap_{x\in A_m} H\text{-}co(\psi_{i(x)}(z)) = \emptyset$; it follows from (iv) that $\cap_{x\in A_m} H\text{-}co(\phi_{i(x)}(z)) = \emptyset$. Thus

$$\begin{aligned} \cap_{x\in A} H\text{-}co(\phi'_{i(x)}(z)) &= \cap_{x\in A} \prod_{j\in I, j\neq i(x)} X_j \otimes H\text{-}co(\phi_{i(x)}(z)) \\ &= \cap_{t=1}^{k} \prod_{j\in I, j\neq i_t} X_j \otimes (\cap_{x\in A_t} H\text{-}co(\phi_{i(x)}(z)) \\ &= \emptyset. \end{aligned}$$

From this fact together with (b), we conclude that

$$\begin{aligned} &\{z \in \cap_{x\in A} N(x) : \cap_{x\in A} H\text{-}co(\psi_{i(x)}(z)) \neq \emptyset\} \\ &\quad = \{z \in \cap_{x\in A} N(x) : \cap_{x\in A} H\text{-}co(\phi_{i(x)}(z)) \neq \emptyset\}. \end{aligned}$$

This shows that P is $\mathcal{L}_F$-majorized. By (4), for each $y \in X\backslash K$, there exists an $x \in L_N$ such that for any nonempty compact subset C of X

$$\begin{aligned} y &\notin \cup_{i\in I} cl_C((X\backslash (H\text{-}co\,P_i)^{-1}(x_i)) \cap C) \\ &= cl_C(\cup_{i\in I} (X\backslash (H\text{-}co\,P_i)^{-1}(x_i)) \cap C) \\ &= cl_C((X\backslash \cap_{i\in I} (H\text{-}co\,P_j)^{-1}(x_i)) \cap C), \end{aligned}$$

and $P_i(y) \neq \emptyset$ for each $y \in X\backslash K$ and $i \in I$. Hence $I(y) = I$ for each $y \in X\backslash K$. By the definition of P'_i and P, we have

$$H\text{-}co(P(y)) = \cap_{i\in I} H\text{-}co(P'_i(y)) = \prod_{i\in I} H\text{-}co(P_i(y)).$$

It follows that

$$\begin{aligned} (H\text{-}co\,P)^{-1}(x) &= \{y \in X : x \in H\text{-}co(P(y))\} \\ &= \{y \in X : x \in \prod_{i\in I} H\text{-}co(P_i(y))\} \\ &= \{y \in X : y \in (H\text{-}co\,P_i)^{-1}(x_i) \text{ for each } i \in I\} \\ &= \cap_{i\in I} (H\text{-}co\,P_i)^{-1}(x_i). \end{aligned}$$

Thus we have that for each $y \in X\backslash K$, there exists an $x \in L_N$ such that $y \notin cl_C((X\backslash(H\text{-}co\,P)^{-1}(x)) \cap C)$ for any nonempty compact subset C of X. It follows that for each $N \in \mathcal{F}(X)$ and for any nonempty compact subset C of X,

$$L_N \cap \cap_{x \in L_N} cl_C((X\backslash(H\text{-}co\,P)^{-1}(x)) \cap C) \subset K.$$

By applying Theorem 5.3, there exists a point $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$. This implies $I(\hat{x}) = \emptyset$ and therefore $P_i(\hat{x}) = \emptyset$ for each $i \in I$, i.e. $\hat{x}$ is an equilibrium point of Γ.

Remark 6.2. Theorem 6.2 improves and generalizes Theorem 5.2 of Ding-Tarafdar [13], Theorem 3 of Ding-Tan [12] Theorem 4.2 of Tan-Yuan [25] and Theorem 7 of Ding-Tan [10] to H-spaces. In Theorem 6.2, if for each $i \in I, (X_i, \{\Gamma_{A_i}\})$ is compact H-space, then $X = \prod_{i \in I} X_i$ is also a compact H-space. By letting $L_N = K = X$ for each $N \in \mathcal{F}(X)$, the condition (4) of Theorem 6.2 is satisfied trivially. Hence Theorem 6.2 also improves and generalizes Theorem 2.4 of Toussaint [34] and Proposition 3 of Tulcea [35] in several aspects which in turn generalize the fixed point theorem of Gale and Mas-Colell [17].

As an application of Theorem 6.2, we shall prove the following equilibrium existence theorem for a noncompact abstract economy in H-spaces.

Theorem 6.3. Letr $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ *be an abstract economy such that* $X = \prod_{i \in I} X_i$ *is paracompact. Suppose that for each* $i \in I$*,*

(1) $(X_i, \{\Gamma_{A_i}\})$ *is an H-space,*
(2) *for each* $x \in X$*,* $A_i(x)$ *is nonempty and* $H\text{-}co(A_i(x)) \subset B_i(x)$*,*
(3) *for each* $y \in X$*,* $A_i^{-1}(y)$ *is compactly open in* X*,*
(4) $A_i \cap P_i$ *is* $\mathcal{L}_F$*-majorized,*
(5) $E_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ *is open in* X*,*
(6) *for each* $N \in \mathcal{F}(X)$*, there exists a compact weakly H-convex subset* L_N *of* X *with* $N \subset L_N$ *and there exists a nonempty compact subset* K *of* X *such that for each* $y \in X\backslash K$*, there is an* $x \in L_N$ *satifying* $y \notin cl_C((X\backslash(H\text{-}co\,A_i \cap P_i)^{-1}(x_i)) \cap C)$ *for each* $i \in I$ *and for any nonempty compact subset* C *of* X*.*

Then Γ *has an equilibrium point in* K*.*

Proof. For each $i \in I$, let $F_i = \{x \in X : x_i \notin \overline{B}_i(x)\}$, then F_i is open in X. For each $i \in I$, define the correspondence $Q_i : X \to 2^{X_i}$ by

$$Q_i(x) = \begin{cases} (A_i \cap P_i)(x) & \text{if } x \notin F_i, \\ A_i(x) & \text{if } x \in F_i. \end{cases}$$

We shall prove that the qualitative game $\Gamma = (X_i, Q_i)_{i\in I}$ satisfies all the hypotheses of Theorem 6.2. For each $i \in I$, the set

$$\begin{aligned}\{x \in X : Q_i(x) \neq \emptyset\} &= \{x \in F_i : Q_i(x) \neq \emptyset\} \cup \{x \in X\backslash F_i : Q_i(x) \neq \emptyset\} \\ &= F_i \cup \{x \in X\backslash F_i : (A_i \cap P_i)(x) \neq \emptyset\} \\ &= F_i \cup [(X\backslash F_i) \cap E_i] \\ &= F_i \cup E_i\end{aligned}$$

is open in X and hence condition (3) of Theorem 6.2 is satisfied. By (4), for each $x \in E_i$, there exist an open neighborhood N_x of x in X and correspondences $\psi_x, \phi_x : X \to 2^{X_i}$ such that

(a) for each $z \in N_x$, $(A_i \cap P_i)(z) \subset \phi_x(z)$ and $z_i \notin H\text{-}co(\phi_x(z))$,

(b) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$,

(c) for each $y \in X$, $\psi_x^{-1}(y)$ is compactly open in X,

(d) for each nonempty finite set $A \subset E_i$,

$$\{z \in \cap_{x\in A} N_x : \cap_{x\in A} H\text{-}co(\psi_x(z)) \neq \emptyset\}$$
$$= \{z \in \cap_{x\in A} N_x : \cap_{x\in A} H\text{-}co(\phi_x(z)) \neq \emptyset\}.$$

Now for each $x \in X$ with $Q_i(x) \neq \emptyset$, let

$$M(x) = \begin{cases} F_i & \text{if } x \in F_i, \\ N_x & \text{if } x \notin F_i, \end{cases}$$

and define the correspondences Φ_x, $\Psi_x : X \to 2^{X_i}$ by

$$\Phi_x(z) = \begin{cases} \phi_x(z) & \text{if } z \notin F_i, \\ A_i(z) & \text{if } z \in F_i, \end{cases}$$

$$\Psi_x(z) = \begin{cases} \psi_x(z) & \text{if } z \notin F_i, \\ A_i(z) & \text{if } z \in F_i. \end{cases}$$

Then for each $x \in X$ with $Q_i(x) \neq \emptyset$, $M(x)$ is an open neighborhood of x in X such that

(i) for each $z \in M(x)$, $Q_i(z) \subset \Phi_x(z)$ and $z_i \notin H\text{-}co(\Phi_x(z))$ by (2) and (a),

(ii) for each $z \in X$, $\Psi_x(z) \subset \Phi_x(z)$ by (b),

(iii) for each $y \in X_i$,

$$\begin{aligned}\Psi_x^{-1}(y) &= \{z \in X\backslash F_i : y \in \Psi_x(z)\} \cup \{z \in F_i : y \in \Psi_x(z)\}\\ &= \{z \in X\backslash F_i : y \in \psi_x(z)\} \cup \{z \in F_i : y \in A_i(z)\}\\ &= [(X\backslash F_i) \cap \psi_x^{-1}(y)] \cup (F_i \cap A_x^{-1}(y))\\ &= [F_i \cup \psi_x^{-1}(y)] \cap A_x^{-1}(y)\end{aligned}$$

is compactly open in X by (3),(c) and F_i being open in X. Now let A be a finite subset of $\{x \in X : Q_i(x) \neq \emptyset\}$.

Then $A = A_1 \cup A_2$ where $A_1 = \{x \in A : x \in F_i\}$ and $A_2 = \{x \in A : x \notin F_i\}$.

Case 1. If $A_1 = \emptyset$, then by (d),

$$\begin{aligned}&\{z \in \cap_{x\in A} M(x) : \cap_{x\in A} H\text{-}co(\Psi_x(z)) \neq \emptyset\}\\ &= \{z \in \cap_{x\in A_2} M(x) \cap F_i : \cap_{x\in A_2} H\text{-}co(\Psi_x(z)) \neq \emptyset\}\\ &\quad \cup \{z \in \cap_{x\in A_2} M(x)\backslash F_i : \cap_{x\in A_2} H\text{-}co(\Psi_x(z)) \neq \emptyset\}\\ &= \{z \in \cap_{x\in A_2} M(x) \cap F_i : A_i(z) \neq \emptyset\}\\ &\quad \cup \{z \in \cap_{x\in A_2} M(x)\backslash F_i : \cap_{x\in A_2} H\text{-}co(\psi_x(z)) \neq \emptyset\}\\ &= \{z \in \cap_{x\in A_2} M(x) \cap F_i : A_i(z) \neq \emptyset\}\\ &\quad \cup \{z \in \cap_{x\in A_2} M(x)\backslash F_i : \cap_{x\in A_2} H\text{-}co(\phi_x(z)) \neq \emptyset\}\\ &= \{z \in \cap_{x\in A_2} M(x) \cap F_i : \cap_{x\in A_2} H\text{-}co(\Phi_x(z)) \neq \emptyset\}\\ &\quad \cup \{z \in \cap_{x\in A_2} M(x)\backslash F_i : \cap_{x\in A_2} H\text{-}co(\Phi_x(z)) \neq \emptyset\}\\ &= \{z \in \cap_{x\in A} M(x) : \cap_{x\in A} H\text{-}co(\Phi_x(z)) \neq \emptyset\}.\end{aligned}$$

Case 2. If $A_1 \neq \emptyset$, then

$$\begin{aligned}&\{z \in \cap_{x\in A} M(x) : \cap_{x\in A} H\text{-}co(\Psi_x(z)) \neq \emptyset\}\\ &= \{z \in \cap_{x\in A_1} M(x) \cap \cap_{x\in A_2} M(x) : \cap_{x\in A} H\text{-}co(\Psi_x(z)) \neq \emptyset\}\\ &= \{z \in F_i \cap \cap_{x\in A_2} M(x) : \cap_{x\in A} H\text{-}co(\Psi_x(z)) \neq \emptyset\}\\ &= \{z \in F_i \cap \cap_{x\in A_2} M(x) : \cap_{x\in A} H\text{-}co(\Phi_x(z)) \neq \emptyset\}\\ &= \{z \in \cap_{x\in A} M(x) : \cap_{x\in A} H\text{-}co(\Phi_x(z)) \neq \emptyset\},\end{aligned}$$

since $\Psi_x(z) = \Phi_x(z) = A_i(z)$ for each $z \in F_i$. This shows that for each $i \in I$, Q_i is $\mathcal{L}_F$-majorized.

Finally, by (6), for each $N \in \mathcal{F}(X)$ there exists a compact weakly H-convex subset L_N of X with $N \subset L_N$ and there exists a nonempty compact subset

K of X such that for each $y \in X\backslash K$, there is an $x \in L_N$ satisfying $y \notin cl_C((X\backslash(H\text{-}co\, Q_i)^{-1}(x_i)) \cap C)$ for each $i \in I$ and for any nonempty compact subset C of X. By Theorem 6.2, there exists a point $\hat{x} \in K$ such that $Q_i(\hat{x}) = \emptyset$ for all $i \in I$. By (2) and the definition of Q_i, this implies that for each $i \in I$, $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Corollary 6.1. *Let $(X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy such that $X = \prod_{i\in I} X_i$ is paracompact. Suppose that for each $i \in I$,*

(1) *X_i is a nonempty convex subset of a topological vector space,*

(2) *for each $x \in X$, $A_i(x)$ is nonempty and $co(A_i(x)) \subset \overline{B}_i(x)$,*

(3) *for each $y \in X_i$, $A_i^{-1}(y)$ is compactly open in X,*

(4) *$A_i \cap P_i$ is $\mathcal{L}_F$-majorized,*

(5) *$E_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X,*

(6) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X\backslash K$, there is an $x \in co(X_0 \cup \{y\})$ satisfying*

$$y \notin cl_C((X\backslash(co\, A_i \cap P_i)^{-1}(x_i)) \cap C)$$

for each $i \in I$ and for any nonempty compact subset C of X.

Then Γ has an equilibrium point in K.

Proof. For each $i \in I$ and for each $A_i \in \mathcal{F}(X_i)$, let $\Gamma_{A_i} = co(A_i)$. Then each $(X_i, \{\Gamma_{A_i}\})$ is an H-space. Since X_0 is a compact convex subset of X, for each $N \in \mathcal{F}(X)$, if $L_N = co(X_0 \cup N)$, then L_N is a nonempty compact H-convex subset of X with $N \subset L_N$. Hence condition (6) implies that the condition (6) of Theorem 6.3 holds. The conclusion of Corollary 6.1 follows from Theorem 6.3.

Remark 6.3. Corollary 6.1 is Theorem 5.3 of Ding-Tarafdar [13] which improves and generalizes Theorem 4 of Ding-Tan [12], Theorem 3 of Tulcea [35] (also see Theorem 4 of Tulcea [36]).

Corollary 6.2. *Under the hypotheses of Corollary* 6.1*, if the coercive condition* (6) *of Corollary* 6.1 *is replaced by the following condition:*

(6)' *there exist a nonempty closed and compact subset K of X and a point $x^0 = (x_i^0)_{i\in I} \in X$ such that $x_i^0 \in co(A_i(y) \cap P_i(y))$ for all $i \in I$ and for all $y \in X\backslash K$.*

Then Γ *has an equilibrium point in* K.

Proof. We claim that the coercive condition (6)' implies the condition (6) of Corollary 6.1. By (6)', for each $i \in I$,

$$X \backslash K \subset (co\, A_i \cap P_i)^{-1}(x_i^0)$$

and hence

$$X \backslash (co\, A_i \cap P_i)^{-1}(x_i^0) \subset K.$$

Since K is closed and compact, we have

$$cl_C((X \backslash (co\, A_i \cap P_i)^{-1}(x_i^0)) \cap C) \subset K \cap C \subset K$$

for any nonempty compact subset C of X. Now let $X_0 = \{x^0\}$. Then X_0 is a nonempty compact convex subset of X and $x^0 \in co(X_0 \cup \{y\})$ for all $y \in X$. It is easy to see that the condition (6) of Corollary 6.1 is satisfied. The conclusion follows from Corollary 6.1.

Remark 6.4. Corollary 6.2 is Theorem 5.4 of Ding-Tarafdar [13] which generalizes Theorem 4.3 of Tan-Yuan [23] to $A_i \cap P_i$ being $\mathcal{L}_F$-majorized for each $i \in I$. Hence Corollary 6.2 positively answers the open question presented by Tan-Yuan in [23]. Corollary 6.2 also generalizes Theorem 8 of Ding-Tan [10], Theorem 3 of Tulcea [33] and Theorem 4 of Tulcea [34] in several aspects. Therefore Theorem 6.3 further improves and generalizes the above results to H-spaces.

Corollary 6.3. *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ *be an abstract economy such that* $X = \prod_{i \in I} X_i$ *is paracompact. Suppose that for each* $i \in I$,

(1) $(X_i, \{\Gamma_{A_i}\})$ *is an H-space,*

(2) *for each* $x \in X$, $A_i(x)$ *is nonempty,* H-$co(A_i(x)) \subset \overline{B}_i(x)$ *and* $x_i \notin H$-$co(P_i(x))$,

(3) *for each* $y \in X_i$, $A_i^{-1}(y)$ *and* $P_i^{-1}(y)$ *are open in* X,

(4) *for each* $N \in \mathcal{F}(X)$, *there exists a compact weakly H-convex subset* L_N *of* X *with* $N \subset L_N$ *and there exists a nonempty compact subset* K *of* X *such that for each* $y \in X \backslash K$ *there is an* $x \in L_N$ satisfying

$$y \notin cl_C((X \backslash (H\text{-}co\, A_i \cap P_i)^{-1}(x)) \cap C)$$

for any nonempty compact subset C *of* X.

Then Γ *has an equilibrium point in* K.

Proof. Since $\{x \in X : (A_i \cap P_i)(x) \neq \emptyset\} = \cup_{y \in X_i}(A_i^{-1}(y) \cap P_i^{-1}(y))$, by (3), the conditions (3) and (5) of Theorem 6.3 are satisfied. Since for each $i \in I$ and for each $y \in X_i$, $(A_i \cap P_i)^{-1}(y) = A_i^{-1}(y) \cap P_i^{-1}(y)$ is open in X, for given any $x \in \{z \in X : (A_i \cap P_i)(z) \neq \emptyset\}$, let $N_x = X$, $\psi_x = \phi_x = A_i \cap P_i$. Then it is easy to see that the condition (4) of Theorem 6.3 is also satisfied. The conclusion follows from Theorem 6.3.

Remark 6.5. Corollary 6.3 generalizes Corollary 5.1 of Ding-Tarafdar [13] to H-spaces. Note that for any $x_i \in X_i$, $(H\text{-}co\, A_i \cap P_i)^{-1}(x_i)$ is open in X by the condition (3) of Corollary 6.3 and Lemma 3.1 and hence for any nonempty compact subset C of X, we have

$$\begin{aligned} &cl_C((X \backslash (H\text{-}co\, A_i \cap P_i)^{-1}(x_i)) \cap C) \\ &= (X \backslash (H\text{-}co\, A_i \cap P_i)^{-1}(x_i)) \cap C. \end{aligned}$$

It follows that $y \notin cl_C((X \backslash (H\text{-}co\, A_i \cap P_i)^{-1}(x_i)) \cap C)$ implies $x_i \in H\text{-}co(A_i(y) \cap P_i(y))$. Hence Corollary 6.3 also generalizes Corollary 1 of Ding-Tan [12] to H-spaces. Corollary 6.3 also in turn generalizes Corollary 4.4 of Tan-Yuan [23], Corollary 1 of Ding-Tan [10], Corollary 2 of Tulcea [35], Theorem 2.5 of Toussaint [34] and Theorem 6.1 of Yannelis-Prabhakar [38] to H-spaces.

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