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BMO SPACES FOR LAGUERRE EXPANSIONS

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Abstract. Let $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ be the Laguerre functions of Hermite type with index α. These are eigenfunctions of the Laguerre differential operator $L_\alpha = \frac{1}{2}(-\frac{d^2}{dy^2}+y^2+\frac{1}{y^2}(\alpha^2-\frac{1}{4}))$. We define and study a BMO-type space BMO_{L_α}, which is identified as the dual space of the Hardy-type space associated with L_α. We characterize BMO_{L_α} by Carlesson measures related to appropriate square functions. Finally, we prove the boundedness on this space of the fractional integral operator and the Riesz transform related to L_α.

1. Introduction and Statement of the Results

Let $n \in \mathbb{N}$, $\alpha > -1$. The Laguerre function of Hermite type φ_α on $(0,\infty)$ is defined as

$$\varphi_n^\alpha(y) = \left(\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\right)^{1/2} e^{-y^2/2} y^\alpha L_n^\alpha(y^2)(2y)^{1/2}, \quad y \in (0,\infty),$$

where $L_n^\alpha(x)$ denotes the Laguerre polynomial of degree n and order α (see [12]). It is well known that for every $\alpha > -1$ the system $\{\varphi_n^\alpha\}_{n=0}^\infty$ forms an orthonormal basis of $L^2(0,\infty)$. Moreover, these functions are eigenfunctions of the Laguerre differential operator

$$L_\alpha = \frac{1}{2}\left(-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\left(\alpha^2 - \frac{1}{4}\right)\right)$$

satisfying $L_\alpha \varphi_n^\alpha = (2n+\alpha+1)\varphi_n^\alpha$. The operator L_α can be extended to a positive self-adjoint operator on $L^2(0,\infty)$ by giving a suitable domain of definition (see [9]). We also denote the extension by L_α. Let $\{T_t^\alpha\}_{t\geq 0}$ be the heat-diffusion semigroup generated by L_α. More precisely, for $f \in L^2(0,\infty)$, we define

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$$T_t^\alpha f(x) = \int_0^\infty W_t^\alpha(x,y) f(y)\,dt, \tag{1.1}$$

where

$$W_t^\alpha(x,y) = \left(\frac{2e^{-t}}{1-e^{-2t}}\right)^{1/2} \left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)$$
$$\exp\left(-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}\left(x^2+y^2\right)\right),$$

I_α is the modified Bessel function of the first kind and order α.

Hardy spaces in the Laguerre setting have been studied by Dziubański [6]. A function $f \in L^1(0,\infty)$ is in the Hardy space $H^1_{L_\alpha}$ if and only if $T^*_\alpha f = \sup_{t>0} |T_t^\alpha f| \in L^1(0,\infty)$. Then we set $\|f\|_{H^1_{L_\alpha}} = \|T^*_\alpha f\|_{L^1}$. Dziubański proved that the spaces $H^1_{L_\alpha}, \alpha > -\frac{1}{2}$, admit atomic decompositions, where the cancellation conditions are only required for atoms with small supports depending on the following auxiliary function

$$\rho_{L_\alpha}(x) = \frac{1}{8}\min(x, 1/x), \qquad x > 0. \tag{1.2}$$

A measurable function $b : (0,\infty) \to \mathbb{C}$ is said to be an $H^1_{L_\alpha}$-atom if there exists a ball $B_r(y_0) = \{y > 0 : |y_0 - y| < r\}$ with $r \le \rho_{L_\alpha}(y_0)$ such that

$$\mathrm{supp\, b} \subset \mathrm{B_r(y_0)},$$
$$\|b\|_\infty \le |B_r(y_0)|^{-1},$$
$$\text{if} \quad \mathrm{r} \le \rho_{\mathrm{L}_\alpha}(\mathrm{y_0})/2, \quad \text{then} \quad \int \mathrm{b(y)\,dy} = 0.$$

$f \in H^1_{L_\alpha}$ if and only if f can be decomposed as $f = \sum_j c_j a_j$, where a_j are $H^1_{L_\alpha}$-atoms and $\sum_j |c_j| < \infty$. Moreover, There exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{H^1_{L_\alpha}} \le \inf\{\sum_j |c_j| : \quad f = \sum_j c_j a_j\} \le C\|f\|_{H^1_{L_\alpha}}.$$

The main result of this paper is to present a BMO-type space identified as the dual space of the Hardy-type $H^1_{L_\alpha}$ and utilize a Carleson measure to characterize the BMO-type space. In the Euclidean space, it is well known that BMO is the dual space of the classical Hardy space H^1 and can be characterized by Carleson measures (see [10], Theorem 3, p.159). In particular, we can choose a special Carleson measure as follow: for $f \in BMO$,

$$d\mu = \left(t^2 \frac{dh_s}{ds}\bigg|_{s=t^2} * f\right)^2 dxdt/t, \quad (x,t) \in \mathbb{R}\times(0,\infty), \tag{1.3}$$

where h_s is the classical heat kernel. Dziubański et al. [7] investigated a similar result for Schrödinger operators with potentials satisfying a reverse Hölder's inequality. They introduce the BMO-type space associated with a Schrödinger operator which is the dual space of the Hardy-type space. They also utilized the heat semigroups to define appropriate square functions, and then give the Carleson measures characterizing the BMO-type space. More precisely,

$$d\mu = \left(t^2 \frac{dT_s}{ds}\bigg|_{s=t^2} f\right)^2 dxdt/t, \quad (x,t) \in \mathbb{R}^d \times (0,\infty), \tag{1.4}$$

where $\{T_s\}_{s>0}$ is the heat semigroup associated with the Schrödinger operator and $d \geq 3$. In addition, they proved that some classical operators in Schrödinger setting (square function, fractional integral operator, and so on) are bounded on their BMO-type space. This kind of BMO-type space on the Heisenberg group $\mathbb{H}^n$ for Schröinger operator was defined and investigated in [8]. To deal with some key estimates, the authors of [8] mainly utilized the perturbation theory for semigroups of operators and consider the difference between the heat kernels associated with the Schröinger operator and the sublaplacian operator on Heisenberg group respectively. Recently, a theory of localized BMO spaces on RD-spaces associated with an admissible function ρ was investigated in [14]. The admissible function ρ in [14] satisfies, as same as [7] and [8],

$$\frac{1}{\rho(x)} \leq C_0 \frac{1}{\rho(y)} \left(1 + \frac{d(x,y)}{\rho(y)}\right)^{k_0}. \tag{1.5}$$

In this paper, we use a similar way to give a Carleson measure characterizing BMO_{L_α} based on the heat semigroup $\{T_t^\alpha\}_{t>0}$ and prove two important operators (fractional integral operator and Riesz transform) are bounded on BMO_{L_α}. The main key in our paper is also to estimate the corresponding integral kernels for Laguerre expansions. However, it is difficult to directly obtain suitable estimates for these kernels. Note that the underlying manifold in our case is $(0,\infty)$ other than $\mathbb{R}$. The new difficulty come from the estimates near to the origin. For example, the admissible function ρ_{L_α} in (1.2) does not satisfy in (1.5). This is obvious when x tends to zero and $y = 1$. In order to overcome this new difficulty and to obtain some key estimates, we will consider the difference of the heat and Riesz kernels, which are associated with Hermite and Laguerre operators respectively(see [1, 3]).

Now we define the BMO_{L_α} associated with the Laguerre operator L_α and state our main theorems.

Definition 1. Let $\alpha > -\frac{1}{2}$, $B_s(y)$ be any ball in $(0,\infty)$ with the center y and the radius s and f be a locally integrable function on $(0,\infty)$. We say $f \in BMO_{L_\alpha}$ if there exists a constant $C \geq 0$ independent of s and y such that

$$\frac{1}{|B_s(y)|}\int_{B_s(y)} |f - f_{B_s(y)}| \leq C, \quad \text{if s} < (\rho_{\mathrm{L}^{\alpha}(y)}),$$
$$\frac{1}{|B_s(y)|}\int_{B_s(y)} |f| \leq C, \quad \text{if s} \geq (\rho_{\mathrm{L}^{\alpha}(y)}).$$

Here, $f_{B_s(y)} = \frac{1}{|B_s(y)|}\int_{B_s(y)} f\,dx$. We let $\|f\|_{BMO_{L_\alpha}}$ denote the smallest C in the two inequalities above.

It is readily seen that BMO_{L_α} is a Banach space with norm $\|.\|_{BMO_{L_\alpha}}$. We will show that BMO_{L_α} is actually the dual space of $H^1_{L_\alpha}$ in the sense of isomorphism.

Let $L^1_{loc}(0,\infty)$ denote the space of all locally integrable functions on $(0,\infty)$ and $L^\infty_c(0,\infty)$ denote the space of all bounded functions with compact supports contained in $(0,\infty)$. It is clear that $L^\infty_c(0,\infty)$ is a dense subspace of $H^1_{L_\alpha}$. Define a linear mapping $\Phi : f \mapsto \Phi_f$ by

$$\Phi_f(g) = \int_0^\infty f(x)g(x)\,dx, \qquad f \in L^1_{loc}(0,\infty), g \in L^\infty_c(0,\infty).$$

Theorem 1. *Φ is a linear isomorphism between BMO_{L_α} and $\left(H^1_{L_\alpha}\right)^*$.*

We will show a characterization of BMO_{L_α} in terms of Carleson measures. More precisely, let

$$Q_t f(x) = t^2 \frac{dT^\alpha_s}{ds}\bigg|_{s=t^2} f(x), \quad (x,t) \in (0,\infty)\times(0,\infty).$$

We define a Carleson measure on $(0,\infty)\times(0,\infty)$ by $d\mu_f := |Q_t f(x)|^2\,dxdt/t$, and the Carleson norm is given by

$$\|d\mu_f\|_{\mathcal{C}} := \sup_{B_r(x)\subset(0,\infty)} \frac{\mu_f(B_r(x)\times(0,r))}{|B_r(x)|} < \infty. \tag{1.6}$$

Theorem 2. *Let $\alpha > -1/2$.*

1. *If $f \in BMO_{L_\alpha}$, then $d\mu_f$ is a Calerson measure.*
2. *Conversely, suppose that f satisfies*

$$\int_0^\infty \frac{|f(x)|}{1+x^2}\,dx < \infty,$$

and $d\mu_f := |Q_t f(x)|^2\,dxdt/t$ satisfies (1.6); then f is in BMO_{L_α}. Moreover, there exists $C > 0$ such that

$$\frac{1}{C}\|f\|^2_{BMO_{L_\alpha}} \leq \|d\mu_f\|_{\mathcal{C}} \leq C\|f\|^2_{BMO_{L_\alpha}}.$$

Then, we consider the mapping properties of the fractional integral operator and the Riesz transform associated with L_α on BMO_{L_α}.

By spectral decomposition and functional calculus of the operator L_α, we introduce the fractional integral operator:

$$\mathcal{I}_\sigma = L_\alpha^{-\sigma/2} = \int_0^\infty e^{-tL_\alpha} t^{\sigma/2-1}\, dt, \qquad 0 < \sigma < 1. \tag{1.7}$$

Theorem 3. *Let $\alpha > -1/2$, $0 < \sigma < 1$ and $\sigma \le 2(\alpha + \frac{1}{2})$. There exists a constant $C > 0$ such that*

$$\|\mathcal{I}_\sigma f\|_{BMO_{L_\alpha}} \le C\|f\|_{\frac{1}{\sigma}}.$$

Recall that the operator L_α can be "factorized" as

$$L_\alpha = \frac{1}{2} D_\alpha^* D_\alpha + \alpha + 1$$

where D_α is the derivative associated with L_α given by

$$D_\alpha = \frac{d}{dx} + x - \frac{\alpha + 1/2}{x},$$

and D_α^* represents its (formal) adjoint operator on $L^2(0,\infty)$. Formally, the Riesz transforms are defined by

$$R_\alpha = D_\alpha L_\alpha^{-1/2},$$

where L_α is considered on $C_c(\mathbb{R}_+)$ as a nature domain. Actually, Nowak and Stempak [9] showed that R_α is a principal value integral operator associated with a kernel

$$R_\alpha(x,y) = \int_0^\infty D_\alpha W_t^\alpha(x,y) \frac{1}{\sqrt{t}}\, dt. \tag{1.8}$$

Also, the authors proved that, for $\alpha > -1/2$, R_α can be extended uniquely to bounded linear operators on $L^p(0,\infty)$ when $1 < p < \infty$ and are of weak type(1,1). Moreover, recently, Betancor et al. [2] proved Riesz transform characterization for Hardy space $H^1_{L_\alpha}$. In the present paper, we will discuss the mapping property of R_α on BMO_{L_α}. More precisely, we shall prove the following theorem.

Theorem 4. *Let $\alpha > -1/2$. There exists a constant $C > 0$ such that*

$$\|R_\alpha f\|_{BMO_{L_\alpha}} \le C\|f\|_{BMO_{L_\alpha}}.$$

Recently a similar result for Riesz transforms associated with Schrödinger operators with potentials belonging to a reverse Hölder class was investigated in [5], in which an estimate for difference between the Riesz kernel of Schrödinger operator and the usual Riesz kernel was utilized to handle the situation of the balls with small radius in the definition of BMO_{L_α}. In our present setting, we need to utilize the difference between the present Riesz kernel and the Riesz kernel related to Hermite operator to obtain some critical estimates.

Throughout this paper by C we always denote a positive constant that may vary at each occurrence; $B_r(y_0)$ stands for a ball with center y_0 and radius r; B^* denotes $B_{2r}(y_0)\cap(0,\infty)$; $A\sim B$ means $\frac{1}{C}A\le B\le CA$ and the notation $X\lesssim Y$ is used to indicate that $X\le CY$ with a positive constant C independent of significant quantities; $\|f\|_p$ denotes the norm $\left(\int_0^\infty |f(x)|^p\,dx\right)^{\frac{1}{p}}$.

2. Preliminaries

For further references, we figure out some properties of the Bessel function I_α (see [12]):

$$I_\alpha(z)\sim z^\alpha,\quad z\to 0, \tag{2.1}$$

$$z^{1/2}I_\alpha(z)=\frac{1}{\sqrt{2\pi}}e^z(1+O(\frac{1}{z})),\quad z\to\infty, \tag{2.2}$$

$$\frac{d}{dz}(z^{-\alpha}I_\alpha(z))=z^{-\alpha}I_{\alpha+1}(z),\quad z\in(0,\infty). \tag{2.3}$$

Set $r=e^{-2t}$. It is known (cf. §2 of [7]) that the heat kernel $W_t^\alpha(x,y)$ can be decomposed as

$$W_t^\alpha(x,y)=H(r,x,y)\Psi(r,x,y)\Phi_\alpha(r,x,y), \tag{2.4}$$

where

$$\begin{aligned}
H(r,x,y)&=\frac{(1+r)^{1/2}}{(1-r)^{1/2}}\exp\left(-\frac{1}{2}\frac{1+r}{1-r}|x-y|^2\right);\\
\Psi(r,x,y)&=\frac{\sqrt{2}r^{1/4}}{(1+r)^{1/2}}\exp\left(-\frac{1-r}{(1+\sqrt{r})^2}xy\right);\\
\Phi_\alpha(r,x,y)&=\left(\frac{2r^{1/2}xy}{1-r}\right)^{1/2}\exp\left(-\frac{2r^{1/2}xy}{1-r}\right)I_\alpha\left(\frac{2r^{1/2}xy}{1-r}\right).
\end{aligned}$$

From (2.1),(2.2), (2.4) it easily follows that there exist constants $C,c_0,c_1,c_2>0$ so that for $t>1$ we have

$$W_t^\alpha(x,y)\le Ce^{-\frac{t}{10}}\exp\left(-c_0|x-y|^2\right), \tag{2.5}$$

and for $0<t\le 1$ we have

$$W_t^\alpha(x,y)\le Ct^{-1/2}\exp\left(-c_1\frac{|x-y|^2}{t}\right)\exp\left(-c_2txy\right). \tag{2.6}$$

Now we give the following covering lemma for $(0,\infty)$ which will be used frequently below. The proof is trivial and left to the reader.

Lemma 1. *Let* $x_0 = 1$, $x_j = x_{j-1} + \rho_{L_\alpha}(x_{j-1})$ *for* $j \geq 1$, *and* $x_j = x_{j+1} - \rho_{L_\alpha}(x_{j+1})$ *for* $j < 0$. *We define the family of "critical balls" of* $\mathcal{B} = \{B_k\}_{k=-\infty}^{\infty}$, *where* $B_k := \{x \in (0,\infty) : |x - x_k| < \rho_{L_\alpha}(x_k)\}$. *Then we have*

(a) $\bigcup_{k=-\infty}^{\infty} B_k = (0,\infty)$.

(b) *For every* $k \in \mathbb{Z}$, $B_k \cap B_j = \emptyset$ *provided that* $j \notin \{k-1, k, k+1\}$.

(c) *For any* $y_0 \in (0,\infty)$, *at most three balls in* $\mathcal{B}$ *have nonempty intersection with* $B(y_0, \rho_{L_\alpha}(y_0))$.

We deduce the following two corollaries which will be used frequently throughout this paper.

Corollary 1. *There exists a constant* $C > 0$ *such that for every* $B_R(x) \subseteq (0,\infty)$ *with* $R > \rho_{L_\alpha}(x)$, *we have*

$$|B_R(x)| \leq \sum_{\{B_k \in \mathcal{B}: B_k \cap B_R(x) \neq \emptyset\}} |B_k| \leq C|B_R(x)|.$$

Corollary 2. *There exists a constant C such that for* $f \in BMO_{L_\alpha}$, *we have*

$$\|f\|_{BMO_{L_\alpha}} \leq C \sup_k (|f|_{B_k} + \|f\|_{BMO(B_k^*)}),$$

where, for any ball B, the norm $\|.\|_{BMO(B)}$ *is given by*

$$\begin{aligned}\|f\|_{BMO(B)} &= \sup_{B_r(x)\subset B} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f - f_{B_r(x)}|\, dy \\ &\sim \sup_{B_r(x)\subset B} \inf_{c\in\mathbb{C}} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f - c|\, dy.\end{aligned}$$

Let H be the Hermite operator

$$H = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2).$$

We consider the heat diffusion semigroup $\{W_t\}_{t>0}$ associated with H and defined by, for every $f \in L^2(\mathbb{R})$,

$$W_t f(x) = \int_{\mathbb{R}} W_t(x,y) f(y)\, dy, \quad x \in \mathbb{R},$$

where for each $x, y \in \mathbb{R}$ and $t > 0$,

$$\text{(2.7)} \quad W_t(x,y) = \left[\frac{e^{-t}}{\pi(1-e^{-2t})}\right]^{1/2} \exp\left(-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2) + 2xy\frac{e^{-t}}{1-e^{-2t}}\right)$$

(see [13]). The Hermite operator can be factorized:

$$H = -\frac{1}{4}\left[\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right)+\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)\right].$$

Motivated by the factorization, formally, the Riesz transform is defined by

$$R^H = (\frac{d}{dx}+x)H^{-1/2},$$

where H is considered on $C_c(\mathbb{R})$. Using the formula $H^{-1/2} = \int_0^\infty W_t t^{-1/2}\,dt$, the Riesz transform R^H can be represented as a principle integral operator of the form

$$R^H f(x) = \lim_{\epsilon\to 0}\int_{|y-x|>\epsilon} R^H(x,y)f(y)\,dy,$$

with the kernel given by

$$R^H(x,y) = \int_0^\infty \left(\frac{d}{dx}+x\right)W_t(x,y)\frac{dt}{\sqrt{t}} \tag{2.8}$$

(see [11]).

Harmonic analysis in the Hermite context has been developed over the past few years. Recently, Betancor et al. ([1, 3]) established the transference between Laguerre and Hermite settings and investigated the Lusin area function associated with Hermite and Laguerre operators. Of particularly interesting in these papers are the differences of heat kernels and Riesz transforms in two settings which are useful in the present paper.

Now we consider estimates for the integral kernels of the operators Q_t :

$$Q_t(x,y) = t^2\frac{\partial W_s^\alpha(x,y)}{\partial s}\Big|_{s=t^2},$$

and similarly consider the integral kernel in the Hermite setting

$$P_t(x,y) = t^2\frac{\partial W_s(x,y)}{\partial s}\Big|_{s=t^2}.$$

Proposition 1. (see [3], (3.4) and (3.6)). *Let $\alpha > -\frac{1}{2}$. There exists $C>0$ such that*

(a) *For every $t,x,y\in(0,\infty)$ such that $\frac{e^{-t^2}xy}{1-e^{-2t^2}} \le 1$,*

$$|Q_t(x,y)| \le Ct^2(xy)^{\alpha+1/2}e^{-\frac{x^2+y^2}{8t^2}}\frac{e^{-(\alpha+1)t^2}}{(1-e^{-2t^2})^{\alpha+2}}.$$

(b) *For every $t,x,y\in(0,\infty)$ such that $\frac{e^{-t^2}xy}{1-e^{-2t^2}} > 1$,*

$$|Q_t(x,y)-P_t(x,y)| \le Ct^2e^{-\frac{(x-y)^2}{2t^2}}\frac{e^{t^2/2}}{xy(1-e^{-t^2})^{1/2}}.$$

Proposition 2. *For any large enough $N > 0$, there exist C and C_N so that*

(a) *For $0 < x < 1$, $\left|\int_{-\infty}^{\infty} P_t(x,y)\,dy\right| \le Ct^2$, C is independent of x.*

(b) *For $x \ge 1$, $\left|\int_{-\infty}^{\infty} P_t(x,y)\,dy\right| \le C_N \frac{t^2x^2}{(1+x^2t^2)^N}$, C_N is independent of x.*

(c) *For $t > 0$ and $x, y \in (0,\infty)$, we have*

$$|P_t(x,y)| \le Ct^2 e^{-\frac{(x-y)^2}{16t^2}} \frac{e^{-t^2/2}}{(1-e^{-2t^2})^{3/2}}.$$

Proof. (c) is contained in [3] (see (2.3)). We only show (a) and (b). Let $\phi_n(y) = \phi(\frac{y}{n})$, $\phi(y)$ is a smooth function satisfying $\triangle\phi(y) \le 1$, $\phi(y) = 1$ for $|y| \le 1$ and $\phi(y) = 0$ for $|y| \ge 2$. Recall (2.7), for fixed s and x, a straightforward manipulation shows that

$$\int_{-\infty}^{+\infty} \left|\frac{\partial W_s(x,y)}{\partial s}\right| dy < \infty.$$

Hence, we have

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} \frac{\partial W_s(x,y)}{\partial s}\,dy\right| &= \left|\lim_{n\to\infty}\int_{-\infty}^{\infty} \frac{\partial W_s(x,y)}{\partial s}\phi_n(y)\,dy\right| \\
&= \left|\lim_{n\to\infty}\int_{-\infty}^{\infty} W_s(x,y) H\phi_n(y)\,dy\right| \\
&\le C\int_{-\infty}^{\infty} W_s(x,y) y^2\,dy.
\end{aligned}$$

Using (2.7) again,

$$\begin{aligned}
I &= \left|\int_{-\infty}^{\infty} \frac{\partial W_s(x,y)}{\partial s}\,dy\right| \\
&\le C\int_{-\infty}^{\infty} \frac{e^{-s/4}}{\sqrt{s}} \exp\left(-\frac{(x-y)^2e^{-s} + (x^2+y^2)(1-e^{-s})^2}{2(1-e^{-2s})}\right) y^2\,dy.
\end{aligned}$$

If $x < 1$, then

$$I \le C\int_{-\infty}^{\infty} \frac{e^{-s/4}}{\sqrt{s}} \exp\left(-c_0\frac{|x-y|^2}{s}\right) y^2\,dy \le C,$$

which implies $\left|\int_{-\infty}^{\infty} P_t(x,y)\,dy\right| \le Ct^2$. If $x \ge 1$, then

$$\begin{aligned}
I &\le C\int_{-\infty}^{\infty} \frac{e^{-s/4}}{\sqrt{s}} \exp\left(-c_0\frac{|x-y|^2}{s}\right) y^2 \frac{1}{(1+x^2s)^N}\,dy \\
&\le C_N \frac{1}{(1+x^2s)^N} \int_{-\infty}^{\infty} \frac{e^{-s/4}}{\sqrt{s}} \exp\left(-c_0\frac{|x-y|^2}{s}\right) \left[(y-x)^2 + x^2\right]\,dy \\
&\le C_N \frac{x^2}{(1+x^2s)^N},
\end{aligned}$$

which shows (b). ∎

Combining Proposition 1 and (c) of Proposition 2, we have the following corollary.

Corollary 3. *There exist* $c, C > 0$ *such that*

$$|Q_t(x,y)| \leq C\frac{1}{t}e^{-\frac{|x-y|^2}{ct^2}}. \tag{2.9}$$

3. Proof of Theorem 1

Now we shall give a proof of Theorem 1. Let $L^2(B)$ denote the space of all square integrable functions supported in B and B^{**} denote the ball with the same center as B and four times the radius of B.

Lemma 2. *For any ball* $B = B_r(y_0)$ *in* $(0,\infty)$ *with* $\rho_{L_\alpha}(y_0)/2 < r \leq \rho_{L_\alpha}(y_0)$, $L^2(B)$ *is contained in* $H^1_{L_\alpha}$. *Moreover, there is a constant* $C > 0$ *such that*

$$\|g\|_{H^1_{L_\alpha}} \leq C|B|^{1/2}\|g\|_{L^2(B)}, \quad \forall g \in L^2(B).$$

Proof. We obviously observe that

$$\|T^*_\alpha g\|_{L^1(B^{**})} \leq C|B|^{1/2}\|T^*_\alpha g\|_{L^2(0,\infty)} \leq C|B|^{1/2}\|g\|_{L^2(B)}.$$

So it suffices to prove that $\|T^*_\alpha g\|_{L^1((B^{**})^c)} \leq C|B|^{1/2}\|g\|_{L^2(B)}$. By Lemma 2.2 in [6], we get

$$\begin{aligned}\int\limits_{\{x>0:x\notin B^{**}\}} \sup_{0<t<1} |T^\alpha_t g(x)|dx &\leq \int\limits_B \int\limits_{\{x>0:x\notin B^{**}\}\}} \sup_{0<t<1} W^\alpha_t(x,y)\,dx|g(y)|dy \\ &\leq C\int\limits_B |g(y)|dy \\ &\leq C|B|^{1/2}\|g\|_{L^2(B)}\end{aligned}$$

Again, using (2.5), we also have

$$\begin{aligned}\int\limits_{\{x>0:x\notin B^{**}\}} \sup_{t\geq 1}|T^\alpha_t g(x)|dx &\leq \int\limits_B \int\limits_{(0,\infty)} \exp\left(-c|x-y|^2\right)|g(y)|\,dxdy \\ &\leq C|B|^{1/2}\|g\|_{L^2(B)}.\end{aligned}$$ ∎

Lemma 3. *Let* H^1 *be the classical Hardy space. If* $f \in H^1$ *and* $\operatorname{supp} f \subset (0,\infty)$, *then* $\|f\|_{H^1_{L_\alpha}} \leq C\|f\|_{H^1}$.

Proof. Since $f \in H^1$ and $\operatorname{supp} f \subset (0,\infty)$, it is well known in [4] that f can be written as $\sum_j = \lambda_j a_i$, where a_j are H^1-atoms supported in $(0,\infty)$. So it suffices to show that, for any H^1-atom a supported in $(0,\infty)$, $\|a\|_{H^1_{L_\alpha}} \le C$.

First assume $\operatorname{supp} a \subset \mathrm{B_r(y_0)} \subset (0,\infty)$ and $r \le \rho_{L_\alpha}(y_0)$, then a also is an $H^1_{L_\alpha}$-atom, and hence $\|a\|_{H^1_{L_\alpha}} \le C$.

Second assume $r > \rho_{L_\alpha}(y_0)$. Recall Corollary 1, there exist two integers i_0 and j_0 such that $B_r(y_0) \subset \bigcup\limits_{k=i_0}^{j_0} B_k$ and $|B_r(y_0)| \le \sum\limits_{k=i_0}^{j_0} |B_k| \le C|B_r(y_0)|$. Thus we can write

$$a = \sum_{k=i_0}^{j_0} \lambda_k b_k, \qquad \lambda_k = \frac{\rho_{L_\alpha}(x_k)}{|B_r(y_0)|} \text{ and } \mathrm{b_k} = \frac{|\mathrm{B_r(y_0)}|}{\rho_{\mathrm{L}_\alpha}(\mathrm{x_k})}\mathrm{a_k},$$

where,

$$a_k = \begin{cases} a\mathbb{1}_{[x_k, x_k+\rho_{L_\alpha}(x_k))}, & \text{if } \mathrm{k} > 0; \\ a\mathbb{1}_{(x_k-\rho_{L_\alpha}(x_k), x_k]}, & \text{if } \mathrm{k} < 0; \\ a\mathbb{1}_{B_k}, \quad \text{if } \mathrm{k} = 0. \end{cases} \tag{3.1}$$

$\mathbb{1}_{\mathbb{E}}$ is the characteristic function of $\mathbb{E}$. It is easy to check that each b_k is an $H^1_{L_\alpha}$-atom, so we get

$$\|a\|_{H^1_{L_\alpha}} \le C \sum_{k=i_0}^{j_0} \lambda_k \le C \sum_{k=i_0}^{j_0} \frac{|B_k|}{|B_r(y_0)|} \le C.$$

∎

Proof of Theorem 1. Assume $f \in BMO_{L_\alpha}$ and a is an $H^1_{L_\alpha}$-atom supported in $B = B_r(y_0)$. If $r < \rho_{L_\alpha}(y_0)/2$, using the cancellation condition, we have

$$\begin{aligned} \left|\int f(x)a(x)\,dx\right| &= \left|\int (f(x) - f_B)a(x)\,dx\right| \\ &\le \frac{1}{|B|}\int_B |f - f_B|\,dx \le \|f\|_{BMO_{L_\alpha}}. \end{aligned}$$

If $\rho_{L_\alpha}(y_0)/2 \le r \le \rho_{L_\alpha}(y_0)$, then

$$\left|\int_B f(x)a(x)\,dx\right| \le \frac{1}{|B|}\int_B |f(x)|\,dx \le \|f\|_{BMO_{L_\alpha}}.$$

Thus, we have seen that each $f \in BMO$ gives a bounded linear functional on the dense subspace of finite linear combinations of $H^1_{L_\alpha}$-atoms, and hence on $H^1_{L_\alpha}$.

Throughout this proof we always assume $B = B_r(y_0)$. Now we come to prove the converse. Let $\Phi \in (H^1_{L_\alpha})^*$. By Lemma 2 with B_k in Lemma 1, there exists a unique f_k such that $\|f_k\|_{L^2(B_k)} \le C|B_k|^{1/2}\|\Phi\|$, and

$$\Phi(g) = \int_{B_k} f_k(x)g(x)\,dx, \quad \forall g \in L^2(B_k).$$

By Lemma 1, only B_{k-1} and B_{k+1} intersect with B_k. Since $L^2(B_k \cap B_{k-1}) \subset L^2(B_k) \subset H^1_{L_\alpha}$, we have $f_k = f_{k-1}$, for a.e. $x \in B_k \cap B_{k-1}$. Similarly, $f_k = f_{k+1}$, for a.e. $x \in B_k \cap B_{k+1}$. Again by (a) of Lemma 1, we define a unique locally square-integrable function f in $(0, \infty)$ so that $f = f_k$, for a.e. $x \in B_k$. Consider any ball $B_r(y_0)$ with $\rho_{L_\alpha}(y_0)/2 \le r \le \rho_{L_\alpha}(y_0)$. By (c) of Lemma 1, we also have

$$\Phi(g) = \int f(x)g(x)\,dx, \quad g \in L^2(B), \quad \text{and}$$

$$\|f\|_{L^2(B)} \le C|B|^{1/2}\|\Phi\|.$$

So

$$\frac{1}{|B|}\int_B |f(x)|\,dx \le \frac{1}{|B|^{1/2}}\left(\int_B |f(x)|^2\,dx\right)^{1/2} \le C\|\Phi\|.$$

For $r < \rho_{L_\alpha}(y_0)/2$, let $L^2_{B,0}$ denote the closed subspace of $L^2(B)$ of functions with zero integral and $H^1(0,\infty)$ denote all functions of H^1 supported in $(0,\infty)$. Notice that $L^2_{B,0} \subset H^1_{L_\alpha} \cap H^1(0,\infty)$ and $\|g\|_{H^1} \le C|B|^{1/2}\|g\|_{L^2}$. Applying Lemma 3, for any $g \in L^2_{B,0}$, we have

$$|\Phi(g)| \le \|\Phi\|\|g\|_{H^1_{L_\alpha}} \le C\|\Phi\||B|^{1/2}\|g\|_{L^2}.$$

By the Riesz representation theorem, there exists an element $h \in L^2_{B,0}$ so that

$$\|h\|_{L^2_{B,0}} \le C\|\Phi\||B|^{1/2},$$

and

$$\Phi(g) =< h, g >=< f, g >,$$

here $f = h + C_B$, indeed, $C_B = \frac{1}{|B|}\int_B f(x)\,dx$. Hence, one easily observes that

$$\frac{1}{|B|}\int_B |f(x) - C_B|\,dx \le \frac{1}{|B|^{1/2}}\|h\|_{L^2} \le C\|\Phi\|.$$

This completes the proof.

Corollary 4. *Let $B = B_r(y_0) \subset (0,\infty)$. There exists $C > 0$ such that, for all $f \in BMO_{L_\alpha}$, we have*

(1) *If $r \ge \rho_{L_\alpha}(y_0)/2$, then $\left(\frac{1}{|B|}\int_B |f(x)|^2\,dx\right)^{1/2} \le C\|f\|_{BMO_{L_\alpha}}$.*

(2) *If $r < \rho_{L_\alpha}(y_0)/2$, then $\left(\frac{1}{|B|}\int_B |f(x) - f_B|^2\,dx\right)^{1/2} \le C\|f\|_{BMO_{L_\alpha}}$.*

Proof. Combining the previous proof and Lemma 1, one easily gets (1). For $f \in BMO_{L_\alpha}$, let f_0 denote the odd extension of f to $\mathbb{R}$. Then it is also easy to see that $\|f_o\|_{BMO} \le 2\|f\|_{BMO_{L_\alpha}}$. Therefore, the John-Nirenberg inequality implies

$$\left(\frac{1}{|B|}\int_B |f(x)-f_B|^2\,dx\right)^{1/2} \le \|f_0\|_{BMO} \le C\|f\|_{BMO_{L_\alpha}}.$$

■

4. Proof of Theorem 2

The proof of the next lemma is absolutely same with that of Lemma 3 in [7].

Lemma 4. *For* $f \in L^2(0,\infty)$, *let* $s_Q f = \left(\int_0^\infty |Q_t f(x)|^2 \frac{dt}{t}\right)^{1/2}$. *Then we have* $\|s_Q f\|_2 = \frac{1}{\sqrt{8}}\|f\|_2$. *Moreover,*

$$f(x) = 8 \lim_{\epsilon\to 0, N\to\infty} \int_\epsilon^N Q_t^2 f(x)\,\frac{dt}{t}, \qquad \text{in } \mathrm{L}^2(0,\infty).$$

Remark 1. The above lemma also holds in the Hermite setting.

Lemma 5. *For all* $f \in BMO_{L_\alpha}$ *and* $B_r(y_0)$ *with* $r < \rho_{L_\alpha}(y_0)$. *There exists* $C > 0$ *so that*

$$|f_{B^*}| \le C(1+\log\frac{\rho_{L_\alpha}(y_0)}{r})\|f\|_{BMO_{L_\alpha}}.$$

The proof is trivial, the reader can refer to [10], p. 141.

4.1. Proof of part 1 of Theorem 2

First of all, because of the kernel decay in Corollary 3 and the integrability of $(1+|x|)^{-2}f(x)$ (see [10], p.141),

$$Q_t f(x) = \int_0^\infty Q_t(x,y) f(y)\,dy$$

is a well defined absolutely convergent integral for all $(x,t) \in (0,\infty)\times(0,\infty)$. For any ball $B = B_r(y_0) \subset (0,\infty)$, we need to show that for $f \in BMO_{L_\alpha}$,

$$\text{(4.2).} \qquad \frac{1}{|B|}\int_0^r \int_B Q_t^2 f(x) \frac{dxdt}{t} \le C\|f\|^2_{BMO_{L_\alpha}}$$

We write

$$\begin{aligned} f &= (f-f_{B^*})\mathbb{1}_{B^*} + (f-f_{B^*})\mathbb{1}_{(B^*)^c\cap(0,\infty)} + f_{B^*}\mathbb{1}_{(0,\infty)} \\ &= f_1 + f_2 + f_3. \end{aligned}$$

Treating f_1 first, we note that

$$\begin{aligned}\frac{1}{|B|}\int_0^r\int_B Q_t^2 f_1(x)\frac{dxdt}{t} &\le \frac{1}{|B|}\int_B |s_Q f_1(x)|^2\,dx\\ &\le \frac{C}{|B|}\int_{B^*}|f-f_{B^*}|^2\,dx\\ &\le C\|f\|_{BMO_{L_\alpha}}^2,\end{aligned}$$

here the second inequality follows from theL^2-boundedness of s_Q in Lemma 4 and the last inequality follows from Corollary 4.

In the following process, we will frequently discuss two cases. For the sake of brevity we introduce the additional notations:

$$\begin{aligned}X_1^t(x) &= \Big\{y\in(0,\infty): \frac{e^{-t^2}xy}{1-e^{-2t^2}}\le 1\Big\},\\ X_2^t(x) &= \Big\{y\in(0,\infty): \frac{e^{-t^2}xy}{1-e^{-2t^2}}> 1\Big\}.\end{aligned}$$

Now we start with estimating (4.2) for f_2.

$$\begin{aligned}|Q_t f_2(x)| \le& \int_{X_1^t(x)}|f_2(y)||Q_t(x,y)|\,dy+\int_{X_2^t(x)}|f_2(y)||Q_t(x,y)-P_t(x,y)|\,dy\\ &+\int_{X_2^t(x)}|f_2(y)||P_t(x,y)|\,dy\\ =& \, I_1^t(x)+I_2^t(x)+I_3^t(x).\end{aligned}$$

For $x\in B$, using (a) and (b) in Proposition 1, we have

$$\begin{aligned}I_1^t(x)+I_2^t(x) &\le C\int_{(B^*)^c}\frac{1}{t}\left(1+\frac{|y-x|}{t}\right)^{-2}f(y)\,dy\\ &\le C\int_{(B^*)^c}\frac{t}{(t+|y-y_0|)^2}f(y)\,dy,\end{aligned}$$

and when $y\in(B^*)^c$ and $t<r$, we have $\frac{t}{(t+|y-y_0|)^2}\le c\frac{t}{(r+|y-y_0|)^2}$. We recall the elementary inequality

$$\int_{\mathbb{R}}|g-g_{B_2}|(1+|x|)^{-2}\,dx\le C\|g\|_{BMO},$$

here $B_2=B_2(0)$. Let f_0 denote the odd extension of f to $\mathbb{R}$. Since the BMO space and its norm are translation and dilation invariant, we easily get

$$\begin{aligned}\int_0^\infty\frac{r}{(r+|y-y_0|)^2}f_2(y)\,dy &\le \int_{\mathbb{R}}\frac{r}{(r+|y-y_0|)^2}|f_0-f_{B^*}|\,dy\\ &\le C\|f_0\|_{BMO}\le C\|f\|_{BMO_{L_\alpha}}.\end{aligned}$$

Inserting this in the above gives us that

$$\begin{aligned}\frac{1}{|B|}\int_0^r\int_B(I_1^t(x)+I_2^t(x))^2\frac{dxdt}{t} &\leq C\frac{1}{|B|}\int_0^r\int_B\left(\frac{t}{r}\right)^2\|f\|_{BMO_{L_\alpha}}^2\frac{dxdt}{t}\\ &\leq C\|f\|_{BMO_{L_\alpha}}^2.\end{aligned}$$

To estimate $I_3^t(x)$, using (c) of Proposition 2 and repeating the above proof, we also get

$$\frac{1}{|B|}\int_0^r\int_B(I_3^t(x))^2\frac{dxdt}{t}\leq C\|f\|_{BMO_{L_\alpha}}^2.$$

Now it remains to show (4.2) holds for the constant term f_3. This part will be proved in several steps. The first step is the case of $r\geq 1$.

$$\begin{aligned}&\frac{1}{|B|}\int_0^r\int_B\left|\int_0^\infty Q_t(x,y)f_3(y)dy\right|^2\frac{dxdt}{t}\\ &=\frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B\left|\int_0^\infty Q_t(x,y)dy\right|^2\frac{dxdt}{t}\\ &\leq\frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B\left|\int_{B^*}Q_t(x,y)dy\right|^2\frac{dxdt}{t}\\ &\quad+\frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B\left|\int_{(0,\infty)/B^*}Q_t(x,y)dy\right|^2\frac{dxdt}{t}\\ &=I_1+I_2.\end{aligned}$$

It is easy to check, by Lemma 4, that

$$I_1\leq C\frac{|f_{B^*}|^2}{|B|}|B^*|\leq C\|f\|_{BMO_{L_\alpha}}^2.$$

To estimate I_2, by Corollary 3, we see that $|Q_t(x,y)|\leq C_N\frac{t}{r}\frac{1}{t}(1+\frac{|x-y|}{t})^{-N}$ when $|x-y|\geq r$ and $r\geq 1$. So

$$I_2\leq C\frac{|f_{B^*}|^2}{|B|}|B^*|\int_0^r(\frac{t}{r})^2\frac{dt}{t}\leq C\|f\|_{BMO_{L_\alpha}}^2.$$

The second step is the case of $r\leq\rho_{L_\alpha}(y_0)$.

$$\begin{aligned}&\frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B\left|\int_0^\infty Q_t(x,y)dy\right|^2\frac{dxdt}{t}\\ &\lesssim\frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B\left|\int_{X_1^t(x)}Q_t(x,y)dy\right|^2\frac{dxdt}{t}\\ &\quad+\frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B\left|\int_{X_2^t(x)}Q_t(x,y)dy\right|^2\frac{dxdt}{t}\\ &=J_1+J_2.\end{aligned}$$

Notice that $x \sim y_0$ and $\rho_{L_\alpha}(y_0) \sim \rho_{L_\alpha}(x)$ when $x \in B_r(y_0)$, and $y \leq \frac{C_0 t^2}{y_0}$ when $\frac{e^{-t^2} xy}{(1-e^{-2t^2})} \leq 1$ and $t \leq 1$. Then by an application of (a) of Proposition 1 it follows that

$$\begin{aligned}
J_1 &\lesssim \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_0^{\frac{C_0 t^2}{y_0}} \frac{1}{t} \left(\frac{x}{t}\right)^{\alpha+1/2} \left(\frac{y}{t}\right)^{\alpha+1/2} e^{-\frac{x^2+y^2}{8t^2}} dy \right|^2 \frac{dxdt}{t} \\
&\lesssim |f_{B^*}|^2 \int_0^r t \frac{1}{(\rho_{L_\alpha}(y_0))^2} dt \\
&\lesssim \|f\|^2_{BMO_{L_\alpha}} \left(1 + \log \frac{\rho_{L_\alpha}(y_0)}{r}\right)^2 \left(\frac{r}{\rho_{L_\alpha}(y_0)}\right)^2 \\
&\lesssim \|f\|^2_{BMO_{L_\alpha}},
\end{aligned}$$

here the third inequality follows from Lemma 5. To estimate J_2, we split it into two parts,

$$\begin{aligned}
J_2 &\lesssim \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_{X_2^t(x)} (Q_t(x,y) - P_t(x,y))\, dy \right|^2 \frac{dxdt}{t} \\
&\quad + \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_{X_2^t(x)} P_t(x,y)\, dy \right|^2 \frac{dxdt}{t} \\
&= J_{21} + J_{22}.
\end{aligned}$$

In order to estimate J_{22}, by the simple fact that $\frac{(1-e^{-2t^2})e^{t^2}}{xt} \lesssim \frac{x}{t}$ when $t \leq r \leq \rho_{L_\alpha}(y_0)$ and $x \in B$, we use Proposition 2 to get

$$\begin{aligned}
\left| \int_{X_2(x)} P_t(x,y)\, dy \right| &\lesssim \left| \int_{-\infty}^{\infty} P_t(x,y)\, dy \right| + \int_{-\infty}^{\frac{(1-e^{-2t^2})e^{t^2}}{x}} \frac{1}{t} e^{-\frac{(x-y)^2}{16t^2}} dy \\
&\lesssim \int_{-\infty}^{\frac{(1-e^{-2t^2})e^{t^2}}{xt} - \frac{x}{t}} e^{-\frac{1}{16}z^2} dz + t^2 + \frac{t^2 x^2}{(1+x^2t^2)^N} \\
&\lesssim e^{-\frac{1}{32}(\frac{(1-e^{-2t^2})e^{t^2}}{xt} - \frac{x}{t})^2} + t^2 + \frac{t^2 x^2}{(1+x^2t^2)^N} \\
&\lesssim \left(\frac{t}{x}\right)^2 + t^2 + t^2 x^2 \lesssim \left(\frac{t}{\rho_{L_\alpha}(y_0)}\right)^2.
\end{aligned}$$

Hence,

$$
\begin{aligned}
J_{22} &\lesssim |f_{B^*}|^2 \int_0^r \left(\frac{t}{\rho_{L_\alpha}(y_0)}\right)^2 \frac{dt}{t} \\
&\lesssim \|f\|^2_{BMO_{L_\alpha}} \left(1+\log\frac{\rho_{L_\alpha}(y_0)}{r}\right)^2 \left(\frac{r}{\rho_{L_\alpha}(y_0)}\right)^2 \\
&\lesssim \|f\|^2_{BMO_{L_\alpha}}.
\end{aligned}
$$

Now we deal with J_{21}. Let $H_t(x,y) = Q_t(x,y) - P_t(x,y)$. Recalling (b) of Proposition 1, we have

$$
|H_t(x,y)| \le Ct^2 e^{-\frac{(x-y)^2}{2t^2}} \frac{e^{t^2/2}}{xyt}. \tag{4.3}
$$

We will consider two cases.

Case 1. $y_0 \ge 1$. Notice that $\rho_{L_\alpha}(y_0) = \frac{1}{8}\frac{1}{y_0}$ and $y > C_0\frac{t^2}{y_0}$ when $y \in X_2^t(x)$.

$$
\begin{aligned}
J_{21} &\lesssim \frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B \left|\int_{\frac{C_0t^2}{y_0}}^\infty H_t(x,y)dy\right|^2 \frac{dxdt}{t} \\
&\lesssim \frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B \left|\int_{\frac{C_0t^2}{y_0}}^1 H_t(x,y)dy\right|^2 \frac{dxdt}{t} + \frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B \left|\int_1^\infty H_t(x,y)dy\right|^2 \frac{dxdt}{t} \\
&= J_{211} + J_{212}.
\end{aligned}
$$

Applying (4.3) and the simple fact $|f_{B^*}| \le C\left(1+\log\frac{\rho_{L_\alpha}(y_0)}{r}\right)\|f\|_{BMO_{L_\alpha}}$ in Lemma 5, we get

$$
\begin{aligned}
J_{211} &\lesssim \frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B \left|\int_{\frac{C_0t^2}{y_0}}^1 \frac{t^2}{x}\frac{1}{y}\frac{1}{t}\,dy\right|^2 \frac{dxdt}{t} \\
&\lesssim |f_{B^*}|^2 \int_0^r \frac{t^2}{y_0^2}\left|\log(c_0\frac{t^2}{y_0})\right|^2 \frac{dt}{t}. \\
&\lesssim \left(1+\log\frac{\rho_{L_\alpha}(y_0)}{r}\right)^2 r^{1/2}\|f\|^2_{BMO_{L_\alpha}} \\
&\lesssim \|f\|^2_{BMO_{L_\alpha}},
\end{aligned}
$$

and

$$
\begin{aligned}
J_{212} &\lesssim \frac{|f_{B^*}|^2}{|B|}\int_0^r\int_B \left|\int_1^\infty \frac{t^2}{x}e^{-\frac{|x-y|^2}{2t^2}}\frac{1}{t}\,dy\right|^2 \frac{dxdt}{t} \\
&\lesssim |f_{B^*}|^2 \int_0^r \left(\frac{t^2}{y_0}\right)^2 \frac{dt}{t}. \\
&\lesssim \|f\|^2_{BMO_{L_\alpha}}.
\end{aligned}
$$

Case 2. $y_0 < 1$. Similar to case 1, we also split J_{21} into two terms:

$$J_{211} = \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_{\frac{C_0 t^2}{y_0}}^2 H_t(x,y)\, dy \right|^2 \frac{dxdt}{t},$$

$$J_{212} = \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_2^\infty H_t(x,y)\, dy \right|^2 \frac{dxdt}{t}.$$

For J_{212} we can repeat the previous proof in case 1 to obtain $J_{212} \le C\|f\|^2_{BMO_{L_\alpha}}$. To deal with J_{211} we need some further decompositions:

$$J_{211} \le \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_{\frac{C_0 t^2}{y_0}}^{\frac{10}{8}y_0} H_t(x,y)\, dy \right|^2 \frac{dxdt}{t}$$
$$+ \frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_{\frac{10}{8}y_0}^2 H_t(x,y)\, dy \right|^2 \frac{dxdt}{t}$$

A new use of (4.3) and Lemma 5 show that the first term is controlled by

$$C\frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_{\frac{C_0 t^2}{y_0}}^{\frac{10}{8}y_0} \frac{t}{xy}\, dy \right|^2 \frac{dxdt}{t} \lesssim |f_{B^*}|^2 \int_0^r \left| \frac{t}{y_0} \log \frac{c_0 y_0^2}{t^2} \right|^2 \frac{dt}{t}$$
$$\lesssim \|f\|^2_{BMO_{L_\alpha}}.$$

Furthermore, combining (4.3) and the simple fact that $y - x \sim y$ when $y > \frac{10}{8}y_0$, the second term is bounded by

$$C\frac{|f_{B^*}|^2}{|B|} \int_0^r \int_B \left| \int_{\frac{10}{8}y_0}^2 \frac{t^2}{y^2 y_0}\, dy \right|^2 \frac{dxdt}{t} \lesssim |f_{B^*}|^2 \int_0^r \left| \frac{t^2}{y_0} \left(\frac{1}{2} - \frac{8}{10 y_0} \right) \right|^2 \frac{dt}{t}$$
$$\lesssim |f_{B^*}|^2 \int_0^r \frac{t^3}{y_0^4} dt$$
$$\lesssim \|f\|^2_{BMO_{L_\alpha}}.$$

Now we consider the final case, that is, $\rho_{L_\alpha}(y_0) < r < 1$. From Lemma 1 we choose a family of critical balls $\{B_l\}_{l=i_0}^{j_0}$ so that $B \subset \bigcup_{l=i_0}^{j_0} B_l$ and $\sum_{l=i_0}^{j_0} |B_l| \sim |B|$. Then the left of (4.2) with f_3 is bounded by

$$\frac{|f_{B^*}|^2}{|B|} \sum_{l=i_0}^{j_0} \int_0^r \int_{B_l} \left| Q_t \mathbb{1}_{(0,\infty)} \right|^2 \frac{dxdt}{t} \le \frac{|f_{B^*}|^2}{|B|} \sum_{l=i_0}^{j_0} \int_0^{\rho_{L_\alpha}(x_l)} \int_{B_l} \left| Q_t \mathbb{1}_{(0,\infty)} \right|^2 \frac{dxdt}{t}$$
$$+ \frac{|f_{B^*}|^2}{|B|} \sum_{l=i_0}^{j_0} \int_{\rho_{L_\alpha}(x_l)}^r \int_{B_l} \left| Q_t \mathbb{1}_{(0,\infty)}(x) \right|^2 \frac{dxdt}{t}.$$

Using the argument in the second step and $|f_{B^*}| \le \|f\|_{BMO_{L_\alpha}}$, the first term of the right is controlled by

$$C\frac{1}{|B|}\sum_{l=i_0}^{j_0}|B_l|\|f\|^2_{BMO_{L_\alpha}} \lesssim \|f\|^2_{BMO_{L_\alpha}}.$$

So it remains to show that

$$\int_{\rho_{L_\alpha}(x_l)}^{r}\int_{B_l}\left|Q_t\mathbb{1}_{(0,\infty)}\right|^2\frac{dxdt}{t} \le C|B_l|. \tag{4.4}$$

In the subsequent proof, we will slightly modify the argument in the second step. In the beginning, we also divide $(0,\infty)$ into two domains $X_1^t(x)$ and $X_2^t(x)$ and break the left of (4.4) into two parts H_1 and H_2. According to Proposition 1, it follows that $H_1 \le C|B_l|$ for $x_l \ge 1$ by a similar proof for J_1. For $x_l < 1$,

$$\begin{aligned} H_1 &\le C\int_{\rho_{L_\alpha}(x_l)}^{r}\int_{B_l}\left|\int_0^{\frac{C_0t^2}{x_l}}\frac{1}{t}\left(\frac{x}{t}\right)^{\alpha+1/2}\left(\frac{y}{t}\right)^{\alpha+1/2}e^{-\frac{x^2+y^2}{8t^2}}dy\right|^2\frac{dxdt}{t} \\ &\le C|B_l|\int_{\rho_{L_\alpha}(x_l)}^{r}\left(\frac{x_l}{t}\right)^{2\alpha+1}\frac{dt}{t} \le C|B_l|. \end{aligned}$$

In order to deal with H_2, like treating J_2 we make difference between $Q_t(x,y)$ and $P_t(x,y)$ to split H_2 as:

$$H_{21} = \int_{\rho_{L_\alpha}(x_l)}^{r}\int_{B_l}\left|\int_{X_2(x)}(Q_t(x,y)-P_t(x,y))\,dy\right|^2\frac{dxdt}{t}$$

and

$$H_{22} = \int_{\rho_{L_\alpha}(x_l)}^{r}\int_{B_l}\left|\int_{X_2(x)}P_t(x,y)\,dy\right|^2\frac{dxdt}{t}.$$

For H_{22}, if $x_l \ge 1$, using the same way to estimate $\left|\int_{X_2(x)}P_t(x,y)\,dy\right|$ in treating J_{22}, then

$$\begin{aligned} H_{22} &\lesssim |B_l|\int_{\rho_{L_\alpha}(x_l)}^{r}\left(\frac{t}{\rho_{L_\alpha}(y_0)}\right)^2\frac{dt}{t} \\ &\lesssim |B_l|. \end{aligned}$$

If $x_l < 1$, then $\rho_{L_\alpha}(x_l) < t < r < 1$ and $x \backsim \rho_{L_\alpha}(x_l)$ give that $\frac{x}{t} \lesssim \frac{t}{x}$. So, using part (c) of Proposition 2 we get

$$\begin{aligned} \left|\int_{X_2(x_l)}P_t(x,y)\,dy\right| &\le \int_{\frac{(1-e^{-2t^2})e^{t^2}}{x}}^{\infty}\frac{1}{t}e^{-\frac{(x-y)^2}{16t^2}}\,dy \\ &\le \int_{\frac{(1-e^{-2t^2})e^{t^2}}{x}-\frac{x}{t}}^{\infty}e^{-\frac{1}{16}z^2}\,dz \le Ce^{-\frac{1}{32}\left(\frac{(1-e^{-2t^2})e^{t^2}}{x}-\frac{x}{t}\right)^2}. \end{aligned}$$

Hence,

$$H_{22} \lesssim |B_l| \int_{\rho_{L_\alpha}(x_l)}^{r} \frac{x_l^2}{t^2}\frac{1}{t}\, dt.$$
$$\lesssim |B_l| x_l^2 \left(\rho_{L_\alpha}(x_l)^{-2} - r^{-2}\right) \lesssim |B_l|.$$

Finally, we estimate H_{21} to complete our proof. To do this, again making use of (4.3), if $x_l < 1$, repeating the argument for H_{22} in the case of $x_l < 1$, we have $H_{21} \lesssim |B_l|$. If $x_l \geq 1$, using (4.3) again, we obtain

$$\begin{aligned} H_{21} &\lesssim \int_{\rho_{L_\alpha}(x_l)}^{r} \int_{B_l} \left| \int_{\frac{c_0 t^2}{x}}^{\infty} t^2 e^{-\frac{(x-y)^2}{2t^2}} \frac{e^{t^2/2}}{xyt}\, dy \right|^2 \frac{dxdt}{t} \\ &\lesssim \int_{\rho_{L_\alpha}(x_l)}^{r} \int_{B_l} \left| \int_{\frac{c_0 t^2}{x}}^{1} \frac{t}{xy}\, dy \right|^2 \frac{dxdt}{t} + \int_{\rho_{L_\alpha}(x_l)}^{r} \int_{B_l} \left| \int_{1}^{\infty} t^2 e^{-\frac{(x-y)^2}{2t^2}}\, dy \right|^2 \frac{dxdt}{t} \\ &\lesssim |B_l| \left| \rho_{L_\alpha}(x_l) \log(\rho_{L_\alpha}(x_l)^3) \right|^2 \int_{\rho_{L_\alpha}(x_l)}^{r} t\, dt + |B_l| \\ &\lesssim |B_l|. \end{aligned}$$

Thus we finish the whole proof.

4.2. Proof of part 2 of Theorem 2

Define

$$\mathcal{G}(f)(x) = \left\{ \int_{\Gamma_+(x)} |Q_t f(y)|^2 \frac{dydt}{t^2} \right\}^{1/2},$$

where $\Gamma_+(x) = \{(y,t) \in (0,\infty) \times (0,\infty) : |x-y| < t\}$, for every $x \in (0,\infty)$.

Lemma 6. ([3], Theorem 1.5). *Let* $\alpha > -1/2$. *If* $f \in H^1_{L_\alpha}(0,\infty)$, *then there exists* $C > 0$ *such that*

$$\|\mathcal{G}(f)\|_{L^1(0,\infty)} \leq C \|f\|_{H^1_{L_\alpha}}.$$

Set

$$\mathcal{I}(f)(x) = \sup_{x \in B \subset (0,\infty)} \left(\frac{1}{|B|} \int_0^{r(B)} \int_B |Q_t f(y)|^2 \frac{dydt}{t} \right)^{1/2},$$

where $r(B)$ denotes the radius of B.

The following lemma is a slightly modified version of Proposition in [10], p.162. We omit the details of proof.

Lemma 7. *If* $\mathcal{G}(g)(x) \in L^1(0,\infty)$ *and* $\mathcal{I}(f)(x) \in L^\infty(0,\infty)$, *then, there exists a* $C > 0$ *so that*

$$\begin{aligned} \int_{(0,\infty)\times(0,\infty)} |Q_t f(x) Q_t g(x)| \frac{dxdt}{t} &\leq C \int_0^\infty \mathcal{I}(f)(x)\mathcal{G}(g)(x)dx \\ &\leq C \|\mathcal{I}(f)(x)\|_{L^\infty(0,\infty)} \|\mathcal{G}(g)(x)\|_{L^1(0,\infty)}. \end{aligned}$$

We need the following identity to finish our proof.

Lemma 8. *Suppose* $\int_0^\infty \frac{|f(x)|}{1+x^2}\,dx < \infty$ *and* g *is an* $H^1_{L_\alpha}$*-atom. Then*

$$\frac{1}{8}\int_0^\infty f(x)\overline{g(x)}\,dx = \int_{(0,\infty)\times(0,\infty)} Q_t f(x)\overline{Q_t g(x)}\frac{dxdt}{t}.$$

Indeed, observe that $\|d\mu\|_{\mathcal{C}} = \|\mathcal{I}(f)(x)\|_{L^\infty(0,\infty)}$. Thus, we have

$$\begin{aligned}\left|\int_0^\infty f(x)\overline{g(x)}\,dx\right| &\le C\|\mathcal{I}(F)(x)\|_{L^\infty(0,\infty)}\|\mathcal{G}(g)\|_{L^1(0,\infty)}\\ &\le C\|d\mu\|_{\mathcal{C}}\|g\|_{H^1_{L_\alpha}}.\end{aligned}$$

Proof of Lemma 8. From Lemma 6 and 7, by the dominated convergence theorem one easily observes that

$$\mathcal{I} = \int_{(0,\infty)\times(0,\infty)} Q_t f(x)\overline{Q_t g(x)}\frac{dxdt}{t} = \lim_{\epsilon\to 0, N\to\infty}\int_\epsilon^N\int_0^\infty Q_t f(x)\overline{Q_t g(x)}\frac{dxdt}{t}.$$

Next, we shall show that, for fixed t,

$$\begin{aligned}\int_0^\infty Q_t f(x)\overline{Q_t g(x)}dx &= \int_0^\infty\int_0^\infty Q_t(x,y)f(y)dy\overline{Q_t g(x)}dx\\ &= \int_0^\infty f(y)\overline{Q_t^2 g(y)}dy,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I} &= \lim_{\epsilon\to 0, N\to\infty}\int_\epsilon^N\left[\int_0^\infty f(y)\overline{Q_t^2 g(y)}dy\right]\frac{dt}{t}\\ &= \int_0^\infty f(y)\left[\lim_{\epsilon\to 0, N\to\infty}\int_\epsilon^N \overline{Q_t^2 g(y)}\frac{dt}{t}\right]dy\\ &= \int_0^\infty f(y)g(y)\,dy.\end{aligned}$$

In order to justify the absolute integrability in the above integrals, by the hypothesis $\int_0^\infty \frac{|f(x)|}{1+x^2}\,dx < \infty$ and the kernel decay in Corollary 3, we only need to show that, for any $H^1_{L\alpha}$-atom g,

$$M_Q g(x) = \sup_{t>0}|Q_t g(x)| \le C_{y_0,r}\left(1+x\right)^{-2}, \quad x>0, \tag{4.5}$$

and

$$\mathcal{M}_Q g(x) = \sup_{\epsilon,N>0}\left|\int_\epsilon^N Q_t^2 g(x)\frac{dt}{t}\right| \le C_{y_0,r}\left(1+x\right)^{-2}, \quad x>0. \tag{4.6}$$

In fact, without loss of generality we assume g supported in $B = B_r(y_0)$ with $r < \rho_{L_\alpha}(y_0)$, first using (a) of Proposition 1 one easily observes

$$\sup_{t\geq 1}\left|\int_0^\infty Q_t(x,y)g(y)\,dy\right| \leq C\int_0^\infty \exp(-c_0|x-y|^2)|g(y)|\,dy \leq C_{y_0,g}(1+x)^{-2}.$$

If $t<1$ and $x\notin B^*$, then for $y\in B$, $|x-y|\sim|x-y_0|$, applying Corollary 3,

$$\begin{aligned}\left|\int_0^\infty Q_t(x,y)g(y)\,dy\right| &\leq C_N\|g\|_1\frac{1}{t}\exp\left(-c_0\frac{|x-y_0|^2}{t^2}\right)\\ &\leq C_{y_0,g}|x-y_0|^{-2}.\end{aligned}$$

Finally, if $x\in B^*$, also by Corollary 3, we have

$$\left|\int_0^\infty Q_t(x,y)g(y)\,dy\right| \leq C\|g\|_{L^\infty},$$

which establishes (4.5). Moreover, using (2.1)-(2.4) we also get

$$\sup_{t>0}|W_t^\alpha g(x)| \leq C_{y_0,r}\,(1+x)^{-2}\,. \tag{4.7}$$

Combining (4.5) and (4.7), we obtain

$$\begin{aligned}\left|\int_\epsilon^N Q_t^2 g(x)\frac{dt}{t}\right| &= \left|\frac{1}{8}\left(W_{2\epsilon^2}^\alpha g(x) - Q_{\sqrt{2}\epsilon}g(x)\right) - \frac{1}{8}\left(W_{2N^2}^\alpha g(x) - Q_{\sqrt{2}N}g(x)\right)\right|\\ &\leq C_{y_0,r}\,(1+x)^{-2}\,,\end{aligned}$$

which establishes (4.6) and hence complete the proof.

5. Proof of Theorem 3

By the definition of BMO_{L_α} and Corollary 2 it suffices to prove the followings: for every fixed "critical ball" $B_k\in\mathcal{B}$ (see Lemma 1) we have

(1) $\frac{1}{|B_k|}\int_{B_k}|\mathcal{I}_\sigma f|\,dx \leq C\|f\|_{\frac{1}{\sigma}}$;

(2) $\|\mathcal{I}_\sigma f\|_{BMO(B_k^*)} \leq C\|f\|_{\frac{1}{\sigma}}$.

Let us treat (1) first. Split

$$\mathcal{I}_\sigma f(x) = \int_0^1 T_t^\alpha f(x)t^{\sigma/2-1}\,dt + \int_1^\infty T_t^\alpha f(x)t^{\sigma/2-1}\,dt = I_1 f(x) + I_2 f(x).$$

By (2.5) one easily gets

$$\begin{aligned}\frac{1}{|B_k|}\int_{B_k}|I_2 f(x)|\,dx &\leq \frac{1}{|B_k|^\sigma}\left(\int_{B_k}|I_2 f(x)|^{\frac{1}{\sigma}}\,dx\right)^\sigma\\ &\leq C\frac{1}{|B_k|^\sigma}\left(\int_{B_k}\left|\int_0^\infty \exp\left(-c_0|x-y|^2\right)f(y)\,dy\right|^{\frac{1}{\sigma}}\,dx\right)^\sigma\\ &\leq C\|f\|_{1/\sigma}.\end{aligned}$$

To deal with $I_1f(x)$, first we consider the case of $x_k \geq 1$. We use (2.6) to obtain

$$\begin{aligned}
&|I_1f(x)| \\
&\lesssim \int_0^1 \int_{\{y>0,|y-x_k|<1/2\}} t^{-1/2} \exp\left(-c_0\frac{|x-y|^2}{t}\right) \exp\left(-c_2txy\right) |f(y)|\, dy t^{\sigma/2-1}\, dt \\
&\quad + \int_0^1 \int_{\{y>0,|y-x_k|\geq 1/2\}} t^{-1/2} \exp\left(-c_0\frac{|x-y|^2}{t}\right) \exp\left(-c_2txy\right) |f(y)|\, dy t^{\sigma/2-1}\, dt \\
&\lesssim Mf(x) \int_0^1 \exp(-c_2x_k^2t)t^{\sigma/2-1}\, dt + \|f\|_{1/\sigma} \\
&\lesssim Mf(x)(\rho_{L_\alpha}(x_k))^\sigma + \|f\|_{1/\sigma},
\end{aligned}$$

where M is the classical Hardy-Littlewood operator and f can be seen as the function defined on $\mathbb{R}$. Then it follows that

$$\frac{1}{|B_k|}\int_{B_k} |I_1f(x)|\, dx \leq C\frac{1}{|B_k|}\int_{B_k} Mf(x)(\rho_{L_\alpha}(x_k))^\sigma\, dx + \|f\|_{1/\sigma} \leq C\|f\|_{1/\sigma}.$$

Next, we treat the case that $x_k < 1$. In this case, $\rho_{L_\alpha}(x_k) \sim x_k$. Repeating a part of the argument above, it is easy to check that

$$\frac{1}{|B_k|}\int_0^1 \int_{B_k} \int_{\{y>0,|y-x_k|>1\}} W_t^\alpha(x,y)|f(y)|\, dydxt^{\sigma/2-1}dt \leq C\|f\|_{1/\sigma}.$$

Now, observe that

$$\begin{aligned}
&\frac{1}{|B_k|}\int_0^1 \int_{B_k} \int_{\{y>0,|y-x_k|<1\}} W_t^\alpha(x,y)|f(y)|\, dydxt^{\sigma/2-1}dt \\
&= \frac{1}{|B_k|}\int_0^1 \int_{B_k} \int_0^{x_k+2\rho_{L_\alpha}(x_k)} W_t^\alpha(x,y)|f(y)|\, dydxt^{\sigma/2-1}dt \\
&\quad + \sum_{j=1}^{j=k_0} \frac{1}{|B_k|}\int_0^1 \int_{B_k} \int_{x_k+2^j\rho_{L_\alpha}(x_k)}^{x_k+2^{j+1}\rho_{L_\alpha}(x_k)} W_t^\alpha(x,y)|f(y)|\, dydxt^{\sigma/2-1}dt \\
&= J_0 + \sum_{j=1}^{j=k_0} J_j,
\end{aligned}$$

where $2^{k_0+1}\rho_{L_\alpha}(x_k) \sim 1$. Using (2.6), J_0 is bounded by

$$
\begin{aligned}
&\frac{1}{|B_k|}\int_0^{\rho_{L_\alpha}(x_k)^2}\int_{B_k}\int_0^{x_k+2\rho_{L_\alpha}(x_k)} W_t^\alpha(x,y)|f(y)|\,dydxt^{\sigma/2-1}dt\\
&+\frac{1}{|B_k|}\int_{\rho_{L_\alpha}(x_k)^2}^1\int_{B_k}\int_0^{x_k+2\rho_{L_\alpha}(x_k)} W_t^\alpha(x,y)|f(y)|\,dydxt^{\sigma/2-1}dt\\
&\lesssim \frac{1}{|B_k|}\int_{B_k} Mf(x)dx\rho_{L_\alpha}(x_k)^\sigma+\int_{\rho_{L_\alpha}(x_k)^2}^1 t^{\sigma/2-1-1/2}\,dt\int_0^{c_0x_k}|f(y)|\,dy\\
&\lesssim \|Mf\|_{1/\sigma}+x_k^{\sigma-1}\int_0^{c_0x_k}|f(y)|\,dy\lesssim \|f\|_{1/\sigma}.
\end{aligned}
$$

In order to analyze J_j , similarly we split J_j as

$$
\begin{aligned}
J_j &= \frac{1}{|B_k|}\int_0^{2^j\rho_{L_\alpha}(x_k)^2}\int_{B_k}\int_{x_k+2^j\rho_{L_\alpha}(x_k)}^{x_k+2^{j+1}\rho_{L_\alpha}(x_k)} W_t^\alpha(x,y)|f(y)|\,dydxt^{\sigma/2-1}dt\\
&\quad+\frac{1}{|B_k|}\int_{2^j\rho_{L_\alpha}(x_k)^2}^1\int_{B_k}\int_{x_k+2^j\rho_{L_\alpha}(x_k)}^{x_k+2^{j+1}\rho_{L_\alpha}(x_k)} W_t^\alpha(x,y)|f(y)|\,dydxt^{\sigma/2-1}dt\\
&= J_{j1}+J_{j2}.
\end{aligned}
$$

Applying (2.6) again, we have

$$
\begin{aligned}
J_{j1} &\lesssim \frac{1}{|B_k|}\int_0^{2^j\rho_{L_\alpha}(x_k)^2}\frac{1}{2^j\rho_{L_\alpha}(x_k)}\int_{B_k}\int_{x_k+2^j\rho_{L_\alpha}(x_k)}^{x_k+2^{j+1}\rho_{L_\alpha}(x_k)}|f(y)|\,dydxt^{\sigma/2-1}dt\\
&\lesssim \frac{1}{(2^j\rho_{L_\alpha}(x_k))^\sigma}\left(2^j\rho_{L_\alpha}(x_k)^2\right)^{\frac{\sigma}{2}}\frac{1}{(2^j\rho_{L_\alpha}(x_k))^{(1-\sigma)}}\int_{x_k+2^j\rho_{L_\alpha}(x_k)}^{x_k+2^{j+1}\rho_{L_\alpha}(x_k)}|f(y)|\,dy\\
&\lesssim (2^j)^{-\frac{\sigma}{2}}\|f\|_{1/\sigma}.
\end{aligned}
$$

For J_{j2}, since $\frac{\sigma}{2}-\alpha-\frac{1}{2}\le 0$, combining (2.1) and (2.4) we obtain

$$
\begin{aligned}
J_{j2} &\lesssim \frac{1}{|B_k|}\int_{2^j\rho_{L_\alpha}(x_k)^2}^1\int_{B_k}\frac{1}{2^j\rho_{L_\alpha}(x_k)}\int_{x_k+2^j\rho_{L_\alpha}(x_k)}^{x_k+2^{j+1}\rho_{L_\alpha}(x_k)}\\
&\qquad |f(y)|\,dydx\left(\frac{2^jx_k^2}{t}\right)^{\alpha+1/2}t^{\sigma/2-1}dt\\
&\lesssim \frac{1}{(2^j\rho_{L_\alpha}(x_k))^\sigma}\left(2^jx_k^2\right)^{\alpha+1/2}\left(2^j\rho_{L_\alpha}(x_k)^2\right)^{\sigma/2-\alpha-1/2}\|f\|_{1/\sigma}\\
&\lesssim (2^j)^{-\sigma/2}\|f\|_{1/\sigma}.
\end{aligned}
$$

Combing these estimates we finish the proof of (1).

Now we pass to prove the assertion (2). As before, we only need to show that

$$
\left\|\int_0^1 T_t^\alpha f(x)t^{\sigma/2-1}\,dt\right\|_{BMO(B_k^*)}\le C\|f\|_{\frac{1}{\sigma}}.
$$

Let $B = B_r(x_0) \subset B_k^*$. Split

$$\int_0^1 T_t^\alpha f(x) t^{\sigma/2-1}\,dt = \int_0^{r^2} T_t^\alpha f(x) t^{\sigma/2-1}\,dt + \int_{r^2}^1 T_t^\alpha f(x) t^{\sigma/2-1}\,dt = J_1 + J_2.$$

For J_1, Hölder's and Minkowski's inequalities give

$$\begin{aligned}\frac{1}{|B|}\int_B J_1\,dx &\le \frac{1}{|B|^\sigma}\left(\int_B J_1^{\frac{1}{\sigma}}\,dx\right)^\sigma \\ &\le \frac{1}{|B|^\sigma}\int_0^{r^2} \|T_t^\alpha f\|_{\frac{1}{\sigma}} t^{\frac{\sigma}{2}-1}\,dt \le C\|f\|_{1/\sigma}.\end{aligned}$$

To deal with J_2, we split J_2 into two terms by decomposing $f = \mathbb{1}_{B^*} f + (\mathbb{1}_{(0,\infty)} - \mathbb{1}_{B^*})f$. Using (2.6) and Hölder's inequality we have

$$\begin{aligned}|J_{21}| = \left|\int_{r^2}^1 T_t^\alpha \mathbb{1}_{B^*} f t^{\sigma/2-1}\,dt\right| &\le C\int_{r^2}^1 \int_{B^*} f(y)\,dy t^{\sigma/2-1-1/2}\,dt \\ &\le C\|f\|_{\frac{1}{\sigma}}.\end{aligned}$$

To complete our proof, by the definition $BMO(B_k^*)$, we only need to find a suitable constant C_B such that

$$\frac{1}{|B|}\int_B \left|\int_{r^2}^1 T_t^\alpha (\mathbb{1}_{(0,\infty)} - \mathbb{1}_{B^*}) f(x) t^{\sigma/2-1}\,dt - C_B\right| dx \le C\|f\|_{1/\sigma}.$$

We set $C_B = \int_{r^2}^1 T_t^\alpha (1 - \mathbb{1}_{B^*}) f(x_0) t^{\sigma/2-1}\,dt$. Using (2.4), we get

$$\left|\int_{r^2}^1 T_t^\alpha (\mathbb{1}_{(0,\infty)} - \mathbb{1}_{B^*}) f(x) t^{\sigma/2-1}\,dt - C_B\right| \le H_1 + H_2 + H_3,$$

where

$$\begin{aligned}H_1 = \int_{r^2}^1 \int_{(B^*)^c\cap(0,\infty)} &|H(x,y,t)\Psi(x,y,t)\Phi_\alpha(x,y,t) \\ &-H(x_0,y,t)\Psi(x,y,t)\Phi_\alpha(x,y,t)|\,|f(y)| t^{\sigma/2-1} dy dt;\\ H_2 = \int_{r^2}^1 \int_{(B^*)^c\cap(0,\infty)} &|H(x_0,y,t)\Psi(x,y,t)\Phi_\alpha(x,y,t) \\ &-H(x_0,y,t)\Psi(x_0,y,t)\Phi_\alpha(x,y,t)|\,|f(y)| t^{\sigma/2-1} dy dt;\\ H_3 = \int_{r^2}^1 \int_{(B^*)^c\cap(0,\infty)} &|H(x_0,y,t)\Psi(x_0,y,t)\Phi_\alpha(x,y,t) \\ &-H(x_0,y,t)\Psi(x_0,y,t)\Phi_\alpha(x_0,y,t)|\,|f(y)| t^{\sigma/2-1} dy dt.\end{aligned}$$

Before estimating H_1, H_2 and H_3, we first give a preliminary inequality: For $x \in B$,

$$\int_{r^2}^1 \int_{(B^*)^c} \frac{|x-x_0|}{\sqrt{t}} \frac{1}{\sqrt{t}} \exp\left\{-c_0 \frac{|y-x_0|^2}{t}\right\} |f(y)| dy t^{\frac{\sigma}{2}-1} dt \leq C\|f\|_{\frac{1}{\sigma}}. \tag{5.1}$$

Indeed, notice that $|x-x_0| \leq r$, then

$$\begin{aligned}
&\int_{r^2}^1 \int_{(B^*)^c \cap (0,\infty)} \frac{|x-x_0|}{\sqrt{t}} \frac{1}{\sqrt{t}} \exp\left(-c_0 \frac{|y-x_0|^2}{t}\right) |f(y)| dy t^{\frac{\sigma}{2}-1} dt \\
&\lesssim \int_{r^2}^1 \sum_{j=1}^{\infty} \int_{\{y>0: |y-x_0| \sim 2^j r\}} \frac{|x-x_0|}{\sqrt{t}} \frac{1}{\sqrt{t}} \exp\left(-c_0 \frac{|y-x_0|^2}{t}\right) |f(y)| dy t^{\frac{\sigma}{2}-1} dt \\
&\lesssim \int_{r^2}^1 \sum_{j=1}^{\infty} \frac{|x-x_0|}{\sqrt{t}} \frac{1}{\sqrt{t}} e^{-c_0 \frac{(2^j r)^2}{t}} \int_{\{y>0: |y-x_0| \sim 2^j r\}} |f(y)| dy t^{\frac{\sigma}{2}-1} dt \\
&\lesssim \int_{r^2}^1 \frac{|x-x_0|}{\sqrt{t}} \frac{1}{\sqrt{t}} \left[\sum_{j=1}^{\infty} \frac{(2^j r)^{1-\sigma}}{(1+\frac{2^j r}{\sqrt{t}})^N}\right] t^{\frac{\sigma}{2}-1} dt \|f\|_{\frac{1}{\sigma}} \\
&\lesssim |x-x_0| \int_{r^2}^1 t^{-\frac{1}{2}-1} dt \|f\|_{\frac{1}{\sigma}} \\
&\lesssim \|f\|_{\frac{1}{\sigma}},
\end{aligned}$$

where we use the simple fact $\sum_{j=1}^{\infty} \frac{(2^j r)^{1-\sigma}}{(1+\frac{2^j r}{\sqrt{t}})^N} \leq c t^{\frac{1}{2}-\frac{\sigma}{2}}$.

Now, we treat H_1 first. Using the mean value theorem and (2.1), (2.2) and (2.4), we have

$$H_1 \lesssim \int_{r^2}^1 \int_{(B^*)^c \cap (0,\infty)} \frac{|x-x_0|}{\sqrt{t}} \frac{1}{\sqrt{t}} \exp\left(-c \frac{|y-x_0|^2}{t}\right) |f(y)| dy t^{\frac{\sigma}{2}-1} dt,$$

and hence (5.1) gives our aim.

For H_2, we use the mean value theorem for Ψ and get

$$H_2 \lesssim \int_{r^2}^1 \int_{(B^*)^c \cap (0,\infty)} |x-x_0| y t \frac{1}{\sqrt{t}} \exp\left(-c \frac{|y-x_0|^2}{t}\right) |f(y)| dy t^{\frac{\sigma}{2}-1} dt.$$

Furthermore, we write

$$\begin{aligned}
H_2 &\lesssim \int_{r^2}^1 \int_{(B^*)^c \cap \{0<y<\min\{x_0+1, 2x_0\}\}} \ldots dy dt + \int_{r^2}^1 \int_{y>\min\{x_0+1, 2x_0\}} \ldots dy dt \\
&\lesssim \int_{r^2}^1 \int_{(B^*)^c} \frac{|x-x_0|}{\rho_{L_\alpha}(x_0)} t \frac{1}{\sqrt{t}} \exp\left(-c \frac{|y-x_0|^2}{t}\right) |f(y)| dy t^{\frac{\sigma}{2}-1} dt \\
&\quad + \int_{r^2}^1 \int_0^{\infty} e^{-c_0 |y-x_0|^2} |f(y)| dy t^{\frac{\sigma}{2}-1} dt \\
&\lesssim \|f\|_{\frac{1}{\sigma}},
\end{aligned}$$

the last inequality follows from the proof of (5.1) and Hölder's inequality.

Finally, it remains to analyze H_3. We start with computing the partial derivative in x of $\Phi_\alpha(x,y,t)$. Using the formula (2.3) and taking $s=\frac{2e^{-t}xy}{1-e^{-2t}}$ we obtain

$$\begin{aligned}\frac{d\Phi_\alpha(x,y,t)}{dx} &= \frac{2e^{-t}y}{1-e^{-2t}}\Big[\frac{1}{2}s^{-1/2}\exp(-s)I_\alpha(s)+\alpha s^{-\frac{1}{2}}\exp(-s)I_\alpha(s)\\ &\qquad +\exp(-s)s^{1/2}(I_{\alpha+1}(s)-I_\alpha(s)).\Big]\end{aligned}$$

We keep in mind that $x\sim x_0$ when $x\in B$. If $\frac{xy}{t}<1$, applying (2.1) and the mean value theorem for Φ_α, the inner of the integration in H_3 is controlled by

$$C\frac{|x-x_0|}{x_0}\Big(\frac{x_0y}{t}\Big)^{\frac{1}{2}+\alpha}\frac{1}{\sqrt{t}}\exp(-c_0\frac{|y-x_0|^2}{t})|f(y)|t^{\frac{\sigma}{2}-1}. \tag{5.2}$$

On the other hand, if $\frac{xy}{t}\geq 1$, using (2.2), the inner function is bounded by

$$C|x-x_0|\frac{y}{t}\Big(\frac{x_0y}{t}\Big)^{-1}\frac{1}{\sqrt{t}}\exp(-c_0\frac{|y-x_0|^2}{t})|f(y)|t^{\frac{\sigma}{2}-1}. \tag{5.3}$$

It is clear that (5.2) and (5.3) are bounded by

$$C\frac{|x-x_0|}{x_0}\frac{1}{\sqrt{t}}\exp(-c_0\frac{|y-x_0|^2}{t})|f(y)|t^{\frac{\sigma}{2}-1}. \tag{5.4}$$

If $x_0\geq 1$, applying (5.1) and (5.4) one easily obtains the desired result. If $x_0<1$, we will split H_3 into the sum of several integrals. we first consider

$$\begin{aligned}&\int_{r^2}^{\rho_{L_\alpha}(x_0)^2}\cdots dt\\ &\lesssim \int_{r^2}^{\rho_{L_\alpha}(x_0)^2}\int_{(B^*)^c\cap(0,\infty)}\frac{|x-x_0|}{x_0}\frac{1}{\sqrt{t}}\exp(-c_0\frac{|y-x_0|^2}{t})|f(y)|t^{\frac{\sigma}{2}-1}dydt\\ &\lesssim \int_{r^2}^{\rho_{L_\alpha}(x_0)^2}\frac{|x-x_0|}{\rho_{L_\alpha}(x_0)}t^{-1}\,dt\|f\|_{\frac{1}{\sigma}}\lesssim\|f\|_{\frac{1}{\sigma}},\end{aligned}$$

here we used the proof in (5.1). Next, we analyze the integral $I=\int_{\rho_{L_\alpha}(x_0)^2}^1\cdots dt$, which is decomposed as

$$\begin{aligned}I &\lesssim \int_{\rho_{L_\alpha}(x_0)^2}^1\int_{(B^*)^c\cap\{y>0,\frac{xy}{t}\geq 1\}}\cdots dydt+\int_{\rho_{L_\alpha}(x_0)^2}^1\int_{(B^*)^c\cap\{y>0,\frac{xy}{t}<1\}}\cdots dydt\\ &= I_1+I_2.\end{aligned}$$

Estimating I_1 we use (5.3)and (5.1) together with $t>\rho_{L_\alpha}(x_0)^2$ to obtain

$$
\begin{aligned}
I_1 &\lesssim \int_{\rho_{L_\alpha}(x_0)^2}^{1} \int_{(B^*)^c} |x-x_0| \frac{|y-x_0|+x_0}{t} \frac{1}{\sqrt{t}} \exp\left(-c_0 \frac{|y-x_0|^2}{t}\right) |f(y)| t^{\frac{\sigma}{2}-1}\, dydt \\
&\lesssim \int_{\rho_{L_\alpha}(x_0)^2}^{1} \int_{(B^*)^c} |x-x_0| \frac{1}{\sqrt{t}} \frac{1}{\sqrt{t}} \exp\left(-c_1 \frac{|y-x_0|^2}{t}\right) |f(y)| t^{\frac{\sigma}{2}-1}\, dydt \\
&\lesssim \|f\|_{\frac{1}{\sigma}}.
\end{aligned}
$$

The final step is to estimate I_2. Applying (5.2) and the proof of (5.1) gives that

$$
\begin{aligned}
I_2 \lesssim & \int_{\rho_{L_\alpha}(x_0)^2}^{1} \int_{(B^*)^c} \frac{|x-x_0|}{x_0} \left(\frac{|y-x_0|x_0}{t}\right)^{\frac{1}{2}+\alpha} \\
& \frac{1}{\sqrt{t}} \exp\left(-c_0 \frac{|y-x_0|^2}{t}\right) |f(y)| t^{\frac{\sigma}{2}-1}\, dydt \\
& + \int_{\rho_{L_\alpha}(x_0)^2}^{1} \int_{(B^*)^c} \frac{|x-x_0|}{x_0} \left(\frac{x_0^2}{t}\right)^{\frac{1}{2}+\alpha} \frac{1}{\sqrt{t}} \exp\left(-c_0 \frac{|y-x_0|^2}{t}\right) |f(y)| t^{\frac{\sigma}{2}-1}\, dydt \\
\lesssim & |x-x_0| x_0^{\alpha-1/2} \int_{\rho_{L_\alpha}(x_0)^2}^{1} t^{-\frac{1}{4}-1-\frac{\alpha}{2}} dt \|f\|_{\frac{1}{\sigma}} \\
& + |x-x_0| x_0^{2\alpha} \int_{\rho_{L_\alpha}(x_0)^2}^{1} t^{-\frac{1}{2}-1-\alpha} dt \|f\|_{\frac{1}{\sigma}} \lesssim \|f\|_{\frac{1}{\sigma}}.
\end{aligned}
$$

Thus we finish the proof.

6. Proof of Theorem 4

Lemma 9. *Let* $\alpha > -1/2$, $R_\alpha(x,y)$ *in (1.8) and* $R_H(x,y)$ *in (2.8). Then:*

(1) $|R_\alpha(x,y)| \le C y^{\alpha+1/2} x^{-\alpha-3/2}, \quad 0 < y < \frac{x}{2}.$

(2) $|R_\alpha(x,y)| \le C x^{\alpha+3/2} y^{-\alpha-5/2}, \quad 2x < y.$

(3) $|R_\alpha(x,y) - R_H(x,y)| \le C\frac{1}{y}\left(1 + \frac{(xy)^{1/4}}{|x-y|^{1/2}}\right), \quad \frac{x}{2} < y < 2x.$

(4) $|R_\alpha(x,y) - R_H(x,y)| \le C\frac{1}{y}\left(1 + \frac{1}{|x-y|(xy)^{1/2}}\right), \quad \frac{x}{2} < y < 2x.$

Proof. (1), (2) and (3) are the contents of Lemma 2.13 in [1]. We only need to prove (4). For $x, y > 0$,

$$
|R_\alpha(x,y) - R_H(x,y)| \le \int_0^\infty |D_t^\alpha(x,y)| \frac{1}{\sqrt{t}}\, dt,
$$

where $D_t^\alpha(x,y) = \left(\frac{d}{dx} + x - \frac{\alpha+\frac{1}{2}}{x}\right) W_t^\alpha(x,y) - (\frac{d}{dx} + x) W_t(x,y)$.

It is proved in [1] that, for $\frac{2e^{-t}xy}{1-e^{-2t}} \le 1$,

$$\int_{\frac{2e^{-t}xy}{1-e^{-2t}}\leq 1} |D_t^\alpha(x,y)| \frac{1}{\sqrt{t}}\, dt \leq C\frac{1}{y},$$

and while for $\frac{2e^{-t}xy}{1-e^{-2t}} > 1$, we have

$$|D_t^\alpha(x,y)| \leq C\frac{x}{1-e^{-2t}}W_t(x,y),$$

and

$$|D_t^\alpha(x,y)| \leq C\frac{1}{y}W_t(x,y).$$

Set $e^{-t} = \frac{1-s}{1+s}$. The proof of Lemma 2.13 in [1] shows that

$$\int_{1/2}^1 \frac{x}{1-e^{-2t}}W_t(x,y)\, ds \leq C\frac{1}{y}.$$

It suffices to show that

$$\int_0^{1/2} \frac{1}{y}W_t(x,y)\, ds \leq \frac{1}{y}\frac{1}{|x-y|(xy)^{1/2}}.$$

Indeed, by (2.7),

$$\int_0^{1/2} \frac{1}{y}W_t(x,y)\, ds \leq C\frac{1}{y}\int_0^{1/2} s^{-1}e^{-\frac{|x-y|^2}{4s}-c_0 sxy}\, ds,$$

which implies the proof. ■

Lemma 10. ([11], Proposition 3.1). *$R_H(x,y)$ is Calderon-Zygmund kernel which satisfies*

$$|R_H(x,y)| \leq C\frac{1}{|x-y|}, \tag{6.1}$$

and, if $|x-y| \geq 2|x-x'|$,

$$|R_H(x,y) - R_H(x',y)| \leq C\frac{|x-x'|}{|x-y|^2}, \tag{6.2}$$

while, if $|x-y| \geq 2|y-y'|$

$$|R_H(x,y) - R_H(x,y')| \leq C\frac{|y-y'|}{|x-y|^2}. \tag{6.3}$$

Notice that

$$R_H(x,y) = \int_0^\infty \frac{d}{dx}W_t(x,y)\frac{dt}{\sqrt{t}} + \int_0^\infty xW_t(x,y)\frac{dt}{\sqrt{t}} = R_1^H(x,y) + R_2^H(x,y).$$

Lemma 11. ([2], (51)and(55)). *Let* $\rho_H(x) = (1+|x|)^{-1}$. *If* $|x-y| > C\rho_H(x)$, *for large enough* $N > 0$, *we have*

$$|R_2^H(x,y)| \le C_N |x| \rho_H(x)^N / |x-y|^N, \tag{6.4}$$

and

$$|R_1^H(x,y)| \le C_N \left(\frac{1}{|x-y|} + |y| \right) \rho_H(x)^N / |x-y|^N. \tag{6.5}$$

Proof of Theorem 4. In a similar way as in the proof of Theorem 3, first we shall show that, for $B_k \in \mathcal{B}$ in Lemma 1,

$$\frac{1}{|B_k|} \int_{B_k} |R_\alpha f| \, dx \le C \|f\|_{BMO_{L_\alpha}}. \tag{6.6}$$

In order to prove (6.6), we will consider two cases. In the first case of $x_k < 1$, Lemma 9, Corollary 4 and L^2-boundedness of R_α lead to

$$\begin{aligned}
&\frac{1}{|B_k|} \int_{B_k} |R_\alpha f| \, dx \\
&\le \frac{1}{|B_k|} \int_{B_k} \Big| \int_{2x_k}^{\infty} R_\alpha(x,y) f(y)\, dy \Big| \, dx + \frac{1}{|B_k|} \int_{B_k} \Big| \int_0^{\frac{x_k}{2}} R_\alpha(x,y) f(y)\, dy \Big| \, dx \\
&\quad + \frac{1}{|B_k|} \int_{B_k} \Big| \int_{\frac{x_k}{2}}^{2x_k} R_\alpha(x,y) f(y)\, dy \Big| \, dx \\
&\lesssim \frac{1}{|B_k|} \int_{B_k} \sum_{n=1}^{\infty} \int_{2^n x_k}^{2^{n+1} x_k} |f(y)| \, dy \frac{1}{(2^n x_k)^{\alpha+\frac{5}{2}}} x^{\alpha+\frac{3}{2}} \, dx + \frac{1}{x_k} \int_0^{\frac{x_k}{2}} |f(y)| \, dy \\
&\quad + \frac{1}{|B_k|^{1/2}} \left(\int_0^\infty \left| R_\alpha \mathbb{1}_{[x_k/2, 2x_k]} f(x) \right|^2 dx \right)^{1/2} \\
&\lesssim \sum_{n=1}^{\infty} \frac{1}{(2^n)^{\alpha+3/2}} \|f\|_{BMO_{L_\alpha}} + \|f\|_{BMO_{L_\alpha}} + \left(\frac{1}{|B_k|} \int_{x_k/2}^{2x_k} f(x)^2 \, dx \right)^{1/2} \\
&\lesssim \|f\|_{BMO_{L_\alpha}}.
\end{aligned}$$

In the second case of $x_k \ge 1$, as in the first case we make the same splitting for the left of (6.6), and the first two parts can be treated similarly. We only need to show

$$I = \frac{1}{|B_k|} \int_{B_k} \left| \int_{\frac{x_k}{2}}^{2x_k} R_\alpha(x,y) f(y) \, dy \right| dx \le C \|f\|_{BMO_{L_\alpha}}. \tag{6.7}$$

To do this, we make the difference between $R_\alpha(x,y)$ and $R_H(x,y)$ and use L^2-boundedness of R_α to get

$$\begin{aligned} I &\le \frac{1}{|B_k|}\int_{B_k}\left|\int_{[\frac{x_k}{2},2x_k]/B_k^*}(R_\alpha(x,y)f(y)-R_H(x,y)f(y))\,dy\right|dx \\ &\quad+\frac{1}{|B_k|}\int_{B_k}\left|\int_{[\frac{x_k}{2},2x_k]/B_k^*}R_H(x,y)f(y)\,dy\right|dx \\ &\quad+\frac{1}{|B_k|}\int_{B_k}\left|\int_{B_k^*}R_\alpha(x,y)f(y)\,dy\right|dx \\ &= I_1+I_2+\|f\|_{BMO_{L_\alpha}}. \end{aligned}$$

For I_1, using (4) of Lemma 9, we obtain

$$I_1 \lesssim \frac{1}{x_k}\int_{x_k/2}^{2x_k}|f(y)|\,dy+\frac{1}{x_k^2}\int_{[\frac{x_k}{2},2x_k]/B_k^*}\frac{1}{|y-x_k|}|f(y)|\,dy \lesssim \|f\|_{BMO_{L_\alpha}}.$$

On the other hand, noticing that $\rho_{L_\alpha}(x)\sim\rho_H(x)$ when $x\ge 1$ and $x_k\sim x$ when $x\in B_k^*$, by Lemma 11, we have

$$\begin{aligned} &I_2 \\ &\lesssim \int_{[\frac{x_k}{2},2x_k]/B_k^*}\left|x_k\rho_H(x_k)^N/|x_k-y|^N+\left(\frac{1}{|x_k-y|}+y\right)\rho_H(x_k)^N/|x_k-y|^N\right|f(y)\,dy \\ &\lesssim \sum_{j=0}^{\infty}\int_{|y-x_k|\sim 2^j\rho_{L_\alpha}(x_k)}\left(\frac{(2x_k+y-x_k)\rho_H(x_k)^N}{(2^j\rho_H(x_k))^N}+\frac{\rho_H(x_k)^N}{(2^j\rho_H(x_k))^{N+1}}\right)|f(y)|\,dy \\ &\lesssim \sum_{j=0}^{\infty}\frac{1}{2^j}\|f\|_{BMO_{L_\alpha}} \lesssim \|f\|_{BMO_{L_\alpha}}. \end{aligned}$$

Next, we turn to proving that

$$\|R_\alpha f\|_{BMO(B_k^*)} \lesssim \|f\|_{BMO_{L_\alpha}}. \tag{6.8}$$

Set $B=B_r(x_0)\subset B_k^*$ and write

$$\begin{aligned} f &= f1_{[2x_0,\infty)}+f1_{(0,x_0/2]}+f1_{(\frac{x_0}{2},2x_0)/B^*_{\rho_{L_\alpha}(x_0)}(x_0)}+f1_{B^*_{\rho_{L_\alpha}(x_0)}(x_0)} \\ &= f_1+f_2+f_3+f_4. \end{aligned}$$

From the above, whenever $x_0>1$ or $x_0\le 1$,

$$\|R_\alpha f_1\|_\infty+\|R_\alpha f_2\|_\infty \lesssim \|f\|_{BMO_{L_\alpha}}.$$

From I_1 and I_2, if $x_0\ge 1$, then

$$\|R_\alpha f_3-R_H f_3\|_\infty \lesssim \|f\|_{BMO_{L_\alpha}}, \tag{6.9}$$

and

$$\|R_H f_3\|_\infty \lesssim \|f\|_{BMO_{L_\alpha}}. \tag{6.10}$$

While, if $x_0 < 1$, using (3) of Lemma 9 and Lemma 10 we get

$$\|R_\alpha f_3 - R_H f_3\|_\infty \lesssim \frac{1}{x_0}\int_{x_0/2}^{2x_0} |f(y)|\, dy \lesssim \|f\|_{BMO_{L_\alpha}}, \tag{6.11}$$

and

$$\|R_H f_3\|_\infty \lesssim \frac{1}{x_0}\int_{x_0/2}^{2x_0} |f(y)|\, dy \lesssim \|f\|_{BMO_{L_\alpha}}. \tag{6.12}$$

Taking into account (6.9),(6.10),(6.11) and (6.12), we get $\|R_\alpha f_3\|_\infty \lesssim \|f\|_{BMO_{L_\alpha}}$.

It remains to show that there exists a constant C_B such that

$$\frac{1}{|B|}\int_B |R_\alpha f_4(x) - C_B|\ dx \lesssim \|f\|_{BMO_{L_\alpha}}. \tag{6.13}$$

The left side of (6.13) is bounded by

$$\frac{1}{|B|}\int_B |R_\alpha f_4(x) - R_H f_4(x)|\ dx + \frac{1}{|B|}\int_B |R_H f_4(x) - C_B|\ dx.$$

Let $x \in B$ and $B_{x,k} = B_{2^{2-k}\rho_{L_\alpha}(x_0)}(x)$, $k = 0, 1 \ldots$. It is clear that

$$\int_{B_{x,k}} |f(x)|\, dx \le C|B_{x,k}|(k+1)\|f\|_{BMO_{L_\alpha}},$$

(see [10], P.141). Using (3) in Lemma 9, we get

$$\begin{aligned}
|R_\alpha f_4(x) - R_H f_4(x)| &\lesssim \int_{B^*_{\rho_{L_\alpha}(x_0)}(x_0)} |R_\alpha(x,y) - R_H(x,y)|\,|f(y)|\, dy \\
&\lesssim \sum_{k=0}^{\infty} \int_{B_{x,k}/B_{x,k+1}} |R_\alpha(x,y) - R_H(x,y)|\,|f(y)|\, dy \\
&\lesssim \sum_{k=0}^{\infty} \left(2^{-k}\right)^{1/2} \frac{1}{2^{-k}\rho_{L_\alpha}(x_0)} \int_{B_{x,k}} |f(y)|\, dy \\
&\lesssim \sum_{k=0}^{\infty} \left(2^{-k}\right)^{1/2} (k+1)\|f\|_{BMO_{L_\alpha}} \\
&\lesssim \|f\|_{BMO_{L_\alpha}}.
\end{aligned}$$

It remains to show that

$$\frac{1}{|B|}\int_B |R_H f_4(x) - C_B|\ dx \lesssim \|f\|_{BMO_{L_\alpha}}.$$

Let $B_k^\sharp = B_{2^{1-k}\rho_{L_\alpha}(x_0)}(x_0)$, $k = 0, 1, \ldots, k_0$, where k_0 satisfies $2^{-k_0-1}\rho_{L_\alpha}(x_0) \leq r < 2^{-k_0}\rho_{L_\alpha}(x_0)$. Set

$$\begin{aligned} f_4 - f_{B_{k_0}^\sharp} &= (f_4 - f_{B_{k_0}^\sharp})\mathbb{1}_{B_{k_0}^\sharp} + (f_4 - f_{B_{k_0}^\sharp})\mathbb{1}_{B_0^\sharp / B_{k_0}^\sharp} - f_{B_{k_0}^\sharp}\mathbb{1}_{(B_0^\sharp)^c} \\ &= f_{41} + f_{42} + f_{43}. \end{aligned}$$

Since R_H is bounded on $L^2(\mathbb{R})$, by Corollary 4, we have

$$\begin{aligned} \frac{1}{|B|}\int_B |R_H f_{41}(x)|\, dx &\lesssim \left(\frac{1}{|B|}\int_B |R_H f_{41}(x)|^2\, dx\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{|B_{k_0}^\sharp|}\int_{B_{k_0}^\sharp} |f(x) - f_{B_{k_0}^\sharp}|^2\, dx\right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{BMO_{L_\alpha}}. \end{aligned}$$

We pass to treat f_{42}. Using (6.2) and the basic fact $|f_{B_k^\sharp} - f_{B_{k_0}^\sharp}| \leq C|k_0 - k|\|f\|_{BMO_{L_\alpha}}$, for $x \in B$, we have

$$\begin{aligned} |R_H f_{42}(x) - R_H f_{42}(x_0)| &\lesssim \sum_{k=0}^{k_0-1} \int_{B_k^\sharp / B_{k+1}^\sharp} |R_H(x,y) - R_H(x_0,y)| \left|f_4(y) - f_{B_{k_0}^\sharp}\right| dy \\ &\lesssim \sum_{k=0}^{k_0-1} \frac{2^{k-k_0}}{|B_k^\sharp|}\int_{B_k^\sharp}\left(\left|f(y) - f_{B_k^\sharp}\right| + |f_{B_k^\sharp} - f_{B_{k_0}^\sharp}|\right) dy \\ &\lesssim \sum_{k=0}^{k_0-1} (k_0 - k + 1)2^{k-k_0}\|f\|_{BMO_{L_\alpha}} \\ &\lesssim \|f\|_{BMO_{L_\alpha}}. \end{aligned}$$

Finally, for the third term, using Lemma 10, we get

$$\begin{aligned} |R_H f_{43}(x) - R_H f_{43}(x_0)| &\lesssim |f_{B_{k_0}^\sharp}| \int_{(B_0^\sharp)^c} |R_H(x,y) - R_H(x_0,y)|\, dy \\ &\lesssim (k_0 + 1)2^{-k_0}\|f\|_{BMO_{L_\alpha}} \\ &\lesssim \|f\|_{BMO_{L_\alpha}}. \end{aligned}$$

The proof of Theorem 4 is completed.

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