TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 6, pp. 2187-2196, December 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

$A_\infty(\mathbb{R}^n)$ WEIGHTS AND THE LOCAL MAXIMAL OPERATOR

Guoen Hu and Wentan Yi

Abstract. Let $s \in (0, 1/2)$, $M_{0,\,s}$ be the local maximal operator of John and Strömberg, and $\mathcal{M}_{0,\,s}$ the multi(sub)linear local maximal operator. In this paper, the authors give some characterizations of the weights $w_1, \dots, w_\ell$ for which the operator $\mathcal{M}_{0,\,s}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times \dots \times L^{p_\ell}(\mathbb{R}^n, w_\ell)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$ with $\nu_{\vec{w}} = \prod_{k=1}^{\ell} w_k^{p/p_k}, p_1, \dots, p_\ell \in (0, \infty)$ and $1/p = \sum_{1\le k\le \ell} 1/p_k$. A new characterization of $A_\infty(\mathbb{R}^n)$ weights and a characterization of weights w which satisfies $w^\theta \in A_\infty(\mathbb{R}^n)$ for some $\theta \in (0, \infty)$, are also obtained.

1. Introduction and Statements of Results

The class of $A_p(\mathbb{R}^n)$ weights was introduced by Muckenhoupt [5], in order to characterize the weight w for which the Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^n, w)$. Let w be a weight, that is, w is a non-negative and locally integrable function. For $p \in [1, \infty)$, a weight w is said to be a $A_p(\mathbb{R}^n)$ weight if

$$\sup_{Q\subset\mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)^{1/p}\Big(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}(x)\,dx\Big)^{1/p'} < \infty,$$

where and in the following, $(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}})^{1/p'}$ in the case of $p = 1$ is understood as $(\inf_{x\in Q} w_k)^{-1}$. As it is well known, the operator M is bounded on $L^p(\mathbb{R}^n, w)$ when $p \in (1, \infty)$ if and only if $w \in A_p(\mathbb{R}^n)$, and is bounded from $L^1(\mathbb{R}^n, w)$ to $L^{1,\infty}(\mathbb{R}^n, w)$ if and only if $w \in A_1(\mathbb{R}^n)$. In the last forty years there has been significant progress in the study of $A_p(\mathbb{R}^n)$ weights and the behavior of classical operators on various weighted spaces with A_p weights, see [1, Chap. 9].

Fairly recently, to study weighted estimates for the multilinear Calderón-Zygmund operators, Lerner et al. [4] introduced the multi(sub)linear Hardy-Littlewood maximal operator $\mathcal{M}$ defined by

Received January 16, 2012, accepted March 30, 2012.
Communicated by Chin-Cheng Lin.
2010 *Mathematics Subject Classification*: 42B25.
Key words and phrases: Weight, $A_\infty(\mathbb{R}^n)$, Weighted norm inequality, Local maximal operator.
The research was supported by the NNSF (Grant No. 10971228) of China.

$$\mathcal{M}(f_1, \ldots, f_\ell)(x) = \sup_{Q\ni x} \prod_{k=1}^{\ell} \Big(\frac{1}{|Q|}\int_Q |f_k(y)|\, dy\Big),$$

and proved that for $p_1, \ldots, p_\ell \in [1, \infty)$, the operator $\mathcal{M}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1)\times \ldots \times L^{p_\ell}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n, \nu_{\vec{w}})$ if and only if $\vec{w} \in A_{\vec{P}}(\mathbb{R}^n)$, namely,

$$\sup_{Q\subset\mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)\, dx\Big)^{1/p} \prod_{k=1}^{\ell} \Big(\frac{1}{|Q|}\int_Q w_k^{-1/(p_k-1)}(x)\Big)^{1/p_k'} < \infty. \tag{1.1}$$

Moreover, in the setting of $\max_{1\le k\le \ell} p_k > 1$, $\vec{w} \in A_{\vec{P}}(\mathbb{R}^n)$ is also the sufficient condition such that $\mathcal{M}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times \ldots \times L^{p_\ell}(\mathbb{R}^n)$ to $L^{p}(\mathbb{R}^n, \nu_{\vec{w}})$, where and in the sequel, for $\vec{P} = (p_1, \ldots, p_\ell)$ and $\vec{w} = (w_1, \ldots, w_\ell)$, we set $p \in (0, \infty)$ such that $1/p = \sum_{1\le k\le \ell} 1/p_k$ and $\nu_{\vec{w}} = \prod_{1\le k\le\ell} w_k^{p/p_k}$. This result is very interesting and leads to the right class of multiple weights for the multilinear Calderón-Zygmund operators.

Now we consider the analogy of the operator $\mathcal{M}$ in the setting of local maximal operator. Let $s \in (0, 1)$ and f be a measurable function in $\mathbb{R}^n$. Set

$$m_{0,s;Q}(f) = \inf\{\lambda > 0 : |\{x \in Q : |f(x)| > \lambda\}| < s|Q|\},$$

and define the local maximal operator $M_{0,s}$ by

$$M_{0,s}f(x) = \sup_{Q\ni x} m_{0,s;Q}(f).$$

This operator is useful in the study of boundedness of some class operators (see [2] and [3]). The multi(sub)linear version of $M_{0,s}$ is defined by

$$\mathcal{M}_{0,s}(f_1, \ldots, f_\ell)(x) = \sup_{Q\ni x} \prod_{k=1}^{\ell} m_{0,s;Q}(f_k).$$

The purpose of this paper is to consider the weighted norm inequalities with multi-weight for the operator $\mathcal{M}_{0,s}$. We will give some characterizations of the weights $w_1, \ldots, w_\ell$ for which $\mathcal{M}_{0,s}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times \ldots \times L^{p_\ell}(\mathbb{R}^n, w_\ell)$ to $L^{p}(\mathbb{R}^n, \nu_{\vec{w}})$ with $p_1, \ldots, p_\ell \in (0, \infty]$, $1/p = \sum_{1\le k\le\ell} 1/p_\ell$, and $\nu_{\vec{w}} = \prod_{k=1}^{\ell} w_k^{p/p_k}$. As usual, set $A_\infty(\mathbb{R}^n) = \cup_{p\ge1} A_p(\mathbb{R}^n)$ (see [1] for the characterizations of $A_\infty(\mathbb{R}^n)$ weights). For $\vec{P} = (p_1, \ldots, p_\ell)$ with $p_1, \ldots, p_\ell \in (0, \infty]$ and $r \in (0, \min_{1\le k\le q\ell} p_k)$, set $\vec{P}/r = (p_1/r, \ldots, p_\ell/r)$ and

$$A_{\vec{P},\infty}(\mathbb{R}^n) = \bigcup_{r:\, 0<r<\min_{1\le k\le\ell} p_k} A_{\vec{P}/r}(\mathbb{R}^n).$$

It is obvious that when $\ell = 1$, $A_{\vec{P},\infty}(\mathbb{R}^n)$ is just the classical $A_\infty(\mathbb{R}^n)$. Our main result can be stated as follows.

Theorem 1.1. *Let* $s \in (0, 1/(2\ell))$, $w_1, \ldots, w_\ell$ *be weights,* $p_1, \ldots, p_\ell \in (0, \infty)$ *with* $1/p = \sum_{k=1}^{\ell} 1/p_k$. *Then the following conditions are equivalent*

(i) *the operator* $\mathcal{M}_{0,s}$ *is bounded from* $L^{p_1}(\mathbb{R}^n, w_1) \times \ldots \times L^{p_\ell}(\mathbb{R}^n, w_\ell)$ *to* $L^p(\mathbb{R}^n, \nu_{\vec{w}})$;

(ii) *the operator* $\mathcal{M}_{0,s}$ *is bounded from* $L^{p_1}(\mathbb{R}^n, w_1) \times \ldots \times L^{p_\ell}(\mathbb{R}^n, w_\ell)$ *to* $L^{p,\infty}(\mathbb{R}^n, \nu_{\vec{w}})$;

(iii) *there exists a constant* $\tau \in (0, 1/(2\ell))$ *such that for any cube* Q *and measurable sets* $E_1, \ldots, E_\ell \subset Q$, *if* $|E_k| > \tau|Q|$ *for any* k *with* $1 \le k \le \ell$, *then* $\prod_{k=1}^{\ell}\{w_k(E_k)\}^{p/p_k} \gtrsim \nu_{\vec{w}}(Q)$;

(iv) *there exists a constant* $\tau \in (0, 1/(2\ell))$, *such that*

$$\sup_{Q\subset\mathbb{R}^n}\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)\,dx\Big)^{1/p}\prod_{k=1}^{\ell}\{m_{0,\tau;Q}(w_k^{-1})\}^{1/p_k} < \infty; \tag{1.2}$$

(v) $\nu_{\vec{w}} \in A_\infty(\mathbb{R}^n)$ *and there exists a constant* $\gamma \in (0, \infty)$ *such that for each* k *with* $1 \le k \le \ell$, $w_k^\gamma \in A_\infty(\mathbb{R}^n)$.

(vi) $\vec{w} \in A_{\vec{P},\infty}(\mathbb{R}^n)$.

Remark 1.1. For the case of $\ell = 1$, Theorem 1.1 tells us that $w \in A_\infty(\mathbb{R}^n)$ if and only if for some $s \in (0, 1/2)$,

$$\sup_{Q\subset\mathbb{R}^n}\Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)m_{0,s;Q}(w^{-1}) < \infty.$$

This is a new characterization of $A_\infty(\mathbb{R}^n)$ weights. Also, Theorem 1.1 implies a characterization of $A_\infty(\mathbb{R}^n)$ weights in terms of the local maximal operator $M_{0,s}$.

Remark 1.2. For the case of $s \in (0, 1)$, the condition (vi) also implies (i) in Theorem 1.1. However, we do not know if (vi) is a necessary condition such that $\mathcal{M}_{0,s}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1)\times\ldots\times L^{p_\ell}(\mathbb{R}^n, w_\ell)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$ when $s \in (1/(2\ell), 1)$.

To prove Theorem 1.1, we will use the following result, which is new and of independent interest.

Theorem 1.2. *Let* w *be a weight,* $s_1, s_2 \in (0, 1/2)$ *with* $s_1 + s_2 < 1/2$. *The following three conditions are equivalent:*

(a) *There exists a constant* $\theta \in (0, \infty)$ *such that* $w^\theta \in A_2(\mathbb{R}^n)$;

(b) *There exists a constant* $\gamma \in (0, \infty)$ *such that* $w^\gamma \in A_\infty(\mathbb{R}^n)$;

(c)

$$\sup_{Q\subset\mathbb{R}^n} m_{0,s_1;Q}(w)m_{0,s_2;Q}(w^{-1}) < \infty. \tag{1.3}$$

We now make some conventions. Throughout this paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constant with subscript such as C_1, does not change in different occurrences. The symbol $A \lesssim B$ means that there exists a positive constant C such that $A \lesssim CB$. Given $\lambda > 0$ and a cube Q, λQ denotes the cube with the same center as Q and whose side length is λ times that of Q.

2. Proof of Theorems

We begin with some preliminary lemmas.

Lemma 2.1. *Let* $s_1, \dots, s_\ell, s \in (0, 1)$*,* $f_1, \dots, f_\ell$ *be measurable functions. Then for any cube* Q*,*

$$m_{0,\,s_1+s_2;\,Q}(f_1 + f_2) \le m_{0,\,s_1;\,Q}(f_1) + m_{0;\,s_2;\,Q}(f_2), \tag{2.1}$$

and

$$m_{0,\,\sum_{1\le k\le \ell} s_k;\,Q}(f_1 \dots f_\ell) \le \prod_{k=1}^{\ell} m_{0,\,s_k;\,Q}(f_k). \tag{2.2}$$

Proof. The proofs for these two inequalities are similar and we only consider (2.2). Without loss of generality, we may assume that

$$m_{0,\,s_1;\,Q}(f_1) = \dots = m_{0,\,s_\ell;\,Q}(f_\ell) = 1.$$

Then for any $\epsilon > 0$ and k with $1 \le k \le \ell$,

$$|\{x \in Q : |f_k(x)| > 1 + \epsilon\}| < s_k|Q|.$$

This in turn implies that

$$|\{x \in Q : |f_1(x) \dots f_\ell(x)| > (1 + \epsilon)^\ell\}| < \sum_{k=1}^{\ell} s_k|Q|,$$

and so

$$m_{0,\,\sum_{1\le k\le \ell} s_k;\,Q}(f_1 \dots f_\ell) \le (1 + \epsilon)^\ell.$$

Our desired conclusion then follows directly.

Lemma 2.2. *Let* w *be a weight. Then* $w \in A_\infty(\mathbb{R}^n)$ *if and only if for some* $s \in (0, 1)$

$$\sup_{Q\subset \mathbb{R}^n} m_{0,\,s;\,Q}(w^{-1})\Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big) < \infty; \tag{2.3}$$

Proof. At first, we claim that if (2.3) is true, then w is doubling. In fact, by the inequality (2.2), we know that for any $\tau \in (s, 1)$ and any $p \in (0, \infty)$,

$$
\begin{aligned}
m_{0,\,\tau;\,Q}(f) &\lesssim \Big\{m_{0,\,\tau-s;\,Q}(f^p w)\Big\}^{1/p}\Big\{m_{0,\,s;\,Q}(w^{-1})\Big\}^{1/p}\\
&\lesssim \Big(\frac{1}{|Q|}\int_Q |f(x)|^p w(x)\,dx\Big)^{1/p}\Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)^{-1/p},
\end{aligned}
\tag{2.4}
$$

where the second inequality follows from the fact that, for any cube Q and any $r \in (0,\,\infty)$,

$$
m_{0,\,\sigma;\,Q}(w^{-1}) \le \sigma^{-1/r}\Big(\frac{1}{|Q|}\int_Q w^{-r}(x)\,dx\Big)^{1/r}.
\tag{2.5}
$$

Choose $f(x) = \chi_{\tau Q}(x)$. Note that $m_{0,\,\tau;\,Q}(f) = 1$. The estimate (2.4) leads to that

$$
w(Q) \lesssim w(\tau Q),
$$

and w is doubling. Also, (2.4) implies that for any $p \in (0,\,\infty)$,

$$
M_{0,\,\tau} f(x) \lesssim \{M_w^c(|f|^p)(x)\}^{1/p}.
$$

where M_w^c is the weighted centered maximal operator with weight w. Since w is doubling, M_w^c is bounded from $L^1(\mathbb{R}^n,\,w)$ to $L^{1,\,\infty}(\mathbb{R}^n,\,w)$. Thus by a simple interpolation argument, we know (2.3) implies that $M_{0,\,\tau}$ is bounded on $L^p(\mathbb{R}^n,\,w)$.

We can now conclude the proof of Lemma 2.2. It is easy to see that $w \in A_\infty(\mathbb{R}^n)$ implies (2.3). On the other hand, if (2.3) is true, as we have pointed out, for $\tau \in (s,\,1)$ and $p \in (0,\,\infty)$,

$$
\|M_{0,\,\tau} f\|_{L^p(\mathbb{R}^n,\,w)} \lesssim \|f\|_{L^p(\mathbb{R}^n,\,w)}.
\tag{2.6}
$$

For each cube Q and measurable set $E \subset Q$, if $|E| \ge \tau|Q|$, choosing $f(x) = \chi_E(x)$ in the inequality (2.6) then yields

$$
w(Q) \lesssim w(E).
$$

This via the characterization of the $A_\infty(\mathbb{R}^n)$ weights tells us that $w \in A_\infty(\mathbb{R}^n)$, see [1, Chap. 9].

The following lemma is a combine of Theorem 3.6 and Theorem 3.7 in [4].

Lemma 2.3. *Let $w_1,\,\ldots,\,w_m$ be weights, $p_1,\,\ldots,\,p_m,\,p \in (0,\,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, $r \in (0,\,\min_{1\le k\le \ell} p_k)$. Then the following three conditions are equivalent*

(i) *The operator $\mathcal{M}_r$ defined by*

$$
\mathcal{M}_r f(x) = \sup_{Q\ni x}\prod_{k=1}^{\ell}\Big(\frac{1}{|Q|}\int_Q |f_k(x)|^r\,dx\Big)^{1/r}
$$

is bounded from $L^{p_1}(\mathbb{R}^n,\,w_1)\times\ldots\times L^{p_\ell}(\mathbb{R}^n,\,w_\ell)$ to $L^p(\mathbb{R}^n,\,\nu_{\vec{w}})$;

(ii) *$\vec{w} \in A_{\vec{P}/r}(\mathbb{R}^n)$;*

(iii) *for any* k *with* $1 \le k \le \ell$, $w_k^{-\frac{1}{\frac{p_k}{r}-1}} \in A_{\frac{\ell p_k}{p_k - r}}(\mathbb{R}^n)$, *and* $\nu_{\vec{w}} \in A_{\ell p/r}(\mathbb{R}^n)$.

To prove Theorem 1.2, we will employ the characterization of $\mathrm{BMO}(\mathbb{R}^n)$ space in terms of John-Strömberg sharp maximal operator, see [7]. Let f be a real-valued measurable function in $\mathbb{R}^n$. For a fixed cube Q, $m_f(Q)$, the median value of f on Q, is defined to be any number such that

$$|\{x \in Q : f(x) > m_f(Q)\}| \le \frac{1}{2}|Q|,\ |\{x \in Q : f(x) < m_f(Q)\}| \le \frac{1}{2}|Q|.$$

If f is complex-valued, the median value of f on Q is defined by $m_f(Q) = m_{\mathrm{Re}(f)}(Q) + i m_{\mathrm{Im}(f)}(Q)$, where $i^2 = -1$.

The following characterization of $\mathrm{BMO}(\mathbb{R}^n)$ can be found in Strömberg [7].

Lemma 2.4. *Let* $s \in (0, 1/2)$ *and* f *be a measurable function. Then*

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|f\|_{\mathrm{BMO}_{0,\,s}(\mathbb{R}^n)},$$

where

$$\|f\|_{\mathrm{BMO}_{0,\,s}(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} m_{0,\,s;\,Q}(f - m_f(Q)).$$

Proof of Theorem 1.2. The implicity (a)⇒(b) and (b)⇒(c) are obvious. To prove that (c) implies (a), we first claim that if w satisfies the estimate (1.3), then there exists a constant C such that for any $\epsilon > 0$,

$$\sup_{Q \subset \mathbb{R}^n} m_{0,\,s;\,Q}(w + \epsilon) m_{0,\,s;\,Q}((w + \epsilon)^{-1}) \le C. \tag{2.7}$$

In fact, for each fixed cube $Q \subset \mathbb{R}^n$, a straightforward computation gives that

$$m_{0,\,s;\,Q}(w + \epsilon) = m_{0,\,s;\,Q}(w) + \epsilon,$$

and

$$\begin{aligned} m_{0,\,s;\,Q}((w+\epsilon)^{-1}) &= \frac{1}{\sup\{\lambda > 0 : |\{x \in Q : w + \epsilon < \lambda\}| < s|Q|\}} \\ &= \frac{1}{\sup\{\lambda > 0 : |\{x \in Q : w < \lambda\}| < s|Q|\} + \epsilon} \\ &= \frac{1}{\{m_{0,\,s;\,Q}(w^{-1})\}^{-1} + \epsilon}. \end{aligned}$$

Therefore,

$$\sup_{Q \subset \mathbb{R}^n} m_{0,\,s;\,Q}(w + \epsilon) m_{0,\,s;\,Q}((w + \epsilon)^{-1}) \le 1 + \sup_{Q \subset \mathbb{R}^n} m_{0,\,s;\,Q}(w) m_{0,\,s;\,Q}(w^{-1}),$$

and (2.7) follows directly.

We now invoke the idea in [1, p. 300] to prove that (c) implies (a). For each positive integer k, set $w_k(x) = w(x) + 1/k$ and let $w_k(x) = \exp\phi_k(x)$. Then $\phi_k(x)$ is finite a. e. $x \in \mathbb{R}^n$. It then follows from (2.7) that for any cube $Q \subset \mathbb{R}^n$,

$$m_{0,\,s_1;\,Q}\Big(\exp(\phi_k(x) - m_{\phi_k}(Q))\Big) m_{0,\,s_2;\,Q}\Big(\exp(-(\phi_k - m_{\phi_k}(Q)))\Big) \lesssim 1. \tag{2.8}$$

Noticing that

$$|\{x \in Q:\ \exp(\phi_k - m_{\phi_k}(Q)) > \frac{1}{2}\}| \geq |\{x \in Q:\ \phi_k(x) \geq m_{\phi_k}(Q)\}| > \frac{1}{2}|Q|,$$

we then know that

$$m_{0,\,s_1;\,Q}\Big(\exp(\phi_k - m_{\phi_k}(Q))\Big) \gtrsim 1,$$

and similarly,

$$m_{0,\,s_2;\,Q}\Big(\exp(-(\phi_k - m_{\phi_k}(Q)))\Big) \gtrsim 1.$$

This, along with (2.8) and the estimate (2.1), leads to that

$$\begin{aligned} m_{0,\,s_1+s_2;\,Q}(|\phi_k - m_{\phi_k}(Q)|) &\lesssim m_{0,\,s_1;\,Q}\Big(\exp(\phi_k - m_{\phi_k}(Q))\Big) \\ &\quad + m_{0,\,s_2;\,Q}\Big(\exp(-(\phi_k - m_{\phi_k}(Q)))\Big) \\ &\lesssim 1. \end{aligned}$$

Lemma 2.4, via the the John-Nirenberg inequality now states that for some positive constants C_1 independent of k,

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q \exp\Big(\frac{|\phi_k(x) - m_Q(\phi_k)|}{C_1}\Big)\,dx < \infty.$$

Therefore, for any cube $Q \subset \mathbb{R}^n$,

$$\Big(\frac{1}{|Q|}\int_Q w_k^{1/C_1}(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w_k^{-1/C_1}(x)\,dx\Big) \lesssim 1.$$

Taking $k \to \infty$ in the last inequality then yields $w^{1/C_1} \in A_2(\mathbb{R}^n)$.

Proof of Theorem 1.1. It suffices to prove that (i)⇒(ii)⇒(iv)⇒(v)⇒(vi)⇒(i), and (ii)⇒(iii) and (iii)⇒(v).

(i)⇒(ii). This is obvious.

(ii)⇒(iv). For each cube $Q \subset \mathbb{R}^n$, set $f_k^j = (w_{k,\,j}^{-1} + 1/j)^{1/p_k}\chi_Q$ with $w_{k,\,j} = w_k + 1/j$ and $1 \leq k \leq \ell$. Also, set $\lambda_0^j = \frac{1}{2}\prod_{k=1}^{\ell} m_{0,\,s;\,Q}(f_k^j)$. It is obvious that $\lambda_0^j \in (0,\,\infty)$. The hypothesis tells us that

$$\nu_{\vec{w}}(\{x \in \mathbb{R}^n:\ \mathcal{M}_{0,\,s}(f_1^j,\,...,\,f_\ell^j)(x) > \lambda_0^j\}) \lesssim (\lambda_0^j)^{-p}\prod_{k=1}^{\ell}\|f_k\|_{L^{p_k}(\mathbb{R}^n,\,w_k)}^p,$$

which, via the fact that $m_{0,\,s;\,Q}(w_{k,j}^{-1}+1/j)=m_{0,\,s;\,Q}(w_{k,j}^{-1})+1/j$, in turn implies that

$$\nu_{\vec{w}}(Q)\prod_{k=1}^{\ell}\{m_{0,\,s;\,Q}(w_{k,j}^{-1})+1/j\}^{p/p_k}\lesssim\prod_{k=1}^{\ell}\Big(\int_Q\frac{w_k(x)}{w_k(x)+1/j}\,dx+w_k(Q)/j\Big)^{p/p_k},$$

Taking $j\to\infty$ then leads to (iv).

(iv)⇒(v). Recall that $2\ell\tau<1$, we can choose a constant $\delta>0$ such that $2\ell\tau+\delta<1$. It follows from the inequality (2.2) that

$$\sup_{Q\subset\mathbb{R}^n}\Big(\frac{1}{|Q|}\int_Q\nu_{\vec{w}}(x)\,dx\Big)m_{0,\,\ell\tau;\,Q}(\nu_{\vec{w}}^{-1})<\infty.$$

This via Lemma 2.2 shows that $\nu_{\vec{w}}\in A_\infty(\mathbb{R}^n)$. On the other hand, for each fixed k with $1\le k\le\ell$, again by (2.2),

$$m_{0,\,(\ell-1)\tau+\delta;\,Q}(w_k^{p/p_k})\le m_{0,\,\delta;\,Q}(\nu_{\vec{w}})\prod_{1\le j\le\ell,\,j\neq k}\{m_{0,\,\tau;\,Q}(w_j^{-1})\}^{p/p_j}.$$

Therefore,

$$m_{0,\,(\ell-1)\tau+\delta;\,Q}(w_k)m_{0,\,\tau;\,Q}(w_k^{-1})\lesssim1,$$

which together with Theorem 1.2 implies that $w_k^{\gamma_k}\in A_\infty(\mathbb{R}^n)$ for some $\gamma_k\in(0,\,\infty)$. Taking $\gamma=\min_{1\le k\le\ell}\gamma_\ell$ then leads to condition (v).

(v)⇒(vi). The case $\ell=1$ is obvious. For the case of $\ell>1$, we know from Theorem 1.2 that there exists a constant $\theta\in(0,\,\infty)$ such that for each k with $1\le k\le\ell$, $w_k^\theta\in A_2(\mathbb{R}^n)$. Thus, we can take some $r\in(0,\,\min_{1\le k\le\ell}p_k)$ which is small enough, such that $\nu_{\vec{w}}\in A_{\ell p/r}(\mathbb{R}^n)$, and for each k with $1\le k\le\ell$, $w_k^{-1/(p_k/r-1)}\in A_2(\mathbb{R}^n)\subset A_{\ell p_k/(p_k-r)}(\mathbb{R}^n)$. This, along with Lemma 2.3, tells us $\vec{w}\in A_{\vec{P}/r}(\mathbb{R}^n)$.

(vi)⇒(i). This is an easy consequence of Lemma 2.3 and the fact that for any $s\in(0,\,1)$ and $r\in(0,\,\infty)$,

$$\mathcal{M}_{0,\,s}(f_1,\,\dots,\,f_\ell)(x)\lesssim\mathcal{M}_r(f_1,\,\dots,\,f_\ell(x).$$

(ii)⇒(iii). Let Q be a cube and $E_1,\,\dots,\,E_\ell\subset Q$ be measurable sets such that $|E_k|>s|Q|$ for k with $1\le k\le\ell$. Since $\mathcal{M}_{0,\,s}$ is bounded from $L^{p_1}(\mathbb{R}^n,\,w_1)\times\dots\times L^{p_\ell}(\mathbb{R}^n,\,w_\ell)$ to $L^{p,\,\infty}(\mathbb{R}^n,\,\nu_{\vec{w}})$, it follows that

$$w(\{x\in\mathbb{R}^n:\,\mathcal{M}_{0,\,s}(\chi_{E_1},\,\dots,\,\chi_{E_\ell})(x)>1/2\})\lesssim\prod_{k=1}^{\ell}\{w_k(E_k)\}^{p/p_k}.$$

Note that for any $x\in Q$, $\mathcal{M}_{0,\,s}(\chi_{E_1},\,\dots,\,\chi_{E_\ell})(x)>1/2$. We thus have that

$$\nu_{\vec{w}}(Q)\lesssim\prod_{k=1}^{\ell}\{w_k(E_k)\}^{p/p_k}.$$

(iii) implies (v). At first, we prove that for $\sigma \in (\tau, 1/(2\ell))$, $\mathcal{M}_{0,\sigma}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times ... \times L^{p_\ell}(\mathbb{R}^n, w_\ell)$ to $L^{p,\infty}(\mathbb{R}^n, \nu_{\vec{w}})$. To see this, for each fixed $\lambda > 0$, set

$$\Omega^R_{\sigma,\lambda} = \{x \in \mathbb{R}^n : \mathcal{M}^R_{0,\sigma}(f_1, ..., f_\ell)(x) > \lambda\},$$

where

$$\mathcal{M}^R_{0,\sigma}(f_1, ..., f_\ell)(x) = \sup_{Q \ni x, |Q| < R^n} \prod_{k=1}^{\ell} m_{0,\sigma;Q}(f_k).$$

For each fixed $x \in \Omega^R_{\sigma,,\lambda}$, we can choose a cube Q_x containing x and satisfies that $\prod_{k=1}^{\ell} m_{0,\sigma;Q_x}(f_k) > \lambda$. Then there exist positive numbers $\lambda_1, ..., \lambda_\ell$ such that

$$\prod_{k=1}^{\ell} \lambda_k > \lambda,\ m_{0,\sigma;Q_x}(f_k) > \lambda_k,\ 1 \le k \le \ell.$$

This in turn implies that

$$|\{y \in Q_x : |f_k(y)| > \lambda_k\}| > \tau \Big|(\frac{\sigma}{\tau})^{1/n} Q_x\Big|.$$

The condition (iii) tells us that

$$\begin{aligned}\nu_{\vec{w}}\Big(\Big(\frac{\sigma}{\tau}\Big)^{1/n} Q_x\Big) &\lesssim \prod_{k=1}^{\ell} \{w_k(\{y \in Q_x : |f_k(y)| > \lambda_k\})\}^{p/p_k}\\ &\lesssim \lambda^{-p} \prod_{k=1}^{\ell} \Big(\int_{Q_x} |f(y)|^{p_k} w_k(y)\,dy\Big)^{p/p_k}.\end{aligned}$$

By the covering lemma of Besicovitch type (see [6]), from the family of cubes $\{Q_x\}_{x \in \Omega^R_{\sigma,\lambda}}$, we can choose N (depending only on n, τ and σ) subfamilies $\mathcal{D}_l = \{Q^l_j\}$, $l = 1, ..., N$, such that

$$\Omega^R_{\sigma,\lambda} \subset \cup_{l=1}^N \cup_j \Big(\frac{\sigma}{\tau}\Big)^{1/n} Q^l_j,$$

and for each fixed l with $1 \le l \le N$, any two cubes $Q^l_{j_1}$ and $Q^l_{j_2}$ are disjoint. We finally have that

$$\begin{aligned}\nu_{\vec{w}}(\Omega^R_{\sigma,\lambda}) &\lesssim \lambda^{-p} \sum_{l=1}^{N} \sum_j \prod_{k=1}^{\ell} \Big(\int_{Q^l_j} |f(y)|^{p_k} w_k(y)\,dy\Big)^{p/p_k}\\ &\lesssim \lambda^{-p} \sum_{l=1}^{N} \prod_{k=1}^{\ell} \Big(\sum_j \int_{Q^l_j} |f_k(y)|^p w_k(y)\,dy\Big)^{p/p_k}\\ &\lesssim \lambda^{-p} \prod_{k=1}^{\ell} \|f_k\|^p_{L^{p_k}(\mathbb{R}^n, w_k)}.\end{aligned}$$

Taking $R \to \infty$ then shows that $\mathcal{M}_{0,\sigma}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times ... \times L^{p_\ell}(\mathbb{R}^n, w_\ell)$ to $L^{p,\infty}(\mathbb{R}^n, \nu_{\vec{w}})$. This, as we have proved, certainly implies the condition (v).

REFERENCES

1. L. Grafakos, *Modern Fourier Analysis*, GTM 250, Springer, New York, 2009.

2. G. Hu, Da Yang and Do Yang, Boundedness of maximal singular integral operators on spaces of homogeneous type and its applications, *J. Math. Soc. Japan,* **59** (2007), 323-349.

3. A. K. Lerner, Weighted rearrangement inequalities for local sharp maximal functions, *Trans. Amer. Math. Soc.,* **357** (2005), 2445-2465.

4. A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, *Adv. Math.,* **220** (2009), 1222-1264.

5. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.,* **165** (1972), 207-226.

6. Y. Sawano, Sharp estimates of the modified Hardy-Littlewood maximal operator on the nonhomogeneous space via covering lemmas, *Hokkaido Math. J.,* **34** (2005), 435-458.

7. J. O. Strömberg, Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, *Indiana Univ. Math. J.,* **28** (1979), 511-544.

Guoen Hu and Wentan Yi
Department of Applied Mathematics
Zhengzhou Information Science and Technology Institute
P. O. Box 1001-747
Zhengzhou 450002
P. R. China
E-mail: guoenxx@yahoo.com.cn