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ON SIMILARITY DEGREES OF FINITE VON NEUMANN ALGEBRAS

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Abstract. In this paper, we showed some results of similarity degrees of von Neumann algebras satisfying co-amenability. We also obtain some results of Christensen's property D_k for such von Neumann algebras.

1. INTRODUCTION

In 1955, Kadison [9] asked whether every bounded homomorphism ϕ from a unital C* algebra $\mathfrak{A}$ into the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on the Hilbert space $\mathcal{H}$ is similar to a *-homomorphism π of $\mathfrak{A}$; i.e. there exist an invertible operator S in $\mathcal{B}(\mathcal{H})$ such that $\pi(X) = S\phi(X)S^{-1}$ for any X in $\mathfrak{A}$. This is the Kadison's similarity problem. When the homomorphism ϕ admits a cyclic vector η in $\mathcal{H}$, Haagerup [7] and Christensen [5] proved that ϕ is similar to a *-homomorphism. More generally when ϕ admits a finite cyclic set $\{\eta_1, \dots, \eta_n\}$, the conjecture is still true. In 1999, G. Pisier [10, 11, 12] introduced the similarity length $l(\mathfrak{A})$ of an operator algebra $\mathfrak{A}$ and the similarity degree $d(\mathfrak{A})$ of $\mathfrak{A}$. Moreover he proved that $l(\mathfrak{A}) = d(\mathfrak{A})$.

Currently, some partial results of the Kadison's similarity problem and corresponding similarity degree are known in the following cases:

(1) $\mathfrak{A}$ is finite dimensional and $d(\mathfrak{A}) = 1$;

(2) $\mathfrak{A}$ is nuclear and infinite-dimensional, such as infinite-dimensional abelian C* algebra, and $\mathcal{K}$, the algebra of all compact operators on an infinite-dimensional separable Hilbert space (see [2]) and $d(\mathfrak{A}) = 2$;

(3) $\mathfrak{A}$ has no tracial state, such as $\mathcal{B}(\mathcal{H})$ (see [7]) and $d(\mathfrak{A}) = 3$;

(4) $\mathfrak{A} = \mathcal{K} \otimes \mathfrak{B}$, where $\mathfrak{B}$ is a unital C* algebra and $2 \le d(\mathfrak{A}) \le 3$;

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(5) $\mathfrak{A} = \mathfrak{N} \otimes \mathfrak{B}$, where $\mathfrak{N}$ is a unital nuclear C* algebra containing unital matrices of any order and $\mathfrak{B}$ is a unital C* algebra and $d(\mathfrak{A}) \leq 5$ (see [16]);

(6) $\mathcal{M}$ is a property Γ factor of type II_1, such as the hyperfinite II_1 factor $\mathcal{R}$, McDuff factor $\mathcal{M} \simeq \mathcal{M}\overline{\otimes}\mathcal{R}$ and in this case the similarity degree $d(\mathcal{M}) = 3$.

In addition to the result (2), G. Pisier [13] showed that an infinite-dimensional C* algebra is nuclear if and only if its similarity degree is equal to 2. The result (6) was first proved by Christensen [1] that property Γ factors have similarity degree ≤ 44. In [12], Pisier improved his result to show that property Γ factors have similarity degree less than or equal to 5. Finally, Christensen [4] proved that the similarity degree of a property Γ factor is equal to 3.

In this paper, we will explore the similarity degrees of some constructions for finite von Neumann algebras. Let G be a discrete group, $(\mathcal{B}_0, \tau_0)$ a finite von Neumann algebra with a normal faithful tracial state and $\sigma : G \mapsto Aut(\mathcal{B}_0, \tau_0)$ a trace preserving cocycle action of G on $(\mathcal{B}_0, \tau_0)$. Let $\mathcal{N} = \mathcal{B}_0 \rtimes_\sigma G$ be the corresponding crossed product von Neumann algebra with its normal faithful tracial state given by $\tau(\sum_{g\in G} B_g U_g) = \tau_0(B_e)$. Let H be a subgroup of G co-amenable in G and $\mathcal{B} = \mathcal{B}_0 \rtimes_\sigma H$. We show that if $\mathcal{N}$ is a factor and $\mathcal{B}$ has similarity degree d, then $\mathcal{N}$ has similarity degree of at most $9d + 8$. In particular, we also obtain some more results on similarity degrees for Jones basic construction when its Jones index is finite. In [1], Christensen introduced property D_k for C* algebras. We will investigate Christensen's property D_k for certain finite von Neumann algebras.

2. Preliminaries

In this section, we will recall some notations and properties for the similarity degrees and similarity length for C* algebras and co-amenability of von Neumann subalgebras.

Let $\mathfrak{A}$ be a unital C* algebra and $\mathcal{H}$ be a Hilbert space. Suppose that $\phi : \mathfrak{A} \mapsto \mathcal{B}(\mathcal{H})$ is a unital homomorphism; i.e. $\phi(I) = I$ and $\phi(X_1X_2) = \phi(X_1)\phi(X_2)$ for all X_1, X_2 in $\mathfrak{A}$. The Kadison's similarity problem is whether the condition that ϕ is bounded implies that ϕ is similar to a *-homomorphism, i.e. there exists an invertible operator S in $\mathcal{B}(\mathcal{H})$ such that $\phi_S : X \mapsto S^{-1}\phi(X)S$ is a *-homomorphism. In [7], Haagerup proved that ϕ is similar to a *-homomorphism if and only if ϕ is completely bounded and

$$\|\phi\|_{cb} = \inf\{\|S^{-1}\| \cdot \|S\| : \phi_S \text{ is a *-homomorphism.}\}$$

An operator algebra $\mathfrak{A}$ has similarity property if any bounded homomorphism $\phi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is completely bounded.

An operator algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is said to be of length $\leq d$ if there is a constant K such that, for any n and any X in $M_n(\mathfrak{A})$, there is a positive integer $N = N(n, X)$ and scalar matrices $\alpha_0 \in M_{n,N}(\mathbb{C})$, $\alpha_1 \in M_N(\mathbb{C})$, …, $\alpha_{d-1} \in M_N(\mathbb{C})$, $\alpha_d \in M_{N,n}(\mathbb{C})$

together with diagonal matrices $D_1, \ldots, D_d$ in $M_N(\mathfrak{A})$ satisfying

$$\begin{cases} X = \alpha_0 D_1 \alpha_1 D_2 \cdots D_d \alpha_d \\ \prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \leq K\|X\|. \end{cases}$$

Denote by $\ell(\mathfrak{A})$ the length of $\mathfrak{A}$; i.e the smallest d for which the two equations above holds. We say the least K satisfying the above condition as length constant for $\mathfrak{A}$.

Let

$$d(\mathfrak{A}) = \inf\{\alpha \geq 0 | \exists K, \forall \phi, \|\phi\|_{cb} \leq K\|\phi\|^\alpha\},$$

where ϕ denotes an arbitrary unital homomorphism from $\mathfrak{A}$ to $\mathcal{B}(\mathcal{H})$. G. Pisier [10, 11, 12] showed that $d(\mathfrak{A}) = \ell(\mathfrak{A})$ for any operator algebra $\mathfrak{A}$ and that $\mathfrak{A}$ has similarity property if and only if $d(\mathfrak{A}) < \infty$. Let $\mathfrak{A}$ be weakly dense C* subalgebra in a factor $\mathcal{M}$ of type II$_1$. Then Pisier showed [12] that $d(\mathcal{M}) = \max\{d(\mathfrak{A}), 3\}$ derived from Remark 7 and Theorem 9. In [6], E. Christensen, A. Sinclair, R. Smith, and S. White showed that similarity property is preserved under perturbation of C*-algebras.

Let ω be a free ultrafilter on $\mathbb{N}$ and $\mathcal{M}$ be a factor of type II$_1$. It is known that the ultrapower $\mathcal{M}^\omega$ of $\mathcal{M}$ is a factor of type II$_1$. By [11, 12], we have that $d(\mathcal{M}^\omega) \leq d(\mathcal{M})$.

A C* algebra $\mathfrak{A}$ is said to have property D_k for some positive real k if for each non degenerate *-representation π of $\mathfrak{A}$ on a Hilbert space we have for any $X \in \mathcal{B}(\mathcal{H})$:

$$\inf\{\|X - Z\| \big| Z \in \pi(\mathfrak{A})'\} \leq k \sup\{\|X\pi(A) - \pi(A)X\| \big| A \in \mathfrak{A}, \|A\| \leq 1\}.$$

In [6], E. Christensen, A. Sinclair, R. Smith, and S. White showed the following relations between the property D_k and the similarity degree (length) for a C* algebra.

Proposition 2.1. *Let $\mathcal{M}$ have property D_k for some k. The the length of $\mathcal{M}$ is $\lfloor 2k \rfloor$, where $\lfloor a \rfloor$ is the integral part of a.*

Proposition 2.2. *Let $\mathcal{M}$ be a C* algebra with length at most ℓ and the length constant at most K. Then $\mathcal{M}$ has property D_k for $k = K\ell/2$.*

Now let us recall the co-amenability of groups and von Neumann algebras. A subgroup H of a group G is called co-amenable in G if there exists a G-invariant mean on the space $\ell^\infty(G/H)$.

Let $\mathcal{N}$ be a finite von Neumann algebra with a faithful normal tracial state τ and $\mathcal{B} \subset \mathcal{N}$ a von Neumann subalgebra. We assume that $\mathcal{N}$ has a separable predual. Let $\mathcal{B} \subset \mathcal{N} \overset{e_{\mathcal{B}}}{\subset} \langle \mathcal{N}, \mathcal{B} \rangle$ be the Jones basic construction for $\mathcal{B} \subset \mathcal{N}$; i.e. $\langle \mathcal{N}, \mathcal{B} \rangle = J\mathcal{B}'J \cap \mathcal{B}(L^2(\mathcal{N}, \tau))$ with J the canonical conjugation on $L^2(\mathcal{N}, \tau)$ and $e_{\mathcal{B}}$ the canonical projection of $L^2(\mathcal{N}, \tau)$ onto $L^2(\mathcal{B}, \tau)$. The subalgebra $\mathcal{B}$ is co-amenable in $\mathcal{N}$ if there exists a norm one projection Ψ of $\langle \mathcal{N}, \mathcal{B} \rangle$ onto $\mathcal{N}$. One also says that $\mathcal{N}$ is amenable relative to $\mathcal{B}$. In particular, if an inclusion $\mathcal{N} \subset \mathcal{M}$ of factors of type II$_1$ has finite Jones index, then $\mathcal{N}$ is co-amenable in $\mathcal{M}$.

3. Basic Construction with Finite Jones Index

In this section we will study similarity degrees for Jones basic construction with finite Jones index.

Lemma 3.1. *Let $\mathcal{M}$ be a factor of type II_1 with similarity property and P be a non zero projection in $\mathcal{M}$. Then $d(P\mathcal{M}P) \leq d(\mathcal{M})$.*

Proof. Let τ be the trace on $\mathcal{M}$, $d = d(\mathcal{M})$ and $\frac{1}{k} \leq \tau(P) < \frac{1}{k-1}$ for some integer $k \geq 2$. We have that there is a constant K such that for any $n \in \mathbb{N}$ and any X in $M_n(P\mathcal{M}P)$, there is an integer $N = N(n, X)$ and scalar matrices $\alpha_0 \in M_{n,N}(\mathbb{C})$, $\alpha_1 \in M_N(\mathbb{C}), \ldots, \alpha_{d-1} \in M_N(\mathbb{C})$, $\alpha_d \in M_{N,n}(\mathbb{C})$ together with diagonal matrices $D_1, \ldots, D_d$ in $M_N(\mathcal{M})$ satisfying

$$\begin{cases} X = \alpha_0 D_1 \alpha_1 D_2 \cdots D_d \alpha_d \\ \prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \leq K\|X\|. \end{cases}$$

Let E be a subprojection of P in $\mathcal{M}$ with $\tau(E) = \frac{1}{k}$ and $\{E_{ij}\}_{i,j=1}^k$ be the system of matrix units in $\mathcal{M}$ such that $E = E_{11}$. We write $X = [X_{ij}]_{i,j=1}^n$, where $X_{ij} \in P\mathcal{M}P$. Then $\widetilde{P}X\widetilde{P} = X$, where $\widetilde{P}$ is the $n \times n$ diagonal matrix $diag(P, \ldots, P)$. For convenience, we denote $n \times n$ or $N \times N$ diagonal matrix $diag(T, \ldots, T)$ by $\widetilde{T}$. Hence

$$\begin{aligned} X = \widetilde{P}X\widetilde{P} &= \alpha_0 \widetilde{P} D_1 \alpha_1 \cdots D_d \widetilde{P} \alpha_d \\ &= \sum_{j_1,\ldots,j_{d-1}=1}^{k} \alpha_0 \widetilde{P} D_1 \widetilde{E}_{j_1 j_1} \alpha_1 \widetilde{E}_{j_1 j_1} D_2 \widetilde{E}_{j_2 j_2} \alpha_2 \\ &\qquad \cdots \widetilde{E}_{j_{d-2} j_{d-2}} D_{d-1} \widetilde{E}_{j_{d-1} j_{d-1}} \alpha_{d-1} \widetilde{E}_{j_{d-1} j_{d-1}} D_d \widetilde{P} \alpha_d \\ &= \sum_{j_1,\ldots,j_{d-1}=1}^{k} \alpha_0 \widetilde{P} D_1 \widetilde{E}_{j_1 1} \alpha_1 \widetilde{E}_{1 j_1} D_2 \widetilde{E}_{j_2 1} \alpha_2 \\ &\qquad \cdots \widetilde{E}_{1 j_{d-2}} D_{d-1} \widetilde{E}_{j_{d-1} 1} \alpha_{d-1} \widetilde{E}_{1 j_{d-1}} D_d \widetilde{P} \alpha_d \end{aligned}$$

Let β be $n \times (nk^{d-1})$ matrix $(I_n \cdots I_n)$, where I_n is the $n \times n$ identity matrix. Let

$$\overline{D} = \begin{pmatrix} \widetilde{P} D_1 \widetilde{E}_{11} & \widetilde{E}_{11} D_2 \widetilde{E}_{11} & \cdots & \widetilde{E}_{11} D_d \widetilde{P} \\ \vdots & \vdots & \cdots & \vdots \\ \widetilde{P} D_1 \widetilde{E}_{j_1 1} & \widetilde{E}_{1 j_1} D_2 \widetilde{E}_{j_2 1} & \cdots & \widetilde{E}_{1 j_{d-1}} D_d \widetilde{P} \\ \vdots & \vdots & \cdots & \vdots \\ \widetilde{P} D_1 \widetilde{E}_{n1} & \widetilde{E}_{1k} D_2 \widetilde{E}_{k1} & \cdots & \widetilde{E}_{1k} D_d \widetilde{P} \end{pmatrix},$$

where $j_1, \ldots, j_{d-1}$ run through $\{1, \ldots, k\}$ and $\overline{D}$ has k^{d-1} rows. Let $\overline{D}_j$ be the diagonal matrix with diagonal obtained from the j-th column in $\overline{D}$. Then

$$X = \beta \widetilde{\alpha_0} \overline{D}_1 \widetilde{\alpha_1} \cdots \widetilde{\alpha_{d-1}} \overline{D}_d \widetilde{\alpha_d} \beta^t,$$

where $\widetilde{\alpha_j}$ is block matrix with block diagonal α_j for $j = 0, \ldots, d$. Moreover,

$$\begin{aligned}\|\beta\widetilde{\alpha_0}\|\|\overline{D}_1\|\cdots\|\overline{D}_d\|\|\widetilde{\alpha_d}\beta^t\| &\leq \|\beta\|\|\alpha_0\|\|D_1\|\cdots\|D_d\|\|\alpha_d\|\|\beta^t\| \\ &\leq \|\beta\|^2 K\|X\| = k^{d-1}K\|X\|.\end{aligned}$$

Therefore $P\mathcal{M}P$ has similarity degree at most d. ■

Proposition 3.2. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of factors of type II_1 and its Jones index $[\mathcal{M} : \mathcal{N}] < \infty$. Suppose that $\mathcal{N}$ has similarity property. Then $d(\mathcal{M}) \leq d(\mathcal{N})+1$.*

Proof. Let $\lambda = [\mathcal{M} : \mathcal{N}]$, $d = d(\mathcal{N})$, and $E_{\mathcal{N}}$ be the trace preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. By [17], Proposition 1.3, there exists a family $\{V_j\}_{1\leq j\leq l+1}$ of elements in $\mathcal{M}$ with l equal to the integer part of λ (i.e. $l = \lfloor\lambda\rfloor$) such that

$$T = \sum_{j=1}^{l+1} V_j E_{\mathcal{N}}(V_j^* T)$$

for all T in $\mathcal{M}$.

For any $n \in \mathbb{N}$ and any X in $M_n(\mathcal{M})$, we write $X = [X_{ij}]_{i,j=1}^n$, where $X_{ij} \in \mathcal{M}$. For each X_{ij}, we have decomposition

$$X_{ij} = \sum_{k=1}^{l+1} V_k E_{\mathcal{N}}(V_k^* X_{ij}) = \sum_{k=1}^{l+1} V_k X_{ij}^{(k)},$$

where $X_{ij}^{(k)} = E_{\mathcal{N}}(V_k^* X_{ij}) \in \mathcal{N}$.

Let $X^{(k)} = [X_{ij}^{(k)}]_{i,j=1}^n$. Then for n and $X^{(k)} \in M_n(\mathcal{N})$, there is a constant K_k such that there is an integer $N(n, X^{(k)})(= N_k)$ and scalar matrices $\alpha_0^{(k)} \in M_{n,N_k}(\mathbb{C})$, $\alpha_1^{(k)} \in M_{N_k}(\mathbb{C}), \ldots, \alpha_{d-1}^{(k)} \in M_{N_k}(\mathbb{C})$, $\alpha_d^{(k)} \in M_{N_k,n}(\mathbb{C})$ together with diagonal matrices $D_1^{(k)}, \ldots, D_d^{(k)}$ in $M_{N_k}(\mathcal{N})$ satisfying

$$\begin{cases} X^{(k)} = \alpha_0^{(k)} D_1^{(k)} \alpha_1^{(k)} D_2^{(k)} \cdots D_d^{(k)} \alpha_d^{(k)} \\ \prod_0^d \|\alpha_i^{(k)}\| \prod_1^d \|D_i^{(k)}\| \leq K_k \|X^{(k)}\|. \end{cases}$$

We may assume that $\|\alpha_i^{(k)}\| = 1$ for $i = 0, \ldots, d-1$ and $\|D_i^{(k)}\| = 1$ for $i = 1, \ldots, d$. Then $\|\alpha_d^{(k)}\| \leq K_k \|X^{(k)}\|$.

Let α_i be the $(l+1) \times (l+1)$ block diagonal matrix with block diagonal $\alpha_i^{(1)}, \ldots, \alpha_i^{(l+1)}$ and D_i be the $(l+1) \times (l+1)$ block diagonal matrix with block diagonal $D_i^{(1)}, \ldots, D_i^{(l+1)}$. Then $\|\alpha_i\| = \max_k \|\alpha_i^{(k)}\| = 1$ and $\|D_i\| = \max_k \|D_i^{(k)}\| = 1$. Let

$\widetilde{V}_k$ be the $n\times n$ diagonal matrix with diagonal $V_k,\ldots,V_k$ and V be the $(l+1)\times(l+1)$ block diagonal matrix with block diagonal $\widetilde{V}_1,\ldots,\widetilde{V}_{l+1}$. Let β be $n\times(n(l+1))$ matrix $(I_n\cdots I_n)$, where I_n is the $n\times n$ identity matrix. Then

$$\begin{aligned}
X &= [X_{ij}]_{i,j=1}^n = [\sum_{k=1}^{l+1} V_k X_{ij}^{(k)}]_{i,j=1}^n \\
&= \sum_{k=1}^{l+1}[V_k X_{ij}^{(k)}]_{i,j=1}^n = \sum_{k=1}^{l+1}\widetilde{V}_k X^{(k)} \\
&= \sum_{k=1}^{l+1}\widetilde{V}_k \alpha_0^{(k)} D_1^{(k)} \alpha_1^{(k)} D_2^{(k)}\cdots D_d^{(k)}\alpha_d^{(k)} \\
&= \beta V \alpha_0 D_1 \alpha_1 D_2 \cdots D_d \alpha_d \beta^t.
\end{aligned}$$

Since $\sum_{k=1}^{l+1} V_k V_k^* = \lambda$, we obtain $\|V_k\|\le\lambda^{1/2}$ and $\|V\| = \max_k\|V_k\|\le\lambda^{1/2}$. On the other hand, we have

$$\begin{aligned}
\|X^{(k)}\| &= \|[X_{ij}^{(k)}]_{i,j=1}^n\| = \|[X_{ij}^{(k)}]_{i,j=1}^n\| \\
&= \|[E_{\mathcal{N}}(V_k X_{ij})]_{i,j=1}^n\| \le \|[V_k X_{ij}]_{i,j=1}^n\| \\
&\le \|\widetilde{V}_k\|\|X\| \le \lambda^{1/2}\|X\|.
\end{aligned}$$

Let $K=\max\{K_1,\ldots,K_{l+1}\}$. Then

$$\begin{aligned}
&\|\beta\|\|V\|\|\alpha_0\|\|D_1\|\cdots\|D_d\|\|\alpha_d\beta^t\| \\
&= \|\beta\|\|V\|\|\alpha_d\beta^t\| \\
&\le (l+1)^{1/2}\lambda^{1/2}\|\alpha_d\|\|\beta^t\| \\
&\le (l+1)\lambda^{1/2}\max\{K_1\|X^{(1)}\|,\ldots,K_{l+1}\|X^{(l+1)}\|\} \\
&\le (l+1)\lambda^{1/2}K\lambda^{1/2}\|X\| \le (l+1)\lambda K\|X\|.
\end{aligned}$$

Therefore $d(\mathcal{M})\le d+1 = d(\mathcal{N})+1$. ■

Let $\mathcal{N}\subset\mathcal{M}$ be an inclusion of factors of type II_1 with finite Jones index; i.e $[\mathcal{M}:\mathcal{N}]<\infty$. If $\mathcal{N}$ has property Γ, then by [17] we have that $\mathcal{M}$ has property Γ. In this case, the similarity degree of the factors are equal; i.e. $d(\mathcal{M}) = d(\mathcal{N}) = 3$. In general, $\mathcal{M}$ preserves most properties of $\mathcal{N}$. Thus, it is natural to ask the question whether it is true that $d(\mathcal{M}) = d(\mathcal{N})$ for the inclusion $\mathcal{N}\subset\mathcal{M}$ of factors of type II_1 with Jones index $[\mathcal{M}:\mathcal{N}]<\infty$.

Corollary 3.3. *Let $\mathcal{M}$ be a factor of type II_1. Suppose that $\mathcal{M}$ has similarity property. Then $\mathcal{M}\otimes M_n(\mathbb{C})$ has similarity property and $d(\mathcal{M}\otimes M_n(\mathbb{C}))\le d(\mathcal{M})+1$.*

Proof. Since $[\mathcal{M} \otimes M_n(\mathbb{C}) : \mathcal{M}] = n^2$ and $\mathcal{M}$ has similarity property, by Proposition 3.2, we have that $d(\mathcal{M} \otimes M_n(\mathbb{C})) \leq d(\mathcal{M}) + 1$. ■

Let $\mathcal{M}$ be a factor of type II$_1$ with the trace τ and $\mathcal{H}$ be an infinite-dimensional Hilbert space. Denote the standard tracial weight on $\mathcal{B}(\mathcal{H})$ by Tr. Suppose that P is a finite projection in $\mathcal{M}\overline{\otimes}\mathcal{B}(\mathcal{H})$; i.e. $(\tau \otimes Tr)(P) = t < \infty$ for some $t \in \mathbb{R}_+$, where $\mathbb{R}_+$ is the group of all positive real numbers. Then $\mathcal{M}_t$ which is defined to be the von Neumann algebra isomorphic to $P\mathcal{M}\overline{\otimes}\mathcal{B}(\mathcal{H})P$ is a t-amplification (contraction) of $\mathcal{M}$. By " $\simeq''$ we mean the *-isomorphism between two C* algebras, for instance, $\mathcal{M}_t \simeq P\mathcal{M}\overline{\otimes}\mathcal{B}(\mathcal{H})P$.

Proposition 3.4. *Let $\mathcal{M}$ be a factor of type II$_1$. Then $d(\mathcal{M}_{t_1}) \leq d(\mathcal{M}_{t_2}) \leq d(\mathcal{M}) + 1$ when $t_1 \leq t_2$, $t_1, t_2 \in \mathbb{R}_+$.*

Proof. By Lemma 3.1 and Corollary 3.3, we have that $\mathcal{M}_t$ has similarity property for any $t \in \mathbb{R}_+$ if $\mathcal{M}$ has similarity property. Therefore $\mathcal{M}_{t_1}$ has similarity property if and only $\mathcal{M}_{t_2}$ has similarity property when $t_1, t_2 \in \mathbb{R}_+$, since $\mathcal{M}_{t_2} = (\mathcal{M}_{t_1})_{t_2/t_1}$ and $\mathcal{M}_{t_1} = (\mathcal{M}_{t_2})_{t_1/t_2}$. Hence if $\mathcal{M}$ does not have similarity property then $d(\mathcal{M}_{t_1}) = \infty$ for all $t \in \mathbb{R}_+$ and the inequality $d(\mathcal{M}_{t_1}) \leq d(\mathcal{M}_{t_2}) \leq d(\mathcal{M}) + 1$ holds. If $\mathcal{M}$ has similarity property, then by Lemma 3.1 and Corollary 3.3 we have $d(\mathcal{M}_{t_1}) \leq d(\mathcal{M}_{t_2}) \leq d(\mathcal{M}) + 1$ again. ■

Corollary 3.5. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of factors of type II$_1$ and its Jones index $[\mathcal{M} : \mathcal{N}] < \infty$. Then $\max\{d(\mathcal{N}) - 1, 3\} \leq d(\mathcal{M}) \leq d(\mathcal{N}) + 1$. In particular, $\mathcal{N}$ does not have similarity property if and only if $\mathcal{M}$ does not have similarity property.*

Proof. Let τ be the trace on $\mathcal{M}$. Suppose that $\mathcal{M}$ acts on the Hilbert space $L^2(\mathcal{M}, \tau)$. Let $e_{\mathcal{N}}$ be the projection of $L^2(\mathcal{M}, \tau)$ onto $L^2(\mathcal{N}, \tau)$. Then $\langle \mathcal{M}, \mathcal{N} \rangle = J\mathcal{N}'J$ and $[\langle \mathcal{M}, \mathcal{N} \rangle : \mathcal{M}] = [\mathcal{M} : \mathcal{N}](= \lambda)$. Suppose $\mathcal{N}$ has similarity property. By Proposition 3.2, we have $d(\langle \mathcal{M}, \mathcal{N} \rangle) \leq d(\mathcal{M}) + 1$. On the other hand $\langle \mathcal{M}, \mathcal{N} \rangle \simeq \mathcal{N}_\lambda$ and $d(\mathcal{N}) \leq d(\mathcal{N}_\lambda)$ since $[\mathcal{M} : \mathcal{N}] = \lambda \geq 1$ by Lemma 3.1. Then

$$d(\mathcal{N}) \leq d(\mathcal{N}_\lambda) = d(\langle \mathcal{M}, \mathcal{N} \rangle) \leq d(\mathcal{M}) + 1.$$

Thus combining the equation $d(\mathcal{M}) \leq d(\mathcal{N}) + 1$ followed by Proposition 3.2, we have that $\mathcal{M}$ does not have similarity property if and only if $\mathcal{N}$ does not have similarity property. Therefore $d(\mathcal{N}) - 1 \leq d(\mathcal{M}) \leq d(\mathcal{N}) + 1$. By [12], we have that $d(\mathcal{M}) \geq 3$. Hence $\max\{d(\mathcal{N}) - 1, 3\} \leq d(\mathcal{M}) \leq d(\mathcal{N}) + 1$. ■

Let $\mathcal{M}$ be a factor of type II$_1$. The subgroup $\mathcal{F}(\mathcal{M}) = \{t \in \mathbb{R}_+ | \mathcal{M}_t \simeq \mathcal{M}\}$ of $\mathbb{R}_+$ is called the fundamental group of $\mathcal{M}$.

Corollary 3.6. *Let $\mathcal{M}$ be a factor of type II$_1$. Suppose that the fundamental group $\mathcal{F}(\mathcal{M})$ of $\mathcal{M}$ is non trivial. Then $d(\mathcal{M}_t) = d(\mathcal{M})$ for any $t \in \mathbb{R}_+$.*

Proof. Since the fundamental group $\mathcal{F}(\mathcal{M})$ is non trivial, there exists a positive number $t_0 \neq 1$ such that $\mathcal{M}_{t_0} \simeq \mathcal{M}$. Without loss of generality, we assume that $t_0 > 1$. Then $\mathcal{M}_{t_0^k} \simeq \mathcal{M}$ for any $k \in \mathbb{Z}$. For any $t \in \mathbb{R}_+$, there exists an integer k such that $t_0^{k-1} \leq t \leq t_0^k$. By Corollary 3.4, we have that

$$d(\mathcal{M}) = d(\mathcal{M}_{t_0^{k-1}}) \leq d(\mathcal{M}_t) \leq d(\mathcal{M}_{t_0^k}) = d(\mathcal{M}).$$

Therefore $d(\mathcal{M}_t) = d(\mathcal{M})$ for $t \in \mathbb{R}_+$. ∎

4. Crossed Product by Amenable Group

In section 3, we have result of similarity degree for inclusion of factors of type II_1 with finite Jones index. In general, for the case , we might not have similar result for the inclusion of factors of type II_1 with infinite Jones index. But for the following case when the Jones index is infinite, we have affirmative answer to Kadison's similarity problem.

Theorem 4.1. *Let G be a discrete group, $(\mathcal{B}_0, \tau_0)$ a finite von Neumann algebra with a normal faithful tracial state and $\sigma : G \mapsto Aut(\mathcal{B}_0, \tau_0)$ a trace preserving cocycle action of G on $(\mathcal{B}_0, \tau_0)$. Let $\mathcal{N} = \mathcal{B}_0 \rtimes_\sigma G$ be the corresponding crossed product von Neumann algebra with its normal faithful tracial state given by $\tau(\sum_{g\in G} B_g U_g) = \tau_0(B_e)$. Let H be a subgroup of G co-amenable in G and $\mathcal{B} = \mathcal{B}_0 \rtimes_\sigma H$. If $\mathcal{N}$ is a factor and $\mathcal{B}$ has similarity degree d, then $\mathcal{N}$ has similarity degree of at most $9d + 8$.*

Proof. Suppose ϕ is a unital bounded representation of $\mathcal{N}$ on a Hilbert space $\mathcal{H}$ such that $\overline{sp}\phi(\mathcal{N})\mathcal{H} = \mathcal{H}$. Then $\phi|_{\mathcal{B}}$ is a bounded representation of $\mathcal{B}$, and so there is an invertible operator S_0 on $\mathcal{H}$ such that $S_0\phi|_{\mathcal{B}}S_0^{-1}$ is a *-representation of $\mathcal{B}$ and $\|S_0^{-1}\|\|S_0\| \leq K\|\phi|_{\mathcal{B}}\|^d$. Let $\phi_0 = S_0\phi S_0^{-1}$. Then ϕ_0 is a bounded representation of $\mathcal{N}$.

We have to estimate the complete bounded norm of ϕ_0. To do this, we may and will assume that the representation has an at most countable cyclic set. In this case [2] there is a *-representation π of $\mathcal{N}$ on $\mathcal{H}$ such that for any vector ξ in $\mathcal{H}$, there exists a bounded injective operator X with dense range and a vector η satisfying

$$\forall Y \in \mathcal{N} : \phi_0(Y)X = X\pi(Y); \|X\| \leq 2\|\phi_0\|^2; X\eta = \xi; \|\eta\| \leq \|\xi\|.$$

The first property admits a homomorphism ψ of $\pi(\mathcal{N})$ into $\mathcal{B}(\mathcal{H})$ by $A \mapsto \overline{XAX^{-1}}$ and $\|\psi\| = \|\phi_0\|$, whereas the second shows that ψ is ultrastrongly continuous since $\psi(A)\xi = XA\eta$. We will denote by ψ again the extension of ψ to the von Neumann algebra generated by $\pi(\mathcal{N})$. In this algebra we will let F denote the maximal finite central projection and let $\mathcal{D}$ be a copy of the compact operators placed inside $(I - F)\pi(\mathcal{N})$, such that $I - F$ belongs to the weak closure of $\mathcal{D}$. Then $\mathcal{D} + \mathbb{C}F$ is a nuclear

C* algebra, by [2], we can perturb ψ with a Z in $GL(\mathcal{H})$ such that $Ad(Z)\circ\psi$ is trivial on $\mathcal{D}\oplus\mathbb{C}F$ and $\|Z^{-1}\|\|Z\|\leq\|\phi_0\|^2$. The new homomorphism $Ad(Z)\circ\psi$ decomposes naturally into an orthogonal direct sum. The restriction to the properly infinite part is by construction completely bounded with complete bounded norm less than $\|\phi_0\|^3$. The restriction to the finite part yields homomorphisms π_F and Δ of the finite von Neumann algebra $\mathcal{N}$ into $\mathcal{B}(F\mathcal{H})$ given by

$$\pi_F(Y)=\pi(Y)|_{F\mathcal{H}} \text{ and } \Delta(Y)=(ZX)F\pi_F(Y)(ZX)^{-1}|_{F\mathcal{H}}.$$

Since a finite representation of a finite representation of a finite factor is ultrastrongly continuous because of the uniqueness of the trace, we see that Δ is ultrastrongly continuous.

Let $F_n\nearrow G/H$ be a net of finite Følner sets, which we identify with some sets of representatives $F_n\subset G$. Since Δ is unital bounded, the set $|F_n|^{-1}\sum_{s\in F_n}\Delta(U_s)^*\Delta(U_s)$ in the von Neumann algebra generated by $\Delta(\mathcal{N})$ has a strong-operator accumulation point. The accumulation point is positive. So let S be the square root of it. We have

$$\|S\xi\|^2=\lim_n\frac{1}{|F_n|}\sum_{s\in F_n}\|\Delta(U_s)\xi\|^2$$

and hence, $\|\Delta\|^{-1}\leq S\leq\|\Delta\|$. For any unitary element U in $\mathcal{B}_0$, let $V_s=U_sUU_s^*$ in $\mathcal{B}_0$. Then

$$\begin{aligned}S^2\Delta(U)\xi &= \lim_n\frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(U_s)\Delta(U)\xi\\ &= \lim_n\frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(V_sU_s)\xi\\ &= \lim_n\frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U)\Delta(U_s)^*\Delta(U_s)\xi\\ &= \Delta(U)S^2\xi.\end{aligned}$$

For any unitary element U_g, $g\in G$ in $\mathcal{N}$, let $h_ss'=sg$ if sg is in F_n. Since F_n is a Følner set and $\Delta(U_{h_s})$ is a unitary, we have that

$$\begin{aligned}S^2\Delta(U_g)\xi &= \lim_n\frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(U_s)\Delta(U_g)\xi\\ &= \lim_n\frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(U_{sg})\xi\\ &= \lim_n\frac{1}{|F_n|}\sum_{s'\in F_ng}\Delta(U_g)\Delta(U_{s'})^*\Delta(U_{s'})\xi\\ &= \Delta(U_g)S^2\xi.\end{aligned}$$

Let $\mathcal{N}_0$ be the *-subalgebra in $\mathcal{N}$ generated by $\mathcal{B}_0$ and $U_g, g \in G$. For any element A_0 in $\mathcal{N}_0$, we have $S^2\Delta(A_0)\xi = \Delta(A_0)S^2\xi$, for all $\xi \in \mathcal{H}$. By the Kaplansky density theorem, for any A in the unit ball of $\mathcal{N}$, there is a net of $\{A_\alpha\}$ in the unit ball of $\mathcal{N}_0$ convergent to A in the strong-operator topology.

Since Δ is strong-operator continuous, $\Delta(A_\alpha)$ converges to $\Delta(A)$, then $\|Ad(S) \circ \Delta\| \leq 1$ and Δ is completely bounded with completely bounded norm $\|\Delta\|_{cb} \leq \|S\|\|S^{-1}\| \leq \|\Delta\|^2$. Thus

$$
\begin{aligned}
\|\phi\|_{cb} &\leq \|S_0^{-1}\|\|S_0\|\|\phi_0\|_{cb} \\
&\leq K\|\phi\|^d\|Z\|\|Z^{-1}\|\|\Delta\|_{cb} \\
&\leq K\|\phi\|^d\|\phi_0\|^2\|\phi_0\|^6 \\
&\leq K^9\|\phi\|^{9d+8},
\end{aligned}
$$

since $\|S_0^{-1}\|\|S_0\| \leq K\|\phi|_{\mathcal{B}}\|^d \leq K\|\phi\|^d$, $\|Z\|\|Z^{-1}\| \leq \|\phi_0\|^2 \leq (K\|\phi\|^{d+1})^2$ and $\|\Delta\| \leq \|Z\|\|Z^{-1}\|\|\phi_0\| \leq \|\phi_0\|^3$. ■

Corollary 4.2. *Let G be a discrete group, $(\mathcal{B}_0, \tau_0)$ a finite von Neumann algebra with a normal faithful tracial state and $\sigma : G \mapsto Aut(\mathcal{B}_0, \tau_0)$ a trace preserving cocycle action of G on $(\mathcal{B}_0, \tau_0)$. Let $\mathcal{N} = \mathcal{B}_0 \rtimes_\sigma G$ be the corresponding crossed product von Neumann algebra with its normal faithful tracial state given by $\tau(\sum_{g\in G} B_g U_g) = \tau_0(B_e)$. If $\mathcal{N}$ is a factor and $\mathcal{B}_0$ has similarity degree d, then $\mathcal{N}$ has similarity degree of at most $9d + 8$.*

Remark . It is known [4] that the similarity degree of a property Γ factor of type II_1 is 3. Hence if the crossed product $\mathcal{M} = \mathcal{B} \rtimes_\sigma G$ of a property Γ factor $\mathcal{B}$ by an amenable group G is a factor, then $\mathcal{M}$ has similarity degree of at most 35. Here we would like to point out that it is open whether the crossed product of a property Γ factor of type II_1 by an amenable group has property Γ.

In Theorem 4.1 and Proposition 2.1, we obtain partial results for similarity problem for von Neumann algebras satisfying co-amenability. The first named author showed that the co-amenability of von Neumann algebra preserves Connes's embedding property [18]. Suppose that $\mathcal{N}$ has similarity property and $\mathcal{N}$ is co-amenable in $\mathcal{M}$. We do not known whether $\mathcal{M}$ have similarity property and what the similarity degree of $\mathcal{M}$ is.

5. Christensen's Property D_k

In this section we will obtain some results on the Christensen's property D_k [1, 2, 3, 4] for some finite von Neumann algebras.

Lemma 5.1. *Let $\mathcal{M}$ be a factor of type II_1 with similarity degree $d < \infty$. Then $\mathcal{M}$ has property $D_{ed/2}$ or $D_{3d/2}$, where e is the Euler's number $2.718\cdots$.*

Proof. Let $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ be a non degenerate *-representation and let X be in $\mathcal{B}(\mathcal{H})$, then we get for any $t \in \mathbb{R}_+$ a homomorphism $\Phi_t : \mathcal{M} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by

$$\Phi_t(Y) = \begin{pmatrix} \pi(Y) & t[X, \pi(Y)] \\ 0 & \pi(Y) \end{pmatrix}.$$

Let $\alpha = \sup\{\|[X, \pi(Y)]\| : Y \in \mathcal{M}, \|Y\| \leq 1\}$ and let $\delta : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ be given by $\delta(Y) = [X, \pi(Y)]$. It is clear that δ is completely bounded, $\|\delta\| = \alpha$, and

$$t\|\delta\|_{cb} \leq \|\Phi_t\|_{cb} \leq \|\Phi_t\|^d \leq (1 + t\|\delta\|)^d.$$

Then

$$\begin{aligned}\|\delta\|_{cb} &\leq \inf\{t^{-1}(1 + t\|\delta\|)^d | t \in \mathbb{R}_+\} \\ &\leq \inf\{s^{-1}(1+s)^d\|\delta\| | s \in \mathbb{R}_+\} \quad (s = t\|\delta\|)\end{aligned}$$

Let $f(s) = \dfrac{(1+s)^d}{s}$. Then $f'(s) = \dfrac{d(1+s)^{d-1}s - (1+s)^d}{s^2} = 0$ implies $s = \dfrac{1}{d-1}$. Hence

$$\begin{aligned}\|\delta\|_{cb} &\leq (d-1)(1 + \frac{1}{d-1})^d\|\delta\| \\ &= d(1 + \frac{1}{d-1})^{d-1}\|\delta\| \\ &\leq de\|\delta\| < 3d\|\delta\|\end{aligned}$$

An application of Corollary 2.2 of [3] yields $\inf\{\|X - Z\| : Z \in \pi(\mathcal{M})'\} = (1/2)\|\delta\|_{cb} \leq (de/2)\|\delta\| < (3d/2)\|\delta\|$ and the Lemma follows. ∎

Corollary 5.2. *Let $\mathcal{M}$ be a factor of type II_1 with property D_k. Then $\mathcal{M}_t$ has property $D_{3k+1.5}$ when $t > 1$; has property D_{3k} when $t < 1$.*

Proof. By the stated hypothesis and Proposition 2.1, we have that $\mathcal{M}$ has similarity degree $\lfloor 2k \rfloor$. Then by Proposition 3.4, we have $\mathcal{M}_t$ has similarity degree at most $\lfloor 2k \rfloor$ when $t < 1$ and $\mathcal{M}_t$ has similarity degree at most $\lfloor 2k \rfloor + 1$ when $t > 1$. Hence by Lemma 5.1, $\mathcal{M}_t$ has property D_{3k} when $t < 1$ and $\mathcal{M}_t$ has property $D_{3k+1.5}$ when $t > 1$. ∎

Corollary 5.3. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of factors of type II_1 with Jones index $[\mathcal{M} : \mathcal{N}] < \infty$. Suppose that $\mathcal{N}$ has property D_k. Then $\mathcal{M}$ has property D_{3k}.*

Proof. It is similar to the proof of Corollary 5.2. ∎

Proposition 5.4. *Let $\mathcal{M}$ be a factor of type II_1 with similarity length $\ell < \infty$ and length constant K. Then $\mathcal{M}$ has property $D_{\min\{K,3\}\ell/2}$.*

Proof. Directly from Proposition 2.2 and Lemma 5.1. ■

Proposition 5.5. *Let G be a discrete group, $(\mathcal{B}_0, \tau_0)$ a finite von Neumann algebra with a normal faithful tracial state and $\sigma : G \mapsto Aut(\mathcal{B}_0, \tau_0)$ a trace preserving cocycle action of G on $(\mathcal{B}_0, \tau_0)$. Let $\mathcal{N} = \mathcal{B}_0 \rtimes_\sigma G$ be the corresponding crossed product von Neumann algebra with its normal faithful tracial state given by $\tau(\sum_{g\in G} B_g U_g) = \tau_0(B_e)$. Let H be a subgroup of G co-amenable in G and $\mathcal{B} = \mathcal{B}_0 \rtimes_\sigma H$. If $\mathcal{N}$ is a factor and $\mathcal{B}$ has property D_k, then $\mathcal{N}$ has similarity degree of at most D_{25k+11}.*

Proof. Since $\mathcal{B}$ has property D_k, $\mathcal{B}$ has similarity degree $\lfloor 2k \rfloor$ by Proposition 2.1. By Theorem 4.1, we have $\mathcal{M}$ has similarity degree $9\lfloor 2k \rfloor + 8$. Then by Lemma 5.1, we obtain that $\mathcal{M}$ has property D_{25k+11}. ■

Remark . By the remark in section 4 and Lemma 5.1, we have that if the crossed product $\mathcal{M} = \mathcal{B} \rtimes_\sigma G$ of a property Γ factor $\mathcal{B}$ by an amenable group G is a factor, then $\mathcal{M}$ has property D_{49}.

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