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RIESZ TRANSFORMS ON Q-TYPE SPACES WITH APPLICATION TO QUASI-GEOSTROPHIC EQUATION

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Abstract. By an equivalent characterization of Morrey space associated with the fractional heat semigroup, we establish a relation between the generalized $Q-$type spaces and Morrey spaces. By this relation, in this paper, we prove the boundedness of the singular integral operatoes on the Q-type spaces $Q^{\beta}_{\alpha}(\mathbb{R}^n)$. As an application, we get the well-posedness and regularity of the quasi-geostrophic equation with initial data in $Q^{\beta,-1}_{\alpha}(\mathbb{R}^n)$.

1. Introduction

In this paper, we consider the boundedness of a class of singular integral operators on the $Q-$type space $Q^{\beta}_{\alpha}(\mathbb{R}^n)$. Here $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is a space defined as the set of all measurable functions with

$$\sup_I (l(I))^{2\alpha-n+2\beta-2}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha-2\beta+2}}dxdy<\infty,$$

where $\alpha\in(0,1),\ \beta\in(1/2,1)$, the supremum is taken over all cubes I with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. This space is introduced in [18] to study the well-posedness of the generalized Naiver-Stokes equations. For $\beta=1$, $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ coincides with the classical space $Q_\alpha(\mathbb{R}^n)$ which is introduced in [13]. Furthermore, if $\alpha=0$, $\beta=1$, $Q^{\beta}_{\alpha}(\mathbb{R}^n)=BMO(\mathbb{R}^n)$.

As a new space between $W^{1,n}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, $Q_\alpha(\mathbb{R}^n)$ has been studied extensively by many authors since 1990s. In 1995, on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, R. Aulaskari, J. Xiao and R. Zhao first introduced a class of Mӧbius invariant analytic function spaces, $Q_p(\mathbb{D}),\ p\in(0,1)$. The class $Q_p(\mathbb{D}),\ p\in(0,1)$ can be

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seen as subspaces and subsets of $BMOA$ and UBC on $\mathbb{D}$. Since then, many studies on $Q_p(\mathbb{D})$ and their characterization have been done. We refer the readers to [1], [2], [21] and [29] and the reference therein. In order to generalize $Q_p(\mathbb{D}),\ p \in (0,1)$ to $\mathbb{R}^n$, in [13], M. Essen, S. Janson, L. Peng and J. Xiao introduced a class of Q-type spaces of several real variables, $Q_\alpha(\mathbb{R}^n), \alpha \in (0,1)$. Later, in [12], G. Dafni and J. Xiao established the Carleson measure characterization of $Q_\alpha(\mathbb{R}^n),\ \alpha \in (0,1)$. For more information of the spaces $Q_\alpha(\mathbb{R}^n)$ and their application, we refer to [28], [12] and [13]. For the generalization of $Q_\alpha(\mathbb{R}^n)$, we refer to [18] and [30].

It is easy to see that a function $f(x)$ belongs to $BMO(\mathbb{R}^n)$ if and only if

$$\sup_I (l(I))^{-2n} \int_I \int_I |f(x) - f(y)|^2\, dxdy < \infty.$$

It can be also proved that if $\alpha \in (-\infty, 0)$ and $\beta = 1$, $Q^\beta_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$. The similarity on the structure of $Q^\beta_\alpha(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ shows that the two spaces share some common properties. It is well-known that the singular integral operators T are bounded on the Hardy space $H^1(\mathbb{R}^n)$. By the duality, the boundedness of T on $BMO(\mathbb{R}^n)$ is obvious. Owing to the relation between $Q^\beta_\alpha(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, it is natural to consider the boundedness of T on $Q^\beta_\alpha(\mathbb{R}^n)$.

Unlike the case of Hardy space $H^1(\mathbb{R}^n)$, the boundedness of T on the dual space of $Q^\beta_\alpha(\mathbb{R}^n)$ is not clear. So we cannot follow the former method to get the boundedness of T on $Q^\beta_\alpha(\mathbb{R}^n)$. Alternatively, we apply an equivalent characterization of $Q^\beta_\alpha(\mathbb{R}^n)$ associated to the fractional heat semigroup $e^{-t(-\Delta)^\beta}$ and establish a relation between $Q^\beta_\alpha(\mathbb{R}^n)$ and some Morrey spaces $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. For $\beta = 1$ and $\alpha \in (0,1)$, such relation was established by Z. Wu and C. Xie in [27]. In [28], J. Xiao gave another proof which is based on the Carleson measure characterization of $Q_\alpha, \alpha \in (0,1)$ and Morrey spaces. Hence our result can be seen as a generalization of those in [27] and [28]. By this relation, the boundedness of T on $Q^\beta_\alpha(\mathbb{R}^n)$ can be deduced by that on $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. See Section 3.

As an application, we consider the well-posedness and regularity of the quasi-geostrophic equations with initial data in $Q^{\beta,-1}_\alpha(\mathbb{R}^n)$. In recent years, Q-type spaces have been applied to the study of the fluid equations by several authors. For example, in [28], J. Xiao introduced a new critical space $Q^{-1}_\alpha(\mathbb{R}^n)$ which is derivatives of $Q_\alpha(\mathbb{R}^n),\ \alpha \in (0,1)$ and got the well-posedness of Naiver-Stokes equations with initial data in $Q^{-1}_\alpha(\mathbb{R}^n)$. When $\alpha = 0$, $Q^{-1}_\alpha(\mathbb{R}^n) = BMO^{-1}(\mathbb{R}^n)$, his result generalized the well-posedness obtained by Koch and Tataru in [17]. In [18], inspiring by [28] and the scaling invariance, we introduced a new Q-type space $Q^\beta_\alpha(\mathbb{R}^n)$ with $\alpha > 0$, $\max\{\frac{1}{2}, \alpha\} < \beta < 1$ such that $\alpha + \beta - 1 \geq 0$. We proved the well-posedness and regularity of the generalized Naiver-Stokes equations with some initial data in the space $Q^{\beta,\ -1}_\alpha(\mathbb{R}^n)$. For $\beta = 1$, our space $Q^{\beta,\ -1}_\alpha(\mathbb{R}^n)$ becomes $Q^{-1}_\alpha(\mathbb{R}^n)$ in [28]. So our result can be regarded as a generalization of those of [17] and [28].

In Section 4, we consider the two-dimensional subcritical quasi-geostrophic dissipative equations $(DQG)_\beta$ with small initial data in $Q_\alpha^{\beta,-1}(\mathbb{R}^n)$,

$$\begin{cases} \partial_t\theta + (-\triangle)^\beta u + (u\cdot\nabla)\theta = 0 & \text{in } \mathbb{R}^2\times\mathbb{R}_+, \alpha>0; \\ u = \nabla^\perp(-\Delta)^{-1/2}\theta; & \\ \theta(0,x) = \theta_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\beta \in (\frac{1}{2}, 1)$, the scalar θ represent the potential temperature, and u is the fluid velocity.

The equations $(DQG)_\beta$ are important models in the atmosphere and ocean fluid dynamics. It was proposed by P. Constantin and A. Majda, etc that the equations $(DQG)_\beta$ can be regarded as low dimensional model equations for mathematical study of singularity in smooth solutions of unforced incompressible three dimensional fluid equations. See e.g. [10, 14, 15, 22, 23] and the references therein.

Owing to the importance in mathematical and geophysical fluid dynamics mentioned above, the equations $(DQG)_\beta$ have been intensively studied. Some important progress has been made. We refer the readers to [4, 5, 6, 7, 8, 11, 16, 25, 26] etc. for details.

In [19], F. Marchand and P. G. Lemarié-Rieusset get the well-posedness of the solutions to the equation $(DQG)_1$ with the initial data in $BMO^{-1}(\mathbb{R}^2)$. However, because the space $BMO^{-1}(\mathbb{R}^2)$ is invariant under the scaling: $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$, we see that under the fractional scaling associated to $0<\beta<1$,

$$\theta_\lambda(t,x) = \lambda^{2\beta-1}\theta(\lambda^{2\beta}t, \lambda x) \text{ and } \theta_{0,\lambda}(x) = \lambda^{2\beta-1}\theta_0(\lambda x), \tag{1.2}$$

the space BMO^{-1} is not invariant.

The above observation implies that if we want to generalize the result in [19] to the general case $\beta<1$, we should choose a new space X^β which satisfies the following two properties. At first, the space X^β should be invariant under the scaling (1.2). Secondly, BMO^{-1} is a "special" case of X^β for $\beta = 1$.

It is proved in [18] that the space $Q_\alpha^{\beta,\,-1}(\mathbb{R}^n)$ is exactly such a space. Therefore we could apply the approach in [18] to the equations $(DQG)_\beta$ and get the well-posedness and regularity of the solution to the equations $(DQG)_\beta$ with $\beta > 1/2$.

It should be pointed out that the scope of β in the equations $(DQG)_\beta$ is depended upon the definition of $Q_\alpha^\beta(\mathbb{R}^n)$. In [18], we proved that the parameters $\{\alpha,\ \beta\}$ should satisfy the condition: $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ and $\alpha<\beta$ with $\alpha+\beta-1\geq 0$. It is easy to see that $\beta > \frac{1}{2}$.

In [24], the authors proved the global existence of the solutions of the subcritical quasi-geostrophic equations with small size initial data in the Besov norms paces $\dot{B}_\infty^{1-2\beta,\infty}$. However our result cannot be deduced by the existence result in [24]. In addition, by the method in [18], we consider the regularity of the solutions to the equations $(DQG)_\beta$.

The organization of this paper is as follows. In Section 2 we state some preliminary knowledge, notation and terminology that will be used throughout this paper. In Section 3 we consider the boundedness of a class of singular integral operators on $Q_\alpha^\beta(\mathbb{R}^n)$. In Section 4 we give a well-posedness of the equations $(DQG)_\beta$ with the initial data in the spaces $Q_\alpha^{\beta,\,-1}(\mathbb{R}^n)$.

2. Preliminaries

In this paper the symbols $\mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ denote the sets of all complex numbers, integers and natural numbers, respectively. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the $n-$dimensional Euclidean space, with Euclidean norm denoted by $|x|$ and the Lebesgue measure denoted by dx. $\mathbb{R}_+^{n+1}$ is the upper half-space $\left\{(t,x) \in \mathbb{R}_+^{n+1} : t > 0, x \in \mathbb{R}^n\right\}$ with Lebesgue measure denoted by $dtdx$.

A ball in $\mathbb{R}^n$ with center x and radius r will be denoted by $B = B(x,r)$; its Lebesgue measure is denoted by $|B|$. A cube in $\mathbb{R}^n$ will always mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. The sidelength of a cube I will be denoted by $l(I)$. Similarly, its volume will be denoted by $|I|$.

The symbol $U \lesssim V$ means that there exists a positive constant C such that $U \le CV$. $U \approx V$ means $U \lesssim V$ and $V \lesssim U$. For convenience, the positive constants C may change from one line to another and usually depend on the dimension n, α, β and other fixed parameters.

The characteristic function of a set A will be denoted by 1_A. For $\Omega \subset \mathbb{R}^n$, the space $C_0^\infty(\Omega)$ consists of all smooth functions with compact support in Ω. The Schwartz class of rapidly decreasing functions and its dual will be denoted by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. For a function $f \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{f}$ means the Fourier transform of f.

The generalized $Q-$type spaces $Q_\alpha^\beta(\mathbb{R}^n)$ are introduce as a substitute of the classical $Q_\alpha(\mathbb{R}^n)$ under the fractional dilation: $f_\lambda(x) = \lambda^{2\beta-1} f(\lambda x),\ 0 < \beta < 1$. This space is defined as follows.

Definition 2.1. Let $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then $f \in Q_\alpha^\beta(\mathbb{R}^n)$ if and only if

$$\sup_I (l(I))^{2\alpha-n+2\beta-2} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha-2\beta+2}} dxdy < \infty,$$

where the supremum is taken over all cubes I with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

For $\beta = 1$ and $\alpha > -\infty$, the above space becomes $Q_\alpha(\mathbb{R}^n)$, which was introduced by M. Essen, S. Janson, L. Peng and J. Xiao in [13]. In 2004, G. Dafni and J. Xiao give the Carleson measure characterization of $Q_\alpha^\beta(\mathbb{R}^n)$ using a new type of tent spaces in [12]. Following the same idea, in order to study the Q_α initial data problem for the generalized Naiver-Stokes equations, we consider the Carleson measure characterization of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ in [18]. Precisely, we get the following result.

Let $\phi(x)$ be a C^{∞} real-valued function on $\mathbb{R}^n$ satisfying the properties

$$(2.1)\ \ \phi(x)\in L^1(\mathbb{R}^n),\ |\phi(x)|\lesssim(1+|x|)^{-(n+1)},\ \int_{\mathbb{R}^n}\phi(x)dx=0,\ \phi_t(x)=t^{-n}\phi(\frac{x}{t}).$$

In [18], we proved that $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ has the following Carleson measure characterization.

Theorem 2.2. ([18, p. 2462]). *Given ϕ be a function satisfying the above conditions (2.1). Let $\alpha>0$ and $\max\{\alpha,1/2\}<\beta<1$ with $\alpha+\beta-1\geq 0$. $f\in Q^{\beta}_{\alpha}(\mathbb{R}^n)$ if and only if*

$$\sup_{x\in\mathbb{R}^n,r\in(0,\infty)} r^{2\alpha-n+2\beta-2}\int_0^r\int_{|y-x|<r}|f*\phi_t(y)|^2t^{-(1+2(\alpha-\beta+1))}dtdy<\infty,$$

that is, $d\mu_{f,\phi,\alpha,\beta}(t,x)=|(f*\phi_t)(x)|^2t^{-1-2(\alpha-\beta+1)}dtdx$ is a $1-2(\alpha+\beta-1)/n-$ Carleson measure.

The main tool for the Carleson measure characterization of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is the following fractional tent spaces.

Definition 2.3. For $\alpha>0$ and $\max\{\alpha,1/2\}<\beta<1$ with $\alpha+\beta-1\geq 0$, we define $T^{\infty}_{\alpha,\beta}$ be the class of all Lebesgue measurable functions f on $\mathbb{R}^{n+1}_+$ with

$$\|f\|_{T^{\infty}_{\alpha,\beta}}=\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|^{1-2(\alpha+\beta-1)/n}}\int_{T(B)}|f(t,y)|^2\frac{dtdy}{t^{1+2(\alpha-\beta+1)}}\right)^{1/2}<\infty.$$

In order to define the dual of $T^{\infty}_{\alpha,\beta}$, we need the following $T^1_{\alpha,\beta}-$atoms.

Definition 2.4. For $\alpha>0$ and $\max\{\alpha,1/2\}<\beta<1$ with $\alpha+\beta-1\geq 0$, a function a on $\mathbb{R}^{n+1}_+$ is said to be a $T^1_{\alpha,\beta}-$atom provided there exists a ball $B\subset\mathbb{R}^n$ such that a is supported in the tent $T(B)$ and satisfies

$$\int_{T(B)}|a(t,y)|^2\frac{dtdy}{t^{1-2(\alpha-\beta+1)}}\leq\frac{1}{|B|^{1-2(\alpha+\beta-1)/n}}.$$

We denote by $d\Lambda^{\infty}_{n-2(\alpha+\beta-1)}$ the $n-2(\alpha+\beta-1)$ dimensional Hausdorff capacity of a set E and refer to [12] for the details of the Hausdorff capacity. For $x\in\mathbb{R}^n$, let $\Gamma(x)=\left\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<t\right\}$ be the cone at x. Define the non-tangential maximal function $N(f)$ of a measurable function f on $\mathbb{R}^{n+1}_+$ by

$$N(f)(x):=\sup_{(y,t)\in\Gamma(x)}|f(y,t)|.$$

The dual of $T^{\infty}_{\alpha,\beta}$ is defined as follows.

Definition 2.5. For $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, the space $T^1_{\alpha,\beta}$ consists of all measurable functions f on $\mathbb{R}^{n+1}_+$ with

$$\|f\|_{T^1_{\alpha,\beta}} = \inf_{\omega} \left(\int_{\mathbb{R}^{n+1}_+} |f(x,t)|^2 \omega^{-1}(x,t) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} < \infty,$$

where the infimum is taken over all nonnegative Borel measurable functions ω on $\mathbb{R}^{n+1}_+$ with

$$\int_{\mathbb{R}^n} N\omega d\Lambda^{\infty}_{n-2(\alpha+\beta-1)} \leq 1$$

and with the restriction that ω is allowed to vanish only where f vanishes.

The above tent spaces and their dualities can be seen as the generalization of the usual one. For $\beta = 1$, $T^{\infty}_{\alpha,\beta}$ and $T^1_{\alpha,\beta}$ coincide with T^{∞}_{α} and T^1_{α}, respectively which are introduced in [12]. For $\alpha = 0$ and $\beta = 1$, $T^{\infty}_{\alpha,\beta}$ becomes the classical tent space T^{∞} in [9].

Let ϕ satisfy the conditions (2.1). For a function F on $\mathbb{R}^{n+1}_+$, denote by Π_ϕ the operator

$$\Pi_\phi(F) = \int_0^{\infty} F(\cdot, t) * \phi_t \frac{dt}{t}. \tag{2.2}$$

In [18], we proved that Π_ϕ is a bounded and surjective operator from $T^{\infty}_{\alpha,\beta}$ to Q^{β}_{α}.

Theorem 2.6. ([18, Theorem 3.20]). *Consider the operator Π_ϕ defined by (2.2). The operator Π_ϕ is a bounded and surjective operator from $T^{\infty}_{\alpha,\beta}$ to $Q^{\beta}_{\alpha}(\mathbb{R}^n)$. More precisely, if $F \in T^{\infty}_{\alpha,\beta}$ then the righthand side of the above integral converges to a function $f \in Q^{\beta}_{\alpha}(\mathbb{R}^n)$, $\|f\|_{Q^{\beta}_{\alpha}} \lesssim \|F\|_{T^{\infty}_{\alpha,\beta}}$, and any $f \in Q^{\beta}_{\alpha}(\mathbb{R}^n)$ can be thus represented.*

3. Boundedness of the Singular Integral Operatorson Q-Type spaces Q^{β}_{α}

In this section, we will prove a class of singular integral operators are bounded on Q-type spaces $Q^{\beta}_{\alpha}(\mathbb{R}^n)$. Our method is based on the characterizations of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ and the Morrey space $\mathcal{L}_{2,\lambda}$ associated to the fractional heat semigroup $e^{-t(-\Delta)^{\beta}}$. Before we state the main results in this section, we give a relation between $Q^{\beta}_{\alpha}(\mathbb{R}^n)$, a class of conformally invariant Sobolev spaces and the fractional BMO type space $BMO^{\beta}(\mathbb{R}^n)$.

Definition 3.1. Let $\beta \in (1/2, 1)$. Then $f \in BMO^{\beta}(\mathbb{R}^n)$ if and only if

$$\sup_I \left((l(I))^{4\beta-4-2n} \int_I \int_I |f(x) - f(y)|^2 dxdy \right)^{1/2} < \infty,$$

where the supremum is taken over all cubes I with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

In [28], J.Xiao proved that $Q_\alpha(\mathbb{R}^n)$ is a space between the Sobolev space $W^{1,n}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. In this section we prove that a similar relation holds for $Q_\alpha^\beta(\mathbb{R}^n)$ and $BMO^\beta(\mathbb{R}^n)$. For this purpose, we introduce a conformally invariant Sobolev space $CIS_\beta(\mathbb{R}^n)$.

Definition 3.2. Let $\beta \in (1/2, 1)$ and $f \in C^1(\mathbb{R}^n)$. $f \in CIS_\beta(\mathbb{R}^n)$ if

$$\|f\|_{CIS_\beta} = \sup_I \left(|I|^{\frac{4\beta-2-n}{n}} \int_I |\nabla f(x)|^2 dx \right)^{1/2} < \infty,$$

where the supremum is taken over all cubes I with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

Theorem 3.3. *Let* $n \geq 2$ *and* $\max\{\alpha.1/2\} < \beta < 1$ *with* $\alpha + \beta - 1 \geq 0$. *If*

$$E_\beta(\mathbb{R}^n) = \left\{ f \in C^1(\mathbb{R}^n) : \ \|f\|_{E_\beta} = \left(\int_{\mathbb{R}^n} |\nabla f(x)|^{\frac{n}{2\beta-1}} dx \right)^{\frac{2\beta-1}{n}} \right\},$$

then

$$E_\beta(\mathbb{R}^n) \subseteq CIS_\beta(\mathbb{R}^n) \subseteq Q_\alpha^\beta(\mathbb{R}^n) \subseteq BMO^\beta(\mathbb{R}^n).$$

Proof. If $n \geq 2$, by Hölder's inequality, we have for any cube $I \subset \mathbb{R}^n$,

$$\int_I |\nabla f(x)|^2 dx \leq \left(\int_I |\nabla f(x)|^{\frac{n}{2\beta-1}} dx \right)^{\frac{4\beta-2}{n}} |I|^{\frac{4\beta-n-2}{n}}.$$

This implies $E_\beta(\mathbb{R}^n) \subseteq CIS_\beta(\mathbb{R}^n)$.

Now we prove $CIS_\beta(\mathbb{R}^n) \subseteq Q_\alpha^\beta(\mathbb{R}^n)$. For a cube $I \subset \mathbb{R}^n$, denote by cI the cube with volume being $c^n|I|$ and the center of I. For $f \in CIS_\beta(\mathbb{R}^n)$, we have

$$|f(z+y) - f(y)| \leq \int_0^1 |\nabla f(y+tz)|\,|z| dt.$$

Hence we can get

$$\begin{aligned}
&\left(\int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha-2\beta+2}} dxdy \right)^{1/2} \\
&= \left(\int_I \int_I \left(\frac{|f(x)-f(y)|}{|x-y|} \right)^2 \frac{1}{|x-y|^{n+2\alpha-2\beta}} dxdy \right)^{1/2} \\
&\leq \left(\int_I \int_{|x-y|<\sqrt{n}|I|^{1/n}} \left(\frac{|f(x)-f(y)|}{|x-y|} \right)^2 |x-y|^{2\beta-n-2\alpha} dxdy \right)^{1/2} \\
&\leq \left(\int_I \int_{|z|<\sqrt{n}|I|^{1/n}} \left(\frac{|f(z+y)-f(y)|}{|z|} \right)^2 |z|^{2\beta-n-2\alpha} dzdy \right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_I\int_{|z|<\sqrt{n}|I|^{1/n}}\left(\int_0^1 |\nabla f(y+tz)|dt\right)^2 |z|^{2\beta-n-2\alpha}dzdy\right)^{1/2}\\
&\leq \int_0^1\left(\int_I\int_{|z|<\sqrt{n}|I|^{1/n}}|\nabla f(y+tz)|^2|z|^{2\beta-n-2\alpha}dzdy\right)^{1/2}dt\\
&\leq \int_0^1\left(\int_{(1+\sqrt{n})I}\int_{|z|<\sqrt{n}|I|^{1/n}}|\nabla f(\omega)|^2|z|^{2\beta-n-2\alpha}dzd\omega\right)^{1/2}dt.
\end{aligned}
$$

Because

$$
\int_{|z|<\sqrt{n}|I|^{1/n}}|z|^{2\beta-2\alpha-n}dz \leq \int_{|z|<\sqrt{n}|I|^{1/n}}|z|^{2\beta-2\alpha-1}d|z| \leq C|I|^{\frac{2\beta-2\alpha}{n}},
$$

we have

$$
\begin{aligned}
\left(\int_I\int_I\frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha-2\beta+2}}dxdy\right)^{1/2} &\leq C\int_0^1\left[\int_{(1+\sqrt{n})I}|\nabla f(\omega)|^2|I|^{\frac{2\beta-2\alpha}{n}}d\omega\right]^{1/2}dt\\
&= C|I|^{\frac{\beta-\alpha}{n}}\left(\int_{(1+\sqrt{n})I}|\nabla f(\omega)|^2d\omega\right)^{1/2}.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
&\left(|I|^{\frac{2\alpha-n+2\beta-2}{n}}\int_I\int_I\frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha-2\beta+2}}dxdy\right)^{1/2}\\
&\leq |I|^{\frac{2\alpha-n+2\beta-2}{2n}}|I|^{\frac{\beta-\alpha}{n}}\left(\int_{(1+\sqrt{n})I}|\nabla f(\omega)|^2d\omega\right)^{1/2}\\
&\leq |I|^{\frac{4\beta-n-2}{2n}}\left(\int_{(1+\sqrt{n})I}|\nabla f(\omega)|^2d\omega\right)^{1/2}.
\end{aligned}
$$

By Definition 2.1, we know that $CIS_\beta(\mathbb{R}^n) \subseteq Q_\alpha^\beta(\mathbb{R}^n)$. This completes the proof of Theorem 3.3. ■

Recall that Morrey space $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ is defined as follows.

$$
\|f\|_{\mathcal{L}_{p,\lambda}} = \sup_I\left((l(I))^{-\lambda}\int_I|f(x)-f_I|^pdx\right)^{1/p} < \infty. \tag{3.1}
$$

We see that if $\lambda = n$, $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ by John-Nirenberg inequality. Owing to $BMO(\mathbb{R}^n)$ is a special case of $Q_\alpha(\mathbb{R}^n)$, it is natural to ask if there exists a general relation between $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $Q_\alpha(\mathbb{R}^n)$. In [28], by a characterization of $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$

associated to the semigroup $e^{-t(-\Delta)}$, J. Xiao established such a relation. Precisely he proved that for $\alpha \in (0,1)$, $Q_\alpha(\mathbb{R}^n) = (-\Delta)^{-\frac{\alpha}{2}}\mathcal{L}_{2,n-2\alpha}(\mathbb{R}^n)$.

Following Xiao's idea in [28], we will prove that a similar result holds for the space $Q_\alpha^\beta(\mathbb{R}^n)$. At first we prove an equivalent characterization of $\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)$ via the semigroup $e^{-t(-\Delta)^\beta}$. Here $e^{-t(-\Delta)^\beta}$ denotes the convolution operator defined by Fourier transform:

$$\widehat{e^{-t(-\Delta)^\beta} f}(\xi) = e^{-t|\xi|^{2\beta}}\widehat{f}(\xi).$$

Lemma 3.4. *Given* $\gamma \in (0,1)$. *Let f be a measurable complex-valued function on* $\mathbb{R}^n$. *Then* $f \in \mathcal{L}_{2,n-\gamma}(\mathbb{R}^n)$ *if and only if*

$$\sup_{x\in\mathbb{R}^n, r\in(0,\infty)} r^{2\gamma-n}\int_0^r\int_{|y-x|<r}\left|\nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y)\right|^2 tdydt < \infty.$$

Proof. Take $(\psi_0)_t(x) = t\nabla e^{-t^{2\beta}(-\Delta)^\beta}(x,0)$ with the Fourier symbol $\widehat{(\psi_0)_t(x)}(\xi) = t|\xi|e^{-t^{2\beta}|\xi|^{2\beta}}$. For a ball $B = \{y \in \mathbb{R}^n : |y-x| < r\}$, the mean of f on $2B$ is defined by $f_{2B} = \frac{1}{|2B|}\int_{2B} f(x)dx$. We split f into $f = f_1 + f_2 + f_3$, where $f_1 = (f - f_{2B})\chi_{2B}$, $f_2 = (f - f_{2B})\chi_{(2B)^c}$ and $f_3 = f_{2B}$. Because

$$\int (\psi_0)_t(x)dx = \int t\nabla e^{-t^{2\beta}(-\Delta)^\beta}(x,0)dx = 0,$$

we have

$$t\nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y) = (\psi_0)_t * f(y) = (\psi_0)_t * f_1(y) + (\psi_0)_t * f_2(y).$$

It is easy to see that

$$\begin{aligned}\int_0^r\int_B |(\psi_0)_t * f_1(y)|^2\frac{dydt}{t} &\lesssim \int_0^r\int_{\mathbb{R}^n} |(\psi_0)_t * f_1(y)|^2\frac{dydt}{t}\\ &= \left\|\left(\int_0^\infty |(\psi_0)_t * f_1(\cdot)|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^2(dy)}.\end{aligned}$$

Because $(\psi_0)_1 = \nabla e^{-(-\Delta)^\beta}$, we have $\int(\psi_0)_1(x)dx = 1$ and $(\psi_0)_1$ belongs to the Schwartz class $\mathcal{S}$. Also the function

$$G(f) = \left(\int_0^\infty |(\psi_0)_t * f_1(y)|^2\frac{dt}{t}\right)^{1/2}$$

is a Littlewood-Paley g-function. So we can get

$$\begin{aligned}\int_0^r\int_B |(\psi_0)_t * f_1(y)|^2\frac{dydt}{t} &\lesssim \int_{2B}|f(y) - f_{2B}|^2 dy\\ &\lesssim r^{n-2\gamma}\|f\|^2_{\mathcal{L}_{2,n-2\gamma}}.\end{aligned}$$

Now we estimate the term associated with $f_2(y)$. Because

$$\begin{aligned}
|(\psi_0)_t * f_2(y)| &= \left|\int_{\mathbb{R}^n} t\nabla e^{-t^{2\beta}(-\Delta)^\beta}(y-z)f_2(z)dz\right| \\
&\lesssim \int_{\mathbb{R}^n\backslash 2B} \left|t\nabla e^{-t^{2\beta}(-\Delta)^\beta}(y-z)\right| |f(z)-f_{2B}|\,dz \\
&\lesssim \int_{\mathbb{R}^n\backslash 2B} \frac{t|f(z)-f_{2B}|}{t^{n+1}(1+t^{-1}|z-y|)^{n+1}}dz,
\end{aligned}$$

where in the last inequality we have used the following estimate:

$$\left|\nabla e^{-t(-\Delta)^\beta}(x,y)\right| \lesssim \frac{1}{t^{\frac{n+1}{2\beta}}}\frac{1}{(1+t^{-\frac{1}{2\beta}}|x-y|)^{n+1}}.$$

Set $B_k = B(x, 2^k)$. For every $(t,y)\in(0,r)\times B(x,r)$, we have $0<t<r$ and $|x-y|<r$. If $z\in B_{k+1}\backslash B_k$, we have $|x-y|<|x-z|/2$ and

$$\begin{aligned}
|(\psi_0)_t * f_2(y)| &\lesssim \int_{\mathbb{R}^n\backslash 2B} \frac{t|f(z)-f_{2B}|}{(t+|z-x|)^{n+1}}dz \\
&\lesssim t\sum_{k=1}^{\infty}\frac{(2^{k+1}r)^n}{(2^kr)^{n+1}}\left(\frac{1}{(2^{k+1}r)^n}\int_{2^{k+1}B}|f(z)-f_{2B}|^2dz\right)^{1/2} \\
&\lesssim t\left[\sum_{k=1}^{\infty}\frac{1}{2^kr}\left(\frac{1}{(2^{k+1}r)^n}\int_{2^{k+1}B}|f(z)-f_{2^{k+1}B}|^2dz\right)^{1/2}\right. \\
&\qquad \left. +\sum_{k=1}^{\infty}\frac{1}{2^kr}\left|f_{2^{k+1}B}-f_{2B}\right|\right] \\
&=: t(S_1+S_2).
\end{aligned}$$

For S_1, we have

$$\begin{aligned}
S_1 &= t\sum_{k=1}^{\infty}\frac{1}{2^kr}\left(\frac{(2^{k+1}r)^{n-2\gamma}}{(2^{k+1}r)^n}\frac{1}{(2^{k+1}r)^{n-2\gamma}}\int_{2^{k+1}B}|f(z)-f_{2^{k+1}B}|^2dz\right)^{1/2} \\
&\lesssim t\sum_{k=1}^{\infty}\frac{1}{2^kr}r^{-\gamma}\|f\|_{\mathcal{L}_{2,n-2\gamma}} \\
&\lesssim tr^{-1-\gamma}\|f\|_{\mathcal{L}_{2,n-2\gamma}}.
\end{aligned}$$

For S_2, we have

$$S_2 \lesssim t\sum_{k=1}^{\infty}\frac{1}{2^kr}\Big[|f_{2B}-f_{4B}|+\cdots+|f_{2^kB}-f_{2^{k+1}B}|\Big].$$

For any j with $2 \leq j \leq k$, it is easy to see that

$$\begin{aligned} |f_{2^jB} - f_{2^{j+1}B}| &\lesssim \frac{1}{|2^jB|}\int_{2^jB}|f(z)-f_{2^{j+1}B}|dz \\ &\lesssim \left(\frac{1}{|2^jB|}\int_{2^jB}|f(z)-f_{2^{j+1}B}|^2dz\right)^{1/2} \\ &\lesssim r^{-\gamma}\|f\|_{\mathcal{L}_{2,n-2\gamma}}. \end{aligned}$$

Then we have

$$S_2 \lesssim t\sum_{k=1}^{\infty}\frac{1}{2^kr}k\cdot r^{-\gamma}\|f\|_{\mathcal{L}_{2,n-2\gamma}} \lesssim tr^{-1-\gamma}\|f\|_{\mathcal{L}_{2,n-2\gamma}}.$$

Therefore, we can get

$$\begin{aligned} \int_0^r\int_B|(\psi_0)_t * f_2(y)|^2\,t^{-1}dydt &\lesssim \int_0^r\int_B t^2r^{-2\gamma-2}\|f\|^2_{\mathcal{L}_{2,n-2\gamma}}dydt \\ &\lesssim \|f\|^2_{\mathcal{L}_{2,n-2\gamma}}r^{-2\gamma-2}|B|\int_0^r tdt \\ &\lesssim r^{n-2\gamma}\|f\|^2_{\mathcal{L}_{2,n-2\gamma}}. \end{aligned}$$

For the converse, let $S(I) = \{(t,x)\in\mathbb{R}^{n+1}_+, 0<t<l(I), x\in I\}$ if f such that

$$\begin{aligned} &\sup_I[l(I)]^{2\gamma-n}\int_{S(I)}\left|t\nabla e^{-t^{2\beta}(-\Delta)^\beta}f(y)\right|^2\frac{dydt}{t} \\ &=\sup_I[l(I)]^{2\gamma-n}\int_{S(I)}\left|\nabla e^{-t^{2\beta}(-\Delta)^\beta}f(y)\right|^2 tdydt<\infty. \end{aligned}$$

Denote

$$\Pi_{\psi_0}F(x) = \int_{\mathbb{R}^{n+1}_+}F(t,y)(\psi_0)_t(x-y)\frac{dydt}{t}.$$

We will prove that if

$$\|F\|_{C_\gamma} = \sup_I\left([l(I)]^{2\gamma-n}\int_{S(I)}|F(t,y)|^2\,\frac{dydt}{t}\right)^{1/2}<\infty,$$

then for any cube $J\subset\mathbb{R}^n$,

$$\int_J|\Pi_{\psi_0}F(x)-(\Pi_{\psi_0}F)_J|^2\,dx \lesssim [l(J)]^{n-2\gamma}\|F\|^2_{C_\gamma}.$$

For this purpose, we split F into $F = F_1+F_2 = F\,|_{S(2J)} + F\,|_{\mathbb{R}^{n+1}\setminus S(2J)}$ and get

$$\begin{aligned}\int_J |\Pi_{\psi_0} F_1(x)|^2\,dx &\le \int_J |\Pi_{\psi_0} F_1(x)|^2\,dx\\ &\le \int_{S(2J)} |F(t,y)|^2\,\frac{dydt}{t}\\ &\lesssim [l(J)]^{n-2\gamma}\|F\|^2_{C_\gamma}.\end{aligned}$$

Now we estimate the term associated with F_2. We have

$$\begin{aligned}\int_J |\Pi_{\psi_0} F_1(x)|^2\,dx &= \int_J \left|\int_{\mathbb{R}^{n+1}_+} (\psi_0)_t(x-y)F_2(t,y)t^{-1}dydt\right|^2 dx\\ &\lesssim \int_J \left(\int_{\mathbb{R}^{n+1}_+\setminus S(2J)} |(\psi_0)_t(x-y)||F_2(t,y)|\frac{dydt}{t}\right)^2 dx\\ &= \int_J \left(\sum_{k=1}^{\infty}\int_{S(2^{k+1}J)\setminus S(2^kJ)} |(\psi_0)_t(x-y)||F_2(t,y)|\frac{dydt}{t}\right)^2 dx.\end{aligned}$$

Because $(\psi_0)_t$ satisfies the estimate

$$|(\psi_0)_t(x-y)| \lesssim \frac{t}{t^{n+1}(1+t^{-1}|x-y|)^{n+1}},$$

we have

$$\begin{aligned}\int_J |\Pi_{\psi_0} F_1(x)|^2\,dx &\lesssim \int_J \left(\sum_{k=1}^{\infty}\int_{S(2^{k+1}J)\setminus S(2^kJ)} \frac{t}{[t+2^kl(J)]^{n+1}}|F_2(t,y)|\frac{dydt}{t}\right)^2 dx\\ &\lesssim \int_J \left(\sum_{k=1}^{\infty}(2^kl(J))^{-(n+1)}\int_{S(2^{k+1}J)\setminus S(2^kJ)} |F_2(t,y)|dydt\right)^2 dx\\ &\lesssim \|F\|^2_{C_\gamma}[l(J)]^{n-2\gamma}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\int_J |\Pi_{\psi_0} F(x) - (\Pi_{\psi_0}F)_J|^2\,dx &\le \int_J |\Pi_{\psi_0} F(x)|^2\,dx\\ &\lesssim \int_J |\Pi_{\psi_0} F_1(x)|^2\,dx + \int_J |\Pi_{\psi_0} F_2(x)|^2\,dx\\ &\lesssim \|F\|^2_{C_\gamma}[l(J)]^{n-2\gamma}.\end{aligned}$$

Because

$$\Pi_{\psi_0}F(x) = \int (\psi_0)_t * (\psi_0)_t * f\,\frac{dt}{t},$$

by Calderón's reproducing formula, we have $\Pi_{\psi_0}F(x) = f(x)$, that is, $f(x) = \Pi_{\psi_0}F(x) \in \mathcal{L}_{2,n-2\gamma}$. This completes the proof of Lemma 3.4. ■

Theorem 3.5. *For* $\alpha > 0$, $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ *with* $\alpha + \beta - 1 \geq 0$, *we have*

$$Q_\alpha^\beta(\mathbb{R}^n) = (-\Delta)^{-\frac{(\alpha-\beta+1)}{2}} \mathcal{L}_{2,\ n-2(\alpha+\beta-1)}(\mathbb{R}^n).$$

Proof. For $f \in \mathcal{L}_{2,\ n-2(\alpha+\beta-1)}$, let $F(t,y) = t^{\alpha-\beta+1} t\nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y)$. By Lemma 3.4, we have

$$\begin{aligned}
&r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |F(t,y)|^2 \frac{dydt}{t^{1+2(\alpha-\beta+1)}} \\
&\lesssim r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |t^{\alpha-\beta+1} t\nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y)|^2 \frac{dydt}{t^{1+2(\alpha-\beta+1)}} \\
&\lesssim r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |t\nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y)|^2 \frac{dydt}{t} \\
&\lesssim \|f\|_{\mathcal{L}_{2,\ n-2(\alpha+\beta-1)}}.
\end{aligned}$$

This implies $F \in T^\infty_{\alpha,\beta}$. By Theorem 2.6, Π_{ψ_0} is bounded from $T^\infty_{\alpha,\beta}$ to $Q_\alpha^\beta(\mathbb{R}^n)$. Therefore we have

$$\|f\|_{Q_\alpha^\beta} = \|\Pi_{\psi_0} F\|_{Q_\alpha^\beta} \lesssim \|F\|_{T^\infty_{\alpha,\beta}}.$$

Because $\widehat{F}(t,\ \xi) = t^{\alpha-\beta+2}|\xi| e^{-t^{2\beta}|\xi|^{2\beta}} \widehat{f}(\xi)$, we have

$$\begin{aligned}
\widehat{\Pi_{\psi_0} F}(\xi) &= \int_0^\infty \widehat{F}(t,\xi) \widehat{(\psi_0)_t}(\xi) \frac{dt}{t} \\
&= \int_0^\infty t^{\alpha-\beta+2}|\xi| e^{-t^{2\beta}|\xi|^{2\beta}} t|\xi| e^{-t^{2\beta}|\xi|^{2\beta}} \widehat{f}(\xi) \frac{dt}{t} \\
&= |\xi|^2 \widehat{f}(\xi) \int_0^\infty t^{\alpha-\beta+2} e^{-t^{2\beta}|\xi|^{2\beta}} dt.
\end{aligned}$$

Set $t^{2\beta} = s$ and $|\xi|^{2\beta} s = u$. We can get

$$\begin{aligned}
\widehat{\Pi_{\psi_0} F}(\xi) &= \int_0^\infty s^{\frac{\alpha-\beta+2}{2\beta}} e^{-2s|\xi|^{2\beta}} s^{\frac{1}{2\beta}-1} ds \widehat{f}(\xi)|\xi|^2 \\
&= \widehat{f}(\xi)|\xi|^2 \int_0^\infty (u|\xi|^{-2\beta})^{\frac{\alpha-\beta+3}{2\beta}-1} e^{-u} |\xi|^{-2\beta} du \\
&= \widehat{f}(\xi)|\xi|^2 |\xi|^{-(\alpha-\beta+3)+2\beta-2\beta} \int_0^\infty u^{\frac{\alpha-\beta+3}{2\beta}-1} e^{-2u} du.
\end{aligned}$$

Because $\frac{1}{2} < \beta < 1$ and $0 < \alpha < \beta$, the integral $\int_0^\infty u^{\frac{\alpha-\beta+3}{2\beta}-1} e^{-2u} du < \infty$. We denote it by $C_{\alpha,\beta}$ and get

$$\widehat{\Pi_{\psi_0} F}(\xi) = C_{\alpha,\beta} \widehat{f}(\xi) |\xi|^{-(\alpha-\beta+1)}.$$

By the inverse Fourier transform, we have

$$\Pi_{\psi_0}F(x) = C_{\alpha,\beta}(-\Delta)^{-\frac{\alpha-\beta+1}{2}}f(x).$$

Conversely, suppose $g \in Q^{\beta}_{\alpha}(\mathbb{R}^n)$. Set $G(t,y) = t^{1-(\alpha-\beta+1)}\nabla e^{-t^{2\beta}(-\Delta)^{\beta}}g(y)$. We have, by the equivalent characterization of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ (see [18] for details),

$$\begin{aligned}&\left([l(I)]^{2(\alpha+\beta-1)-n}\int_{S(I)}\left|t^{1-2(\alpha-\beta+1)}\nabla e^{-t^{2\beta}(-\Delta)^{\beta}}g(y)\right|^2\frac{dydt}{t}\right)^{1/2}\\ &=\left([l(I)]^{2(\alpha+\beta-1)-n}\int_{S(I)}\left|t\nabla e^{-t^{2\beta}(-\Delta)^{\beta}}g(y)\right|^2\frac{dydt}{t^{1+2(\alpha-\beta+1)}}\right)^{1/2}\\ &\lesssim \|g\|_{Q^{\beta}_{\alpha}(\mathbb{R}^n)},\end{aligned}$$

that is, $G(t,y) \in \mathcal{C}_{\alpha+\beta-1}$. By Lemma 3.4, we have $\Pi_{\psi_0}G(t,y) \in \mathcal{L}_{2,\ n-2(\alpha+\beta-1)}$. Hence we get

$$\begin{aligned}\widehat{f}(\xi) &= \widehat{\Pi_{\psi_0}G}(t,\xi)\\ &= \int_0^{\infty} t|\xi|e^{-t^{2\beta}|\xi|^{2\beta}}t^{1-(\alpha-\beta+1)}|\xi|e^{-t^{2\beta}|\xi|^{2\beta}}\widehat{g}(\xi)\frac{dt}{t}\\ &= C_{\alpha,\beta}|\xi|^{1+(\alpha-\beta)}\widehat{g}(\xi)\\ &= C_{\alpha,\beta}(\widehat{(-\Delta)^{\frac{\alpha-\beta+1}{2}}g})(\xi).\end{aligned}$$

Then $f(x) = C_{\alpha,\beta}(-\Delta)^{\frac{\alpha-\beta+1}{2}}g$. This completes the proof of this theorem. ∎

Based on the above theorem, we can deduce the boundedness of the convolution singular integral operators on $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ directly and state this result as the following theorem.

Theorem 3.6. *Let T be a singular operator defined by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy,$$

where the kernel $K(x)$ satisfies

$$|\partial_x^{\gamma}K(x)| \le A_{\gamma}|x|^{-n-\gamma}, \quad (\gamma > 0).$$

Or equivalently, let $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$, where the symbol $m(\xi)$ satisfies

$$|\partial_{\xi}^{\gamma}m(\xi)| \le A_{\gamma'}|\xi|^{-\gamma}$$

for all γ. Suppose $\alpha > 0$, $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ with $\alpha+\beta-1 \ge 0$. We have T is bounded on the $Q-$type spaces $Q^{\beta}_{\alpha}(\mathbb{R}^n)$.

Proof. It is well-known that the singular integral operator T is bounded on the Morrey space $\mathcal{L}_{2,\ n-2(\alpha+\beta-1)}(\mathbb{R}^n)$. Moreover as a convolution operator, T can commutate with the fractional Laplace operator $(-\Delta)^{-\frac{(\alpha-\beta+1)}{2}}$. By Theorem 3.5, we complete the proof of this theorem. ■

Specially, taking $T = R_j,\ j = 1, 2, \cdots, n$ as the Riesz transforms, we have the following corollary.

Corollary 3.7. *Suppose* $\alpha > 0$, $\max \alpha, \frac{1}{2} < \beta < 1$ *with* $\alpha + \beta - 1 \geq 0$. *For* $j = 1, 2, \ldots, n$, *the Riesz transforms* $R_j = \partial_j(-\Delta)^{-1/2}$ *are bounded on the* $Q-$*type spaces* $Q^{\beta}_{\alpha}(\mathbb{R}^n)$.

Remark 3.8. There exists another method to prove Theorem 3.6. In fact we can get the boundedness of T on $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ directly by its characterization associated to $e^{-t(-\Delta)^{\beta}}$. In Section 4, this method can be applied to study the well-posedness of the equations $(DQG)_{\beta}$ with the initial data in $Q^{\beta,-1}_{\alpha}(\mathbb{R}^n)$. See Lemma 4.5.

4. Well-posedness and Regularity of Quasi-geostrophic Equation

In this section, we study the well-posedness and regularity of quasi-geostrophic equation with initial data in the space $Q^{\beta}_{\alpha}(\mathbb{R}^2)$. We introduce the definition of $X^{\beta}_{\alpha}(\mathbb{R}^n)$.

Definition 4.1. The space $X^{\beta}_{\alpha}(\mathbb{R}^n)$ consists of the functions which are locally integrable on $(0, \infty) \times \mathbb{R}^2$ such that $\sup\limits_{t>0} t^{1-\frac{1}{2\beta}} \|f(t, \cdot)\|_{\dot{B}^{0,1}_{\infty}} < \infty$ and

$$\sup_{x\in\mathbb{R}^2,\ r>0} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |f(t, y)|^2 + |R_1 f(t, y)|^2 + |R_2 f(t, y)|^2 \frac{dydt}{t^{\alpha/\beta}} < \infty,$$

where $R_j,\ j = 1, 2$ denote the Riesz transforms in $\mathbb{R}^2$.

For the quasi-geostrophic dissipative equations

$$\text{(4.1)} \qquad \begin{cases} \partial_t \theta = -(-\Delta)^{\beta} + \partial_1(\theta R_2 \theta) - \partial_2(\theta R_1 \theta), \\ \theta(0, x) = \theta_0(x), \end{cases}$$

where $\beta \in (\frac{1}{2}, 1)$. The solution to equations (4.1) can be represented as

$$u(t, x) = e^{-t(-\Delta)^{\beta}} u_0 + B(u, u),$$

where the bilinear form $B(u, v)$ is defined by

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^{\beta}} (\partial_1(vR_2u) - \partial_2(vR_1u)) ds.$$

In order to prove the well-posedness, we need the following preliminary lemmas. For their proofs, we refer the readers to Lemma 4.8 and Lemma 4.9 in [18].

Lemma 4.2. ([18, Lemma 4.8]). *Given* $\alpha \in (0,1)$. *For a fixed* $T \in (0,\infty]$ *and a function* $f(t,x)$ *on* $\mathbb{R}^{1+n}_+$, *let* $A(t) = \int_0^t e^{-(t-s)(-\triangle)^\beta}(-\triangle)^\beta f(s,x)ds$. *Then*

$$\int_0^T \|A(t,\cdot)\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^T \|f(t,\cdot)\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}}. \tag{4.2}$$

Lemma 4.3. ([18, Lemma 4.9]). *For* $\beta \in (1/2,1)$ *and* $N(t,x)$ *defined on* $(0,1)\times \mathbb{R}^n$, *let* $A(N)$ *be the quantity*

$$A(\alpha,\beta,N) = \sup_{x\in\mathbb{R}^n, r\in(0,1)} r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x|<r}|f(t,x)|\frac{dxdt}{t^{\alpha/\beta}}.$$

Then for each $k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ *there exists a constant* $b(k)$ *such that the following inequality holds:*

$$\begin{aligned}(4.3)\qquad &\int_0^1 \left\| t^{\frac{k}{2}}(-\triangle)^{\frac{k\beta+1}{2}} e^{-\frac{t}{2}(-\triangle)^\beta}\int_0^t N(s,\cdot)ds\right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}}\\ &\le b(k)A(\alpha,\beta,N)\int_0^1\int_{\mathbb{R}^n}|N(s,x)|\frac{dxds}{s^{\alpha/\beta}}.\end{aligned}$$

Remark 4.4. Similarly when $k=0$, we can prove the following inequality:

$$\begin{aligned}(4.4)\qquad &\int_0^1 \left\| (-\triangle)^{\frac{1}{2}} e^{-t(-\triangle)^\beta}\int_0^t N(s,\cdot)ds\right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}}\\ &\lesssim A(\alpha,\beta,N)\int_0^1\int_{\mathbb{R}^n}|N(s,x)|\frac{dxds}{s^{\alpha/\beta}}.\end{aligned}$$

Lemma 4.5. *Assume* $\alpha > 0$ *and* $\max\{\alpha,\ 1/2\} < \beta < 1$ *with* $\alpha+\beta-1\ge 0$. *Let* $R_j, j=1,2$ *be the Riesz transforms. Then for any* $x_0 \in \mathbb{R}^n$,

$$\begin{aligned}&\left(\sup_{r>0} r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|R_jf(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right)^{1/2}\\ &\lesssim \left(\sup_{x\in\mathbb{R}^n, r>0} r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|f(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right)^{1/2}.\end{aligned}$$

Proof. We split $f(t,y)$ into

$$f(t,y) = f_0(t,y) + \sum_{k=1}^{\infty} f_k(t,y),$$

where $f_0(t,y) = f(t,y)\chi_{B(x_0,2r)}(y)$ and $f_k(t,y)\chi_{B(x_0,2^{k+1}r)\setminus B(x_0,2^kr)}(y)$. We have

$$
\begin{aligned}
&\left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|R_jf(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right)\\
\le\ &\left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|R_jf_0(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right)\\
&+\sum_{k=1}^{\infty}\left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|R_jf_k(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right)\\
=:\ &M_0+\sum_{k=1}^{\infty}M_k.
\end{aligned}
$$

By the L^2 boundedness of Riesz transforms $R_j,\ j=1,2$, we have

$$
\begin{aligned}
M_0 &\lesssim \left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|f(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right)\\
&\lesssim C\sup_{x\in\mathbb{R}^n,r>0}\left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|f(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right).
\end{aligned}
$$

Now we estimate the terms M_k. We only need to estimate the integral as follows.

$$
I=\int_{|y-x_0|<r}|R_jf_k(t,y)|^2dy.
$$

As a singular integral operator,

$$
R_jg(x)=\int_{\mathbb{R}^n}\frac{x_j-y_j}{|x_j-y_j|^{n+1}}g(y)dy.
$$

By Hölder's inequality, we can get

$$
\begin{aligned}
I &= \int_{|y-x_0|<r}\left|\int_{2^kr\le|z-x_0|<2^{k+1}r}\frac{y_j-z_j}{|y-z|^{n+1}}f(t,z)dz\right|^2dy\\
&\lesssim \int_{|y-x_0|<r}\left(\frac{1}{(2^kr)^n}\int_{|z-x_0|<2^{k+1}r}|f(t,z)|dz\right)^2dy\\
&\lesssim \frac{1}{2^{kn}}\int_{|z-x_0|<2^{k+1}r}|f(t,z)|^2dz.
\end{aligned}
$$

So we have

$$
M_k=\left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r}|R_jf_k(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\right)^{1/2}
$$

$$\lesssim \left(2^{-k(2\alpha-n+2\beta-2)}\frac{1}{2^{kn}}(2^kr)^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|z-x_0|<2^{k+1}r}|f(t,z)|^2\frac{dydt}{t^{\alpha/\beta}}\right)^{1/2}$$

$$\lesssim \left(2^{-k(2\alpha-n+2\beta-2)}\frac{1}{2^{kn}}\right)^{1/2}\sup_{x_0\in\mathbb{R}^n,r>0}\left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|z-x_0|<r}|f(t,z)|^2\frac{dzdt}{t^{\alpha/\beta}}\right)^{1/2}.$$

Therefore we can get

$$\begin{aligned} &M_0+\sum_{k=1}^{\infty}M_k\\ &\lesssim \left[1+\sum_{k=1}^{\infty}2^{-k(\alpha+\beta-1)}\right]\sup_{x_0\in\mathbb{R}^n,r>0}\left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|z-x_0|<r}|f(t,z)|^2\frac{dydt}{t^{\alpha/\beta}}\right).\end{aligned}$$

This completes the proof of Lemma 4.5. ■

Now we give the main result of this section.

Theorem 4.6. (Well-posedness).

(i) *The subcritical quasi-geostrophic equation* (4.1) *has a unique small global mild solution in* $(X^{\beta}_{\alpha}(\mathbb{R}^2))^2$ *for all initial data* θ_0 *with* $\nabla\cdot\theta=0$ *and* $\|u_0\|_{Q^{\beta,-1}_{\alpha}}$ *being small.*

(ii) *For any* $T\in(0,\infty)$, *there is an* $\varepsilon>0$ *such that the quasi-geostrophic equation* (4.1) *has a unique small mild solution in* $(X^{\beta}_{\alpha}(\mathbb{R}^2))^2$ *on* $(0,T)\times\mathbb{R}^2$ *when the initial data* u_0 *satisfies* $\nabla\cdot u_0=0$ *and* $\|u_0\|_{(Q^{\beta,-1}_{\alpha,\ T})^2}\le\varepsilon$. *In particular, for all* $u_0\in\overline{(VQ^{\beta,-1}_{\alpha})^2}$ *with* $\nabla\cdot u_0=0$, *there exists a unique small local mild solution in* $(X^{\beta}_{\alpha,T})^2$ *on* $(0,T)\times\mathbb{R}^2$.

Proof. By the Picard contraction principle we only need to prove the bilinear form $B(u,v)$ is bounded on X^{β}_{α}. We split the proof into two parts.

Part I. $\dot{B}^{0,1}_{\infty}-$boundedness. The proof of this part has been given in [19]. For completeness, we give the details. We have

$$\begin{aligned}\|B(u,v)\|_{\dot{B}^{0,1}_{\infty}} &\lesssim \int_0^t\|e^{-(t-s)(-\Delta)^{\beta}}(\partial_1(gR_2f)-\partial_2(gR_1f))\|_{\dot{B}^{0,1}_{\infty}}ds\\ &\lesssim \int_0^t\frac{C_\beta}{(t-s)^{\frac{1}{2\beta}}s^{1+(1-\frac{1}{\beta})}}s^{1-\frac{1}{2\beta}}\|u\|_{\dot{B}^{0,1}_{\infty}}s^{1-\frac{1}{2\beta}}\|v\|_{\dot{B}^{0,1}_{\infty}}ds\\ &\lesssim \|u\|_{X^{\beta}_{\alpha}}\|v\|_{X^{\beta}_{\alpha}}\int_0^t\frac{ds}{(t-s)^{\frac{1}{2\beta}}s^{1+(1-\frac{1}{\beta})}}.\end{aligned}$$

Because when $\frac{1}{2} < \beta < 1$,

$$\int_0^{t/2} \frac{1}{(t-s)^{\frac{1}{2\beta}} s^{1+(1-\frac{1}{\beta})}} ds \lesssim t^{\frac{1}{2\beta}-1}$$

and

$$\int_{t/2}^{t} \frac{1}{(t-s)^{\frac{1}{2\beta}} s^{1+(1-\frac{1}{\beta})}} ds \lesssim t^{-2+\frac{1}{\beta}} \int_{t/2}^{t} \frac{1}{(t-s)^{\frac{1}{2\beta}}} ds \lesssim t^{\frac{1}{2\beta}-1}.$$

Then we can get

$$t^{1-\frac{1}{2\beta}} \|B(u,v)\|_{\dot{B}^{0,1}_\infty} \lesssim \|u\|_{X^\beta_\alpha} \|v\|_{X^\beta_\alpha},$$

where in the above estimates we have used the fact that $\|R_j f\|_{\dot{B}^{0,1}_\infty} \lesssim \|f\|_{\dot{B}^{0,1}_\infty}$ for $f \in \dot{B}^{0,1}_\infty$. In fact by Bernstein's inequality, we have

$$\begin{aligned} \sum_l \|\Delta_l R_j f\|_{L^\infty} &= \sum_l \|\partial_j (-\Delta)^{-1/2} \Delta_l f\|_{L^\infty} \\ &\lesssim \sum_l 2^l \|(-\Delta)^{-1/2} \Delta_l f\|_{L^\infty} \\ &\lesssim \sum_l 2^l 2^{-l} \|\Delta_l f\|_{L^\infty} \\ &\leq \|f\|_{\dot{B}^{0,1}_\infty}. \end{aligned}$$

On the other hand, by Young's inequality, we have

$$t^{1-\frac{1}{2\beta}} \|e^{-t(-\Delta)^\beta} u_0\|_{\dot{B}^{0,1}_\infty} \lesssim \|u_0\|_{\dot{B}^{1-2\beta,\infty}_\infty} \leq \|u_0\|_{Q^{\beta,-1}_\alpha}.$$

Part II. L^2-boundedness. This part contributes to the operation of $B(u,v)$ on the Carleson part of X^β_α. We split again the estimate into two steps.

Step I. We want to prove the following estimate:

$$r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |B(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X^\beta_\alpha} \|v\|_{X^\beta_\alpha}.$$

By symmetry, we only need to deal with the term

$$\int_0^t e^{-(t-s)(-\Delta)^\beta} [\partial_1 (v R_1 u)] ds = B_1(u,v) + B_2(u,v) + B_3(u,v),$$

where

$$B_1(u,v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \partial_1 [(1 - 1_{r,x}) v R_1 u] ds,$$

$$B_2(u,v) = (-\Delta)^{-1/2} \partial_1 \int_0^t e^{-(t-s)(-\Delta)^\beta} (-\Delta)((-\Delta)^{1/2} (I - e^{-s(-\Delta)^\beta})(1_{r,x}) v R_1 u) ds$$

and

$$B_3(u,v)=(-\Delta)^{-1/2}\partial_1(-\Delta)^{1/2}e^{-t(-\Delta)^\beta}\int_0^t(1_{r,x})vR_1uds.$$

For B_1, it can be proved that the fractional heat kernel satisfies the following estimate ([20]):

$$|\nabla e^{-t(-\Delta)^\beta}(x,y)|\lesssim\frac{1}{t^{\frac{n+1}{2\beta}}}\frac{1}{\left(1+\frac{|x-y|}{t^{1/2\beta}}\right)^{n+1}}\lesssim\frac{1}{(t^{\frac{1}{2\beta}}+|x-y|)^{n+1}}. \tag{4.5}$$

For $0<t<r^{2\beta}$, taking $n=2$ in (4.5), we have

$$\begin{aligned}&|B_1(u,v)(t,x)|\\&\lesssim\int_0^t\int_{|z-x|\geq 10r}\frac{|R_1u(s,z)||v(s,z)|}{|x-z|^{2+1}}dzds\\&\lesssim\left(\int_0^{r^{2\beta}}\int_{|z-x|\geq 10r}\frac{|R_1u(s,z)|^2}{|x-z|^3}dzds\right)^{1/2}\left(\int_0^{r^{2\beta}}\int_{|z-x|\geq 10r}\frac{|v(s,z)|^2}{|x-z|^3}dzds\right)^{1/2}\\&:=I_1\times I_2.\end{aligned}$$

For I_1, we have

$$\begin{aligned}I_1&\lesssim\left(\sum_{k=3}^{\infty}\frac{1}{(2^kr)^3}\int_0^{r^{2\beta}}\int_{|x-z|\leq 2^{k+1}r}|R_1u(s,x)|^2dsdx\right)^{1/2}\\&\lesssim\left(\sum_{k=3}^{\infty}\frac{1}{(2^kr)^3}(2^kr)^{2\alpha+2\beta-2}(2^kr)^{2-2\beta}\int_0^{r^{2\beta}}\int_{|x-z|\leq 2^{k+1}r}|R_1u(s,x)|^2\frac{dsdx}{s^{\alpha/\beta}}\right)^{1/2}\\&\lesssim\|u\|_{X_\alpha^\beta}\left(\sum_{k=3}^{\infty}\frac{1}{2^{k(2\beta-1)}}\frac{1}{r^{2\beta-1}}\right)^{1/2}\\&\lesssim\left(\frac{1}{r^{2\beta-1}}\right)^{1/2}\|u\|_{X_\alpha^\beta}.\end{aligned}$$

Similarly, we can get $I_2\lesssim\left(\frac{1}{r^{2\beta-1}}\right)^{1/2}\|v\|_{X_\alpha^\beta}$ and $|B_1(u,v)|\lesssim\frac{1}{r^{2\beta-1}}\|u\|_{X_\alpha^\beta}\|v\|_{X_\alpha^\beta}$. Then we have

$$\begin{aligned}\int_0^{r^{2\beta}}\int_{|x-y|<r}|B_1(u,v)|^2\frac{dydt}{t^{\alpha/\beta}}&\lesssim\frac{1}{r^{4\beta-2}}r^2\int_0^{r^{2\beta}}\frac{dt}{t^{\alpha/\beta}}\|u\|^2_{X_\alpha^\beta}\|v\|^2_{X_\alpha^\beta}\\&\lesssim\frac{1}{r^{4\beta-2}}r^2r^{2\beta-2\alpha}\|u\|^2_{X_\alpha^\beta}\|v\|^2_{X_\alpha^\beta}\\&\lesssim r^{2-2\alpha-2\beta+2}\|u\|^2_{X_\alpha^\beta}\|v\|^2_{X_\alpha^\beta},\end{aligned}$$

where in the second inequality we have used the fact $0<\alpha<\beta$. That is to say

$$r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-y|<r}|B_1(u,v)(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\lesssim\|u\|^2_{X^\beta_\alpha}\|v\|^2_{X^\beta_\alpha}.$$

For B_2, by the L^2-boundedness of Riesz transform, we have

$$\begin{aligned}
&\int_0^{r^{2\beta}}\int_{|x-y|<r}|B_2(u,v)|^2\frac{dydt}{t^{\alpha/\beta}}\\
\lesssim&\int_0^{r^\beta}\left\|\int_0^t e^{-(t-s)(-\Delta)^\beta}(-\Delta)((-\Delta)^{-1/2}(I-e^{-s(-\Delta)^\beta})(1_{r,x})vR_1u)ds\right\|^2_{L^2}\frac{dt}{t^{\alpha/\beta}}\\
\lesssim&\int_0^{r^\beta}\left\|\int_0^t e^{-(t-s)(-\Delta)^\beta}(-\Delta)^\beta((-\Delta)^{1/2-\beta}(I-e^{-s(-\Delta)^\beta})(1_{r,x})vR_1u)ds\right\|^2_{L^2}\frac{dt}{t^{\alpha/\beta}}\\
\lesssim&\int_0^{r^{2\beta}}t^{2-\frac{1}{\beta}}\int_{|y-x|<r}|R_1u(t,y)|^2|v(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\\
\lesssim&\left(\sup_{t>0}t^{1-\frac{1}{2\beta}}\|R_1u(t,\cdot)\|_{L^\infty}\right)\left(\sup_{t>0}t^{1-\frac{1}{2\beta}}\|v(t,\cdot)\|_{L^\infty}\right)\\
&\quad\int_0^{r^{2\beta}}\int_{|y-x|<r}|R_1u(t,y)||v(t,y)|\frac{dtdy}{t^{\alpha/\beta}}.
\end{aligned}$$

On one hand, by Bernstein's inequality, we have

$$\|R_1u(t,\cdot)\|_{L^\infty}\le\|R_1u(t,\cdot)\|_{\dot B^{0,1}_\infty}\lesssim\|u(t,\cdot)\|_{\dot B^{0,1}_\infty}.$$

Then we get

$$\sup_{t>0}t^{1-\frac{1}{2\beta}}\|R_1u(t,\cdot)\|_{L^\infty}\lesssim\sup_{t>0}t^{1-\frac{1}{2\beta}}\|u(t,\cdot)\|_{\dot B^{0,1}_\infty}.$$

On the other hand, we have, by Hölder's inequality,

$$\begin{aligned}
&\int_0^{r^{2\beta}}\int_{|x-y|<r}|R_1u(t,y)||v(t,y)|\frac{dtdy}{t^{\alpha/\beta}}\\
\lesssim&\left(\int_0^{r^{2\beta}}\int_{|y-x|<r}|R_1u(t,y)|^2\frac{dtdy}{t^{\alpha/\beta}}\right)^{1/2}\left(\int_0^{r^{2\beta}}\int_{|y-x|<r}|v(t,y)|^2\frac{dtdy}{t^{\alpha/\beta}}\right)^{1/2}\\
\lesssim&\ r^{2-2\alpha-2\beta+2}\|u\|^2_{X^\alpha_\alpha}\|v\|^2_{X^\alpha_\alpha}.
\end{aligned}$$

Hence we get

$$\int_0^{r^{2\beta}}\int_{|x-y|<r}|B_2(u,v)(t,y)|^2\frac{dydt}{t^{\alpha/\beta}}\lesssim r^{2-2\alpha-2\beta+2}\|u\|^2_{X^\beta_\alpha}\|v\|^2_{X^\beta_\alpha}.$$

For $B_3(u,v)$, we have

$$
\begin{aligned}
&\int_0^{r^{2\beta}} \int_{|y-x|<r} |B_3(u,v)(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \\
&= \int_0^{r^{2\beta}} \int_{|y-x|<r} \left| (-\Delta)^{-1/2} \partial_1 (-\Delta)^{1/2} e^{-t(-\Delta)^\beta} \left(\int_0^t (1_{r,x}) v R_1 u dh \right) \right|^2 \frac{dydt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^{2\beta}} \left\| (-\Delta)^{1/2} e^{-t(-\Delta)^\beta} \left(\int_0^t (1_{r,x}) v R_1 u dh \right) \right\| \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim r^{2-2\alpha+6\beta-2} \left(\int_0^1 \|M(r^{2\beta}s,\ r\cdot)\|_{L^1} \frac{ds}{s^{\alpha/\beta}} \right) C(\alpha,\beta,f) \\
&\lesssim r^{2-2\alpha+6\beta-2} r^{2-4\beta} r^{2-4\beta} \|u\|_{X^\beta_\alpha} \|v\|_{X^\beta_\alpha} \\
&\lesssim r^{2-2\alpha-2\beta+2} \|u\|_{X^\beta_\alpha} \|v\|_{X^\beta_\alpha}.
\end{aligned}
$$

Step II. For $j = 1, 2$, we want to prove

$$
r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |R_j B(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X^\beta_\alpha} \|v\|_{X^\beta_\alpha}, \tag{4.6}
$$

where R_j are the Riesz transforms $\partial_j(-\Delta)^{-1/2}$. Similar to Step I, we can split $B(u,v)$ into $B_i(u,v),\ i = 1,2,3$. We denote by $A_i, i = 1,2,3$

$$
A_i := r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |R_j B_i(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X^\beta_\alpha} \|v\|_{X^\beta_\alpha}. \tag{4.7}
$$

In order to estimate the term A_1, we need the following lemma.

Lemma 4.7. *For $\beta > 0$, if we denote by K^β_j the kernel of the operator $e^{-t(-\Delta)^\beta} R_j$, we have*

$$
(1+|x|)^{n+|\alpha|} \partial^\alpha e^{-t(-\Delta)^\beta} R_j \in L^\infty.
$$

Proof. By the Fourier transform, we have $K^\beta_j = \mathcal{F}^{-1}(\frac{\xi_j}{|\xi|} e^{-|\xi|^{2\beta}})$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Because

$$
\left[\partial^\alpha K^\beta_j(x) \right]^\wedge (\xi) = \frac{\xi_j}{|\xi|} |\xi|^\alpha e^{-|\xi|^{2\beta}} \in L^1,
$$

we have

$$
|\partial^\alpha K^\beta_j(x)| \le \int_{\mathbb{R}^2} \left| \frac{\xi_j}{|\xi|} |\xi|^\alpha e^{-|\xi|^{2\beta}} \right| d\xi \le C.
$$

Then $\partial^\alpha K^\beta_j(x) \in L^\infty$. If $|x| \le 1$, we have

$$
(1+|x|)^{n+|\alpha|} |K^\beta_j(x)| \lesssim C_\alpha |K^\beta_j(x)| \lesssim C.
$$

If $|x| > 1$, by Littlewood-Paley decomposition and write

$$K_j^\beta(x) = (Id - S_0)K_j^\beta + \sum_{l<0} \Delta_l K_j^\beta,$$

where $(Id - S_0)K_j^\beta \in \mathcal{S}(\mathbb{R}^n)$ and $\Delta_l K_j^\beta = 2^{2l}\omega_{j,l}(2^l x)$ where $\widehat{\omega_{j,l}}(\xi) = \psi(\xi)\frac{\xi_j}{|\xi|}e^{-|2^l\xi|^{2\beta}} \in L^1$. Then $\omega_{j,l}(x)_{(l<0)}$ are a bounded set in $\mathcal{S}(\mathbb{R}^n)$. So we have

$$(1 + 2^l|x|)^N 2^{l(2+|\alpha|)}|\partial^\alpha \Delta_l K_j^\beta(x)| \lesssim C_N$$

and

$$\begin{aligned} |\partial^\alpha S_0 K_j^\beta(x)| &\lesssim C \sum_{2^l|x|\le 1} 2^{l(2+|\alpha|)} + \sum_{2^l|x|>1} 2^{l(2+|\alpha|-N)}|x|^{-N} \\ &\lesssim C|x|^{-(2+|\alpha|)}. \end{aligned}$$

This completes the proof of Lemma 4.7 ∎

Now we complete the proof of Theorem 4.6. In Lemma 4.7, we take $\alpha = 1$ and get

$$\left|\partial_x R_j e^{-t(-\Delta)^\beta}(x,y)\right| \lesssim \frac{1}{(t^{\frac{1}{2\beta}} + |x-y|)^{n+1}}.$$

Similar to the proof in Part I, we can get

$$A_1 := r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-y|<r} |R_j B_1(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^\beta}\|v\|_{X_\alpha^\beta}.$$

By Lemma 4.5, we know

$$\begin{aligned} & r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r} |R_j f(t,y)|^2 \,\frac{dydt}{t^{\alpha/\beta}} \\ \lesssim & \sup_{r>0, x_0\in\mathbb{R}^n} r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r} |f(t,y)|^2 \,\frac{dydt}{t^{\alpha/\beta}}. \end{aligned}$$

By the above estimate, we have

$$\begin{aligned} A_i &:= r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-y|<r} |R_j B_i(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \\ &\lesssim r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-y|<r} |B_i(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}}, \end{aligned}$$

where $i = 2, 3$. Following the estimate to $B_i, i = 2, 3$, we can get

$$A_i := r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-y|<r} |R_j B_i(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^\beta}\|v\|_{X_\alpha^\beta}.$$

This completes the proof of Theorem 4.6. ■

Following the method applied in Section 5 of [18], we can easily get the regularity of the solution to the quasi-geostrophic equations (4.1). So we only state the result and omit the details of the proof. For convenience of the study, we introduce a class of spaces $X_\alpha^{\beta,k}$ as follows.

Definition 4.8. For a nonnegative integer k and $\beta \in (1/2, 1]$, we introduce the space $X_\alpha^{\beta,k}$ which is equipped with the following norm:

$$\|u\|_{X_\alpha^{\beta,k}} = \|u\|_{N_{\alpha,\infty}^{\beta,k}} + \|u\|_{N_{\alpha,C}^{\beta,k}},$$

where

$$\begin{aligned}
\|u\|_{N_{\alpha,\infty}^{\beta,k}} &= \sup_{\alpha_1+\cdots+\alpha_n=k} \sup_t t^{\frac{2\beta-1+k}{2\beta}} \|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} u(\cdot,t)\|_{\dot{B}_\infty^{0,1}},\\
\|u\|_{N_{\alpha,C}^{\beta,k}} &= \sup_{\alpha_1+\cdots+\alpha_n=k} \sup_{x_0,r}\\
&\quad \left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r} |t^{\frac{k}{2\beta}}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} u(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}}\right)^{1/2}\\
&\quad + \sum_{j=1}^{2} \sup_{\alpha_1+\cdots+\alpha_n=k} \sup_{x_0,r}\\
&\quad \left(r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|y-x_0|<r} |R_j t^{\frac{k}{2\beta}}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} u(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}}\right)^{1/2}.
\end{aligned}$$

Now we state the regularity result.

Theorem 4.9. *Let $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha+\beta-1 \geq 0$. There exists an $\varepsilon = \varepsilon(n)$ such that if $\|u_0\|_{Q_{\alpha;\infty}^{\beta,-1}} < \varepsilon$, the solution u to equations (4.1) verifies:*

$$t^{\frac{k}{2\beta}}\nabla^k u \in X_\alpha^{\beta,0} \text{ for any } k \geq 0.$$

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