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EXISTENCE OF PERIODIC SOLUTIONS FOR HIGHER ORDER DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we study a sufficient condition for the existence of a periodic solution for the nth order dynamic equations on time scales. The results are shown by the use of coincidence degree theory. The necessary a priori bounds are based on Wirtinger type inequality established as Lemma B.

1. Introduction

The theory of dynamic systems on time scales $\mathbb{T}$(closed subsets of the reals) provides a framework for dealing with both continuous and discrete dynamic systems simultaneously so as to bring out a new insight of subtle differences for these two types of systems. More and more integrated results spring up in recent years, for example, [3, 4, 10, 11, 13]. Qualitative properties including stability, oscillation theory and asymptotic behavior of the solutions are also widely discussed.

The basic tool used in this article is the Mawhin's coincidence degree theory [9], which can be directly applied to study the periodic boundary value problems. Many researchers have already focused on this topic for a long period of time and plenty of essential papers are worked out, see [2, 7, 8, 17, 23, 24]. Recently, they are generalized naturally on the so-called field, time scales. This consideration can be related with many interesting biological issues including predator-prey and competition dynamic systems, population models, or other mutualism models, for example [5, 6, 14, 25, 26, 27]. We also refer more detailed treatment to more references [1, 12, 15, 16, 19, 20, 21, 22]. However, one can see that, until now, there are relatively few conclusions for higher order dynamic systems.

We briefly introduce the problem considered along the article. Let $\mathbb{T}$ be a time scale with period $\sigma(T)$, where $0, T \in \mathbb{T}$. The purpose is to consider the existence of solutions of the nonlinear periodic boundary value problem($n \geq 2$)

$$x^{\Delta^n} = f(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t),\ t \in \mathbb{T}, \tag{1.1}$$

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$$x^{\Delta^i}(0) = x^{\Delta^i}(\sigma(T)),\ 0 \le i \le n-1. \tag{1.2}$$

Throughout we assume that f is σ_{n+1}-completely delta differentiable on $D := \mathbb{R}^n \times [0, \sigma(T)]_{\mathbb{T}}$ (see Definition 2, [18]), its partial derivatives $f_i (1 \le i \le n)$ are continuous on D and f is $\sigma(T)$-periodic in t.

This paper is organized as follows. In next section, some fundamental concepts and inequalities on time scales are exposed. Section 3 is devoted to derive the necessary a priori bounds which are based on Wirtinger type inequality (Lemma B). In Section 4 we develop our main existence result (that is, Theorem A) of solutions for the higher order periodic boundary value problem $(1.1), (1.2)$.

2. Preliminaries on Time Scales

In this section, we give some of the basics of the time scale theory and refer to [3, 4] for more details. Let $\mathbb{T}$ be a time scale which means any closed subset of $\mathbb{R}$ and the interval $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. The embedding of $\mathbb{T}$ in $\mathbb{R}$ gives rise to the order and topological structure of the time scale in a canonical way.

Definition 1. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense* (otherwise: *right-scattered*), and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense* (otherwise: *left-scattered*). The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu(t) = \sigma(t) - t$. We denote $u^\sigma(t) \equiv u(\sigma(t))$ for $t \in \mathbb{T}$ and $\mathbb{T}^\kappa$ be the set of points of $\mathbb{T}$ except for a maximal element which is also left scattered.

Definition 2. The mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at each right-dense or maximal point $t \in \mathbb{T}$ and if the left-sided limit exists at each left-dense points. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T})$.

Definition 3. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number(provided it exists) with the property that, for any given $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)|\sigma(t) - s|| \le \epsilon|\sigma(t) - s| \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t. If $F^\Delta = f$, then we define the *Cauchy integral* by

$$\int_r^s f(\tau)\Delta\tau = F(s) - F(r) \text{ for } r, s \in \mathbb{T}.$$

Lemma 1. *If $t \in \mathbb{T}^\kappa$ and $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t, then*

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t).$$

Lemma 2. *If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t)$.*

Lemma 3. *If $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^\kappa$, then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

Lemma 4. *Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by $F(t) := \int_{t_0}^t f(\tau)\Delta\tau$ for all $t \in \mathbb{T}$ is an antiderivative of f, that is, $F^\Delta = f$.*

Definition 4. Let $\omega > 0$. A time scale $\mathbb{T}$ is called ω-periodic if $t+\omega \in \mathbb{T}$ whenever $t \in \mathbb{T}$. A function p is said to be ω-periodic on $\mathbb{T}$ if $p(t+\omega) = p(t)$ for all $t \in \mathbb{T}$.

Next, two well-known conclusions, the Cauchy-Schwarz inequality and one type of Wirtinger's inequality are stated.

Theorem 1. *Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b]_\mathbb{T} \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \le \sqrt{\Big\{\int_a^b |f(t)|^2\Delta t\Big\}\Big\{\int_a^b |g(t)|^2\Delta t\Big\}}.$$

Theorem 2. *Let M be positive and strictly monotone such that $M \in C^1_{rd}$. Then we have*

$$\int_a^b |M^\Delta(t)|(y^\sigma(t))^2\Delta t \le \Psi \int_a^b \frac{M(t)M^\sigma(t)}{|M^\Delta(t)|}(y^\Delta(t))^2\Delta t$$

for any $y \in C^1_{rd}$ with $y(a) = y(b) = 0$, where

$$\Psi = \Big\{\sqrt{\sup_{t\in[a,b]_\mathbb{T}} \frac{M(t)}{M^\sigma(t)}} + \sqrt{\sup_{t\in[a,b]_\mathbb{T}} \frac{\mu(t)|M^\Delta(t)|}{M^\sigma(t)} + \sup_{t\in[a,b]_\mathbb{T}} \frac{M(t)}{M^\sigma(t)}}\Big\}^2.$$

3. A Priori Estimates

In this section, let $x(t)$ be a $\sigma(T)$-periodic solution of the problem $(1.1), (1.2)$ and we shall get the priori bounds for the derivatives of $x(t)$ up to $(n-1)$th order by imposing certain conditions on the partial derivatives of f. Such priori estimates will be used to construct a bounded open set in the next section. Firstly, we need two crucial lemmas showed as follows:

Lemma A. *Let M be positive and strictly monotone such that $M \in C^1_{rd}$. Then we have*

$$\int_a^b |M^\Delta(t)|(y(t))^2\Delta t \leq \Phi \int_a^b \frac{M^\sigma(t)}{|M^\Delta(t)|}(y^\Delta(t))^2\Delta t$$

for any $y \in C^1_{rd}$ with $y(a) = y(b) = 0$, where

$$\Phi = \Big\{\sqrt{\sup_{t\in[a,b]_{\mathbb{T}}} M^\sigma(t)} + \sqrt{\sup_{t\in[a,b]_{\mathbb{T}}} \mu(t)|M^\Delta(t)| + \sup_{t\in[a,b]_{\mathbb{T}}} M^\sigma(t)}\Big\}^2.$$

Proof. We follow the steps of the proof of Theorem 2. For convenience we omit the argument (t) in the following arguments. Let

$$J = \int_a^b M^\Delta y^2\Delta t,\ V = \int_a^b \frac{M^\sigma}{|M^\Delta|}(y^\Delta)^2\Delta t,$$

$$\alpha = \sqrt{\sup_{[a,b]_{\mathbb{T}}} M^\sigma},\ \beta = \sup_{[a,b]_{\mathbb{T}}} \mu|M^\Delta|.$$

Without loss of generality we assume that M^Δ is of positive sign. Then we apply the Cauchy-Schwarz inequality(Theorem 1), Lemma 1, 3 and $y(a) = y(b) = 0$ to estimate

$$\begin{aligned}
J &= \int_a^b M^\Delta y^2\Delta t \\
&= \int_a^b (My^2)^\Delta\Delta t - \int_a^b M^\sigma y^\Delta(y+y^\sigma)\Delta t \\
&= -\int_a^b M^\sigma y^\Delta(y+y^\sigma)\Delta t \\
&\leq \int_a^b M^\sigma|y^\Delta||2y+\mu y^\Delta|\Delta t \\
&\leq 2\int_a^b M^\sigma|y^\Delta||y|\Delta t + \int_a^b \mu M^\sigma(y^\Delta)^2\Delta t \\
&= 2\int_a^b \sqrt{\frac{M^\sigma}{|M^\Delta|}}|y^\Delta|\sqrt{M^\sigma|M^\Delta|}|y|\Delta t + \int_a^b \frac{\mu M^\Delta M^\sigma}{|M^\Delta|}(y^\Delta)^2\Delta t \\
&\leq 2\Big\{\int_a^b \frac{M^\sigma}{|M^\Delta|}|y^\Delta|^2\Delta t\Big\}^{\frac{1}{2}}\Big\{\int_a^b M^\sigma|M^\Delta||y|^2\Delta t\Big\}^{\frac{1}{2}} + \beta V \\
&\leq 2\alpha\sqrt{JV} + \beta V.
\end{aligned}$$

Therefore, by denoting $H = \sqrt{\frac{J}{V}}$, we find that $H^2 - 2\alpha H - \beta \leq 0$, and solving for $H \geq 0$ we obtain

$$\frac{J}{V} = H^2 \leq (\alpha + \sqrt{\alpha^2+\beta})^2 = \Phi$$

so that the proof is complete. ■

Lemma B. *For any* $y \in C^1_{rd}$ *with* $y(a) = y(b)$, $y^\Delta(a) = y^\Delta(b)$ *and* $\int_a^b y(t)\Delta t = 0$, *we have*

$$\int_a^b (y(t))^2 \Delta t \le K^2 \int_a^b (y^\Delta(t))^2 \Delta t,$$

where

$$K := \sigma(b) - a + \sqrt{(\sigma(b)-a)^2 + (\sigma(b)-a) \sup_{t\in[a,b]_{\mathbb{T}}} \mu(t)}.$$

Proof. Let $f(t) = \int_a^t y^\Delta(s)\Delta s$, $t \in [a,b]_{\mathbb{T}}$, then $f(a) = f(b) = 0$. Choose $M(t) = t - a$ and apply Lemma A to $f(t)$, then we can conclude this lemma. ■

In what follows, let $\mathbb{T}$ be a $\sigma(T)$-periodic time scale containing $\{0, T\}$ and denote $[0, \sigma(T)]_{\mathbb{T}} \equiv I$. If h is a real-valued function which is bounded on the set W, we put $|h|_W = \sup_{z\in W} |h(z)|$. If h is square integrable over I, we denote $||h|| = \sqrt{\int_0^{\sigma(T)} |h(z)|^2 \Delta z}$. Let

$$Z = \{z \in C(\mathbb{T}) : z \text{ is } \sigma(T)\text{-periodic}\}$$

with norm $|z|_Z = \max_{t\in I} |z(t)|$ and

$$X = \{x \in C^{\Delta^{n-1}}(\mathbb{T}) : x \text{ is } \sigma(T)\text{-periodic with } x^{\Delta^i}(0) = x^{\Delta^i}(\sigma(T)), 0 \le i \le n-1\}$$

with norm $|x|_X = \Sigma_{i=0}^{n-1} \max_{t\in I} |x^{\Delta^i}(t)|$. We see that Z and X are real Banach spaces. Our useful estimates are immediately listed as follows:

Lemma C. *Let*

$$C = \sigma^2(T) + \sqrt{[\sigma^2(T)]^2 + \sigma^2(T) \sup_{t\in I} \mu(t)}.$$

When $n \ge 3$, *we assume that there exists a constant* $\eta > 0$ *such that*

$$f_2 \ge -\eta > -\frac{1}{CC_1} \text{ on } D, \tag{3.1}$$

$$\sum_{i=1, i\neq n-2}^{n-1} C^{n-i-1} C_1 |f_{n-i}|_D + C^{n-2} C_1 |f_n|_D \sigma(T) < 1 - \eta C C_1, \tag{3.2}$$

where $C_1 = C + \sup_{t\in I} \mu(t)$, *are satisfied; when* $n = 2$, *assume that there exists a constant* $\zeta < \frac{1}{\sigma(T)^2}$ *such that*

$$f_2 \ge -\zeta \text{ on } D, \tag{3.3}$$

$$|f_1|_D < \frac{1}{\sigma(T)} - \zeta\sigma(T). \tag{3.4}$$

If there exists $R > 0$ *such that*

$$\int_0^{\sigma(T)} f(x^{\Delta^{n-1}}(t), \cdots, x(t), t)\Delta t \neq 0 \tag{3.5}$$

for each $x \in X$ *such that*

$$\inf_{t\in\mathbb{T}} |x(t)| \geq R. \tag{3.6}$$

Then there is a positive constant B *such that*

$$||x^{\Delta^i}|| \leq M_i,\ 0 \leq i \leq n-1, \tag{3.7}$$

and

$$|x^{\Delta^i}(t)| \leq \{\frac{1}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}}{C}\}M_i,\ 0 \leq i \leq n-2, \tag{3.8}$$

where

$$M_i = C^{n-1-i}B,\ \textit{for}\ 1 \leq i \leq n-2,$$
$$M_0 = \sqrt{\sigma(T)}\{R + C^{n-2}B\sqrt{\sigma(T)}\}.$$

Furthermore, there exists a constant L *such that*

$$|x^{\Delta^{n-1}}(t)| \leq N^*, \tag{3.9}$$

here $N^* = 2\sqrt{\sigma(T)}L + \frac{B}{\sqrt{\sigma(T)}}$, *$L$ is given as in* (3.26).

Proof. Since f is σ_{n+1}-completely delta differentiable on $D := \mathbb{R}^n \times I$, according to Theorem 9 [18], we rewrite (1.1) in the form

$$x^{\Delta^n} = \sum_{i=0}^{n-1} x^{\Delta^i} F_{n-i}(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t) + g(t) \tag{3.10}$$

where

$$F_{n-i}(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t) = \int_0^1 f_{n-i}(\xi x^{\Delta^{n-1}}, \cdots, \xi x, t)d\xi,$$
$$g(t) = f(0, 0, \cdots, 0, t).$$

Multiplying by $x^{\Delta^{n-2}\sigma}$ on both sides of (3.10) and integrating from 0 to $\sigma(T)$, we have

$$\begin{aligned}\int_0^{\sigma(T)}[x^{\Delta^{n-1}}]^2\Delta t = &-\sum_{i=0}^{n-1}\int_0^{\sigma(T)} x^{\Delta^{n-2}\sum}x^{\Delta^i}F_{n-i}(x^{\Delta^{n-1}},\cdots,x,t)\Delta t\\ &-\int_0^{\sigma(T)} x^{\Delta^{n-2}\sigma}(t)g(t)\Delta t.\end{aligned} \tag{3.11}$$

Note that we also have

$$\begin{aligned}&-\int_0^{\sigma(T)} x^{\Delta^{n-2}\sigma}(t)x(t)F_n(x^{\Delta^{n-1}},\cdots,x,t)\Delta t\\ &\quad\le |F_n|_D|x|_I\sqrt{\sigma(T)}||x^{\Delta^{n-2}\sigma}||, \qquad \text{when } n\ge 3\\ &\qquad\le \zeta|x|_I^2\sigma(T), \qquad \text{when } n=2.\end{aligned} \tag{3.12}$$

By integrating (1.1) over $\sigma(T)$ period, we then obtain

$$\int_0^{\sigma(T)} f(x^{\Delta^{n-1}}(t),\cdots,x(t),t)\Delta t = 0,$$

when $x(t)$ is a possible $\sigma(T)$-periodic solution of (1.1). Hence, by means of (3.5), (3.6), this implies that there exists $\xi\in I$ such that $|x(\xi)|<R$. For $t\in I$, we have

$$|x(t)-x(\xi)|\le\int_\xi^t|x^\Delta(s)|\Delta s\le\int_0^{\sigma(T)}|x^\Delta(s)|\Delta s\le||x^\Delta||\sqrt{\sigma(T)}.$$

Therefore, it follows from the above inequality and Lemma B(setting $b=\sigma(T)$, $a=0$) that

$$|x|_I\le R+||x^\Delta||\sqrt{\sigma(T)}, \text{ when } n=2, \tag{3.13}$$

$$\le R+C^{n-2}||x^{\Delta^{n-1}}||\sqrt{\sigma(T)}, \text{ when } n\ge 3. \tag{3.14}$$

For the case $n=2$, from (3.11), (3.12), we have

$$||x^\Delta||^2\le\zeta|x|_I^2+|F_1|_D|x|_I||x^\Delta||\sqrt{\sigma(T)}+|x|_I||g||\sqrt{\sigma(T)}.$$

Using (3.13), we obtain

$$a||x^\Delta||^2-b||x^\Delta||-c\le 0,$$

where

$$a:=1-\zeta\sigma(T)^2-|F_1|_D\sigma(T)>0 \text{ by (3.4)},$$
$$b:=2\zeta R\sigma(T)^{\frac{3}{2}}+|F_1|_DR\sqrt{\sigma(T)}+||g||\sigma(T),$$

$$c := \zeta R^2\sigma(T) + R||g||\sqrt{\sigma(T)}.$$

Hence, we have

$$||x^{\Delta}|| \le S, \tag{3.15}$$

here

$$S = \frac{b+\sqrt{b^2+4ac}}{2a}.$$

For the case $n \ge 3$, from (3.11), using (3.1), the Schwarz inequality (Theorem 1) and Wirtinger type inequality (Lemma B) established, we get

$$\begin{aligned}||x^{\Delta^{n-1}}||^2 \le &-\int_0^{\sigma(T)} x^{\Delta^{n-2}\sigma} xF_n(x^{\Delta^{n-1}}, \cdots, x, t)\Delta t\\ &+\sum_{i=1, i\neq n-2}^{n-1} \{C^{n-i} + C^{n-i-1}\sup_{t\in I}\mu(t)\}||x^{\Delta^{n-1}}||^2|F_{n-i}|_D\\ &+\eta\{C^2 + C\sup_{t\in I}\mu(t)\}||x^{\Delta^{n-1}}||^2 + ||g||\{C+\sup_{t\in I}\mu(t)\}||x^{\Delta^{n-1}}||.\end{aligned}$$

Since, by (3.14),

$$\begin{aligned}&-\int_0^{\sigma(T)} x^{\Delta^{n-2}\sigma} xF_n(x^{\Delta^{n-1}}, \cdots, x, t)\Delta t\\ &\quad\le \{R + C^{n-2}||x^{\Delta^{n-1}}||\sqrt{\sigma(T)}\}|F_n|_D\{C+\sup_{t\in I}\mu(t)\}||x^{\Delta^{n-1}}||\sqrt{\sigma(T)},\end{aligned}$$

one can immediately get

$$\begin{aligned}||x^{\Delta^{n-1}}||^2 \le &C^{n-2}C_1||x^{\Delta^{n-1}}||^2|F_n|_D\sigma(T) + \sum_{i=1, i\neq n-2}^{n-1} C^{n-i-1}C_1||x^{\Delta^{n-1}}||^2|F_{n-i}|_D\\ &+\eta CC_1||x^{\Delta^{n-1}}||^2 + C_1\{||g|| + R|F_n|_D\sqrt{\sigma(T)}\}||x^{\Delta^{n-1}}||,\end{aligned}$$

that is,

$$\begin{aligned}&\left[1 - \sum_{i=1, i\neq n-2}^{n-1} C^{n-i-1}C_1|F_{n-i}|_D - C^{n-2}C_1|F_n|_D\sigma(T) - \eta CC_1\right]||x^{\Delta^{n-1}}||\\ &\qquad\le C_1\{||g|| + R|F_n|_D\sqrt{\sigma(T)}\}.\end{aligned}$$

Note that the term in the bracket of the above inequality is positive by the hypothesis (3.2). Thus we have

$$||x^{\Delta^{n-1}}|| \le K\{||g|| + R|F_n|_D\sqrt{\sigma(T)}\}, \tag{3.16}$$

where

$$K = \{C_1^{-1} - \sum_{i=1, i\neq n-2}^{n-1} C^{n-i-1}|F_{n-i}|_D - C^{n-2}|F_n|_D\sigma(T) - \eta C\}^{-1}.$$

Combing (3.15) and (3.16), we have for $n \geq 2$, we have

$$||x^{\Delta^{n-1}}|| \leq B, \tag{3.17}$$

where

$$B = \max\{S, K[||g|| + R|F_n|_D\sqrt{\sigma(T)}]\}.$$

By (3.17) and the Wirtinger type inequality (Lemma B) again, we have

$$||x^{\Delta^i}|| \leq C^{n-1-i}||x^{\Delta^{n-1}}|| \leq M_i \tag{3.18}$$

for $1 \leq i \leq n-1$ and by (3.13), (3.14), $||x|| \leq M_0$. Also, (3.8) can derived from (3.18). For $t \in I$, $0 \leq i \leq n-2$, we have

$$|x^{\Delta^i}(t) - x^{\Delta^i}(0)| \leq \int_0^t |x^{\Delta^{i+1}}(s)|\Delta s \leq ||x^{\Delta^{i+1}}||\sqrt{\sigma(T)} \leq C^{-1}\sqrt{\sigma(T)}M_i. \tag{3.19}$$

From the inequality (3.19), we have for $0 \leq i \leq n-2$,

$$|x^{\Delta^i}(t)| \leq |x^{\Delta^i}(0)| + C^{-1}\sqrt{\sigma(T)}M_i, \tag{3.20}$$

and

$$|x^{\Delta^i}(t)| \geq |x^{\Delta^i}(0)| - C^{-1}\sqrt{\sigma(T)}M_i. \tag{3.21}$$

If $|x^{\Delta^i}(0)| - C^{-1}\sqrt{\sigma(T)}M_i \leq 0$, it follows immediately that

$$|x^{\Delta^i}| \leq 2C^{-1}\sqrt{\sigma(T)}M_i \text{ on } I. \tag{3.22}$$

If $|x^{\Delta^i}(0)| - C^{-1}\sqrt{\sigma(T)}M_i > 0$, we integrate (3.21) on both sides from 0 to $\sigma(T)$ and get

$$||x^{\Delta^i}|| \geq \{|x^{\Delta^i}(0)| - C^{-1}\sqrt{\sigma(T)}M_i\}\sqrt{\sigma(T)}. \tag{3.23}$$

From (3.18) and (3.23), we have

$$|x^{\Delta^i}(0)| \leq \left\{\frac{1}{\sqrt{\sigma(T)}} + \frac{\sqrt{\sigma(T)}}{C}\right\}M_i. \tag{3.24}$$

Hence, it follows from (3.20) that

$$|x^{\Delta^i}(t)| \leq \{\frac{1}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}}{C}\}M_i, \ t \in I. \tag{3.25}$$

Combining (3.22) and (3.25), we obtain (3.8). On the other hand, rewrite (1.1) in the alternative form

$$x^{\Delta^n} = \sum_{i=0}^{n-1} x^{\Delta^i} G_{n-i}(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t) + g(t),$$

where for $0 \leq i \leq n-1$,

$$G_{n-i}(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t) = \int_0^1 f_{n-i}(0, \cdots, 0, \xi x^{\Delta^i}, x^{\Delta^{i-1}}, \cdots, x, t)d\xi.$$

Note that for any $t_1, t_2 \in I$, we have

$$\begin{aligned}|x^{\Delta^{n-1}}(t_1) - x^{\Delta^{n-1}}(t_2)| &\leq \int_{t_1}^{t_2} |x^{\Delta^n}(s)|\Delta s\\ &\leq \sum_{i=0}^{n-1} \int_{t_1}^{t_2} |x^{\Delta^i} G_{n-i}(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t)|\Delta t + \int_{t_1}^{t_2} |g(t)|\Delta t.\end{aligned}$$

By the Schwarz inequality we have, for $n = 2$,

$$|x^{\Delta^{n-1}}(t_1) - x^{\Delta^{n-1}}(t_2)| \leq \sqrt{t_1 - t_2}\{|f_1|_D M_1 + M_0|f_2|_E + ||g||\}.$$

or for $n \geq 3$,

$$\begin{aligned}&|x^{\Delta^{n-1}}(t_1) - x^{\Delta^{n-1}}(t_2)|\\ &\qquad \leq \sqrt{t_1 - t_2}\{\sum_{i=1, i\neq n-2}^{n-1} |f_{n-i}|_D M_i + M_{n-2}|f_2|_E + |f_n|_D M_0 + ||g||\},\end{aligned}$$

that is,

$$|x^{\Delta^{n-1}}(t_1) - x^{\Delta^{n-1}}(t_2)| \leq L\sqrt{t_1 - t_2},$$

where

$$\begin{aligned}L = \max\{&(1-\zeta)C^{n-2}B + |f_2|_E\big[R\sqrt{\sigma(T)} + C^{n-2}B\sigma(T)\big] + ||g||,\\ &(\frac{1}{C_1} - \eta C)B + |f_2|_E CB + |f_n|_D\big[R\sqrt{\sigma(T)} + C^{n-2}B\sigma(T)\big] + ||g||\}\end{aligned} \tag{3.26}$$

and

$$E = \{0\} \times \Pi_{i=0}^{n-2} \big[-(\frac{1}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}}{C}) M_{n-2-i}, (\frac{1}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}}{C}) M_{n-2-i} \big] \times I.$$

In particular, for $t \in I$, we obtain

$$|x^{\Delta^{n-1}}(t) - x^{\Delta^{n-1}}(0)| \leq L\sqrt{\sigma(T)}.$$

Following the calculations as we did from $(3.19) - (3.25)$, we get

$$|x^{\Delta^{n-1}}(t)| \leq N^*. \qquad \blacksquare$$

4. Existence Theorem

In this section we are going to show the existence of periodic solutions for the problem (1.1), (1.2) via coincidence degree theory associated with Lemma C. Our proof utilizes a continuation theorem of Mawhin [9] which we state here for the reader's convenience.

Let X and Z be real Banach spaces and let

$$L : domL \subset X \to Z$$

be a linear Fredholm mapping, that is, imL(the range of L) is closed and the dimension of $kerL$(the kernel of L) and codimension of imL are finite. Let Ω be a bounded open subset of X and let

$$N : \bar{\Omega} \subset X \to Z$$

be a (not necessarily linear) mapping which is L-compact on $\bar{\Omega}$, that is, if $P : X \to X$ and $Q : Z \to Z$ denote bounded linear projections such that $imP = kerL$, $imL = kerQ$ and if

$$K_{P.Q} = (L \mid kerP \cap domL)^{-1}(I - Q),$$

where I is the identity map on Z, then $QN : \bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N : \bar{\Omega} \to X$ is completely continuous. The following continuation theorem has been established.

Theorem 3. *Let the above assumptions hold and let $indL(= dim\ kerL - codim\ imL) = 0$. Further assume*

(1) *for each $\lambda \in (0, 1)$ and each $x \in domL \cap \partial\Omega$, $Lx \neq \lambda Nx$,*

(2) *for each $x \in kerL \cap \partial\Omega$, $Nx \notin imL$, that is, $QNx \neq 0$,*

(3) *$d_B(JQN \mid kerL, \Omega \cap kerL, 0) \neq 0$, where d_B denotes the Brouwer degree and $J : imQ \to kerL$ is any isomorphism.*

Then there exists at least one $x \in domL \cap \bar{\Omega}$ *such that* $Lx = Nx$.

With the aid of Theorem 3, we shall now construct our main result as follows.

Theorem A. *Suppose that* $(3.1) - (3.2)$ $($*or* $(3.3) - (3.4)$ *as* $n = 2)$ *and* (3.5) *hold. Assume that*

$$\int_0^{\sigma(T)} f(0, \cdots, 0, R_0, t)\Delta t \int_0^{\sigma(T)} f(0, \cdots, 0, -R_0, t)\Delta t < 0, \tag{4.1}$$

where $R_0 = \{\frac{1}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}}{C}\}M_0 + 1$. *Then the problem* $(1.1), (1.2)$ *has a* $\sigma(T)$-*periodic solution.*

Proof. Define $L : domL \subset X \to Z$ by $Lx = x^{\Delta^n}$, where $domL = \{x \in X : x \in C^{\Delta^n}(\mathbb{T})\}$. Define $N : X \to Z$ by $Nx(t) = f(x^{\Delta^{n-1}}(t), \cdots, x^{\Delta}(t), x(t), t)$. We see that $x \in kerL$ if and only if $x(t) = x(0)$ for all $t \in I$ if and only if $x(t) = \frac{1}{\sigma(T)} \int_0^{\sigma(T)} x(s)\Delta s$, for all $t \in I$. Hence $dim\ kerL = 1$. And we also note that $z \in Z$ is in imL if and only if there exists a solution $x \in X$ satisfying $x^{\Delta^n} = z(t)$ if and only if $\int_0^{\sigma(T)} z(s)\Delta s = 0$. Thus we have $imL = \{z \in Z : \int_0^{\sigma(T)} z(s)\Delta s = 0\}$. It is obvious that imL is closed in Z.

Note that $dim(cokerL) = dim(Z/imL) = 1$. Indeed, let $[z_1]$ and $[z_2]$ be two equivalent classes in $cokerL$ other than imL; then $\int_0^{\sigma(T)} z_i(s)\Delta s \neq 0$, $i = 1, 2$. Hence there exists a real constant $c \neq 0$ such that $z_1 - cz_2 \in imL$. Thus $dim(Z/imL) = 1$. Therefore, $indL = 0$. Define

$$\begin{aligned} \Omega = \{x \in X : |x^{\Delta^i}|_I < \frac{M_i}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}M_i}{C} + 1, \\ 0 \le i \le n-2,\ |x^{\Delta^{n-1}}|_I < N^* + 1\}, \end{aligned}$$

where M_i and N^* are given in the Lemma C. Then Ω is a bounded open subset of X such that $\bar{\Omega} \cap domL \neq \emptyset$. Note that N is continuous on $\bar{\Omega}$ and $N(\bar{\Omega})$ is bounded in Z.

Define $Q : Z \to Z$ by $Qx = \frac{1}{\sigma(T)} \int_0^{\sigma(T)} x(s)\Delta s$. Then Q is a continuous projection with $imL = im(I-Q)$, $imQ = kerL$. Define $T : imL \to X$ to be a right inverse of L so that $LTz = z$ for every $z \in imL$ and $PT = 0$, where $P : X \to X$ is some projection with $imP = kerL$. By Arzela-Ascoli theorem, we see that $K_{P,Q}N \equiv T(I - Q)N$ is completely continuous on $\bar{\Omega}$.

Next we claim that $Lx \neq \lambda Nx$ for every $x \in \partial\Omega \cap domL$ and $\lambda \in (0, 1)$. Suppose not; then there exists a function $x(t)$ satisfying $x^{\Delta^n} = \lambda f(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t)$ with $x^{\Delta^i}(0) = x^{\Delta^i}(\sigma(T))$, $0 \le i \le n-1$ for some $\lambda \in (0, 1)$ and $(x^{\Delta^{n-1}}(t), \cdots, x^{\Delta}(t),$

$x(t), t) \in G$, where

$$G = \{(x^{\Delta^{n-1}}, \cdots, x^{\Delta}, x, t) : |x^{\Delta^i}|_I \leq \frac{M_i}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}M_i}{C} + 1,$$
$$0 \leq i \leq n-2,\ |x^{\Delta^{n-1}}|_I \leq N^* + 1, t \in I\}$$

and $(x^{\Delta^{n-1}}(t_0), \cdots, x^{\Delta}(t_0), x(t_0), t_0) \in \partial G$ for some $t_0 \in I$. This is impossible, since by arguments similar to those in the Lemma C, one can see that $|x^{\Delta^i}|_I \leq \frac{M_i}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}M_i}{C}$, $0 \leq i \leq n-2$ and $|x^{\Delta^{n-1}}|_I \leq N^*$.

We also see that $QNa \neq 0$ for every $a \in \partial\Omega \cap kerL$. Indeed, $a(t) = \pm(\frac{M_i}{\sqrt{\sigma(T)}} + \frac{2\sqrt{\sigma(T)}M_i}{C} + 1) = \pm R_0$ and by the hypothesis (4.1),

$$QNa = \frac{1}{\sigma(T)} \int_0^{\sigma(T)} f(0, \cdots, 0, \pm R_0, t)\Delta t \neq 0.$$

Finally we claim $d_B(JQN \mid kerL, \Omega \cap kerL, 0) \neq 0$. Here we take J to be an identity operator in Z since $imQ = kerL$. From the hypothesis (4.1) we see that

$$d_B(QN \mid kerL, \Omega \cap kerL, 0)$$
$$= d_B\left(\frac{1}{\sigma(T)} \int_0^{\sigma(T)} f(0, \cdots, 0, \bullet, t)\Delta t, (-R_0, R_0), 0\right) \neq 0.$$

By Theorem 3, $Lx = Nx$ has at least one solution $x \in \bar{\Omega}$. Therefore the problem $(1.1), (1.2)$ has a periodic solution. ∎

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