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NEW EXACT SOLUTIONS TO THE MODIFIED FORNBERG-WHITHAM EQUATION

Ameina S. Nuseir

Abstract. In this paper a unified approach for finding soliton solutions is applied to the modified Fornberg-Whitham equation. Variety of peakon, periodic, and solitary new exact solutions are constructed.

1. INTRODUCTION

The study of nonlinear wave equations and their solutions is of great importance in many areas of physics. Travelling wave solutions are among the interesting types of solutions for the nonlinear partial differential equations. On the other hand, many nonlinear partial differential equations have been found to have a variety of travelling wave solutions. An example, is the well-known Korteweg–de Vries equation

$$u_{\mathrm{t}} + 6uu_x + uu_{xxx} = 0 \tag{1}$$

which is a mathematical model of waves on shallow water surfaces that has smooth solitary wave solutions [8]. Also, the Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \tag{2}$$

proposed by Camassa and Holm (1993)[1] is a model equation for the unidirectional nonlinear dispersive waves in shallow water. Due to its interesting mathematical properties, this equation has gained a lot of interest over the past decade. This equation has been found to have peakons, stumpons, cuspons, and composite wave solutions [9]. Nevertheless, it also has compactons [11]. Liu and coworkers found a new type of travelling wave solutions for the Camassa–Holm equation [12], which are defined on some semifinal bounded domains that posses properties of kink or antikink waves

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which they called kink-like waves and antikink-like waves. Later on, it has been found that the CH-γ equation

$$u_t + c_0 u_x + 3uu_x - \alpha^2(u_{xxt} + uu_{xxx} + 3u_x u_{xx}) + \gamma u_{xxx} = 0 \tag{3}$$

posses kink-like waves when $\alpha \neq 0$[4,15]. Furthermore, the Fornberg–Whitham equation which is given as

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx} \tag{4}$$

was used to study the qualitative behavior of wave-breaking [8,9]. It admits a wave of greatest height, as a peaked limiting form of the travelling wave solution

$$u(x,t) = A\exp(\frac{1}{2}|x - \frac{4}{3}t|),$$

where A is an arbitrary constant [3].

In 2006, Wazwaz [17] suggested studying the modified forms of the Camassa–Holm and the Degasperis–Procesi equations

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx}; \tag{5}$$

and

$$u_t - u_{xxt} + 4u^2 u_x = 3u_x u_{xx} + uu_{xxx}. \tag{6}$$

Then, many travelling wave solutions of Eqs. 5 and 6 have been obtained using tanh method, sine–cosine method and extended tanh method [17,18]. Using the bifurcation method of planar systems and numerical simulation of differential equations, Liu and Ouyang [13] obtained some peakon and solitary wave solutions. Moreover, Wang and Tang [16] obtained some new peakon and solitary wave solutions through some special phase orbits. Another study by Rui et al. [14] obtained some exact travelling wave solutions using the integral bifurcation method. Using the Homotopy perturbation method, Zhang et al. [21] solved Eqs. 5 and 6 to obtain some exact solutions.

He [5] used the bifurcation theory and the method of phase portraits analysis [6,7,10] to investigate the modified Fornberg–Whitham (mFW) equation:

$$u_t - u_{xxt} + u_x = uu_{xxx} - u^2 u_x + 3u_x u_{xx}, \tag{7}$$

and obtained some explicit peakon and solitary wave solutions.

In this article, a unified approach to find more explicit solutions for the mFW equation is used.

2. The Unified Approach for Finding Soliton Solutions

Based on the randomness of the Painlevé analysis manifold, E. Fan [2] developed a method to find a rich variety of exact travelling wave solutions of a nonlinear evolution equation. At the same time, G. Xu and coworkers [20] published another article describing the same method and applied it to some nonlinear PDE's. In this approach, they considered nonlinear the evolution equations (NEE's) in two variables,

(8) $$F(u;\ u_t;\ u_x;\ u_{xt};\ u_{xx};\ \dots) \ = \ 0$$

where the subscripts denote partial derivatives, and F is a polynomial in unknown function $u(x, t)$ and its derivatives. The travelling wave solutions to 8 can be written as

(9) $$u(x;\ t) \ = \ u(\xi)\ ;\ \xi \ = x - c\,t$$

in which k is the wave number, and c is the wave speed to be determined. Using the transformation in 9, Eq. 8 can be transformed to an ordinary differential equation of the type:

(10) $$G(u;\ u';\ u'';\ \dots) \ = \ 0;$$

where $u' = \frac{du}{d\xi}$.

To introduce the concept of "rank", the term in the reduced ordinary differential equations will be written as:

$$u^{k_0}\ (u')^{k_1}\ (u'')^{k_2}\ \dots\dots\ (u^{(m)})^{k_m}$$

where $kj(j = 0; \dots\dots ;m)$ are real constants. Thus the rank of this term is defined as the number:

$$0\ \cdot\ k_0\ +\ 1 \cdot k_1\ +\ 2 \cdot\ k_2\ +\ \dots\dots\dots\ +\ m \cdot\ k_m.$$

Considering that the travelling wave solutions of many NEEs derived from soliton theory can be expressed as polynomials of special functions such as sech, tanh, sin, cos and the like. Now, if the rank of every term in 10 is even or odd, then the following truncated expansion

(11) $$u(\xi) = \sum_{j=0}^{m} a_j \phi^j,$$

can be taken, and the expansion variable $\phi = \phi(\xi)$ satisfies

(12) $$(\phi')^2 = c_0 + c_1\phi + c_2\phi^2 + \dots + c_r\phi^r,$$

where *m* and *r* are positive integers, and $\mathrm{a_j}$ $(\mathrm{j} \;=\; 0,\; 1, ..., \mathrm{m})$, $\mathrm{c_i}(\mathrm{i} \;=\; 0,\; ...,\; \mathrm{r})$ are constants the to be determined. To find the travelling wave solutions of Eq. 8, the following steps will be followed:

Step 1. To determine *m* and *r*, one may substitute 11 into Eq. 10 and balance the highest derivative terms with the most nonlinear term in Eq.10. By doing so, a relation for *m* and *r* will be obtained. From this relation different possible values of *m* and *r* can be found.

Step 2. Substitute the series 11 along with 12 into Eq. 10; to get $P(\phi) = 0$, where $P(\phi)$ is a polynomial in ϕ. By equating the coefficients of each power of ϕ in $P(\phi)$ to zero, an algebraic system involving $a_j(j \;=\; 0,\; \dots, m)$, $c_i(i \;=\; 0, ...,\; r)$ and c is obtained.

Step 3. By solving Eq. 12 and then substituting the solution in Eq. 8, some kinds of interesting travelling wave solutions will be obtained.

3. Solving the Modified Fornberg-whitham Equation

To apply this method, the modified Fornberg-Whitham equation

$$u_t - u_{xxt} + u_x = uu_{xxx} - u^2 u_x + 3u_x u_{xx}, \tag{13}$$

should be transformed into an ODE. Let $\xi \; = x - c\, t$ and $u(x;\; t) \;=\; u(\xi)$, then the equation will transform into

$$cu''' - uu''' - 3u'u'' + u^2 u' + (1-c)u' = 0. \tag{14}$$

The ranks of the terms in Eq. 14 are $3, 3, 3, 1$ and 1 respectively. Therefore, we can use the above method.

First, find all derivatives of u of orders $1, 2$ and 3 in terms of ϕ

$$u' = a_1\phi' + 2a_2\phi\phi' + \ldots\ldots + ma_m\phi^{m-1}\phi' = \phi'(a_1 + 2a_2\phi + \ldots\ldots + ma_m\phi^{m-1}), \tag{15}$$

and

$$u'' = a_1\phi'' + 2a_2(\phi\phi'' + \phi'^2) + \ldots\ldots + ma_m(\phi^{m-1}\phi'' + (m-1)\phi^{m-2}\phi'^2). \tag{16}$$

To find ϕ'' we differentiate Eq. 12 and get

$$\phi'' = \frac{1}{2}(c_1 + 2c_2\phi + \ldots\ldots + rc_r\phi^{r-1}). \tag{17}$$

Therefore,

$$\begin{aligned} u'' = &\frac{1}{2}(c_1 + 2c_2\phi + \ldots\ldots + rc_r\phi^{r-1})(a_1 + 2a_2\phi + \ldots\ldots + ma_m\phi^{m-1}) \\ &+(2a_2 + \ldots\ldots + m(m-1)a_m\phi^{m-2})(c_0 + c_1\phi + c_2\phi^2 + \ldots + c_r\phi^r) \end{aligned} \tag{18}$$

and

$$u''' = \frac{1}{2}\phi' \tag{19}$$
$$\begin{bmatrix} (2c_2+6c_3\phi+\cdots+r(r-1)c_r\phi^{r-2})(a_1+2a_2\phi+\cdots+ma_m\phi^{m-1}) \\ +(c_1+2c_2\phi+\cdots+rc_r\phi^{r-1})(6a_2+18a_3\phi+\cdots+3m(m-1)a_m\phi^{m-2}) \\ +2(c_0+c_1\phi+c_2\phi^2+\cdots+c_r\phi^r)(6a_3+\cdots+m(m-1)(m-2)a_m\phi^{m-3}) \end{bmatrix}$$

Substitute Eqs. 15, 16, 18, and 19 into Eq. 14 and take ϕ' as common factor, then the highest nonlinear terms in uu''' is ϕ^{2m+r-3}; while that for the term u^2u' is ϕ^{3m-1}.

Now, if we balance the powers, we obtain the relation $r=m+2$ between m and r. If we take $m = 1$, then $r = 3$, so

$$u(\xi) = a_0 + a_1\phi(\xi)$$

and

$$(\phi')^2 = c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3. \tag{20}$$

The solutions for Eq. 20 are:

$$\begin{aligned} \phi &= \frac{-c_2}{c_3}Sec^2(\frac{\sqrt{-c_2}}{2}\xi), \text{ if } c_0 = c_1 = 0, c_2 < 0, \text{ and} \\ \phi &= \frac{-c_2}{c_3}Sech^2(\frac{\sqrt{c_2}}{2}\xi), \text{ if } c_0 = c_1 = 0, c_2 > 0. \end{aligned}$$

So the following two solutions for $u(x,t)$ will be obtained:

$$\begin{aligned} u &= \frac{1}{4}(10c_2 - 2(4+\sqrt{3}\sqrt{4-5c_2^2})) \\ &-\frac{15}{2}c_2Sec\left(\frac{1}{2}\sqrt{-c_2}(-2(4+\sqrt{3}\sqrt{4-5c_2^2})t+x)\right)^2, c_2 > 0 \end{aligned}$$

and

$$\begin{aligned} u &= \frac{1}{4}(10c_2 - 2(4+\sqrt{3}\sqrt{4-5c_2^2})) \\ &-\frac{15}{2}c_2Sech\left(\frac{1}{2}\sqrt{-c_2}(-2(4+\sqrt{3}\sqrt{4-5c_2^2})t+x)\right)^2, c_2 < 0 \end{aligned}$$

Now take $m = 2$, then $r = 4$, and

$$
\begin{aligned}
u(\xi) &= a_0 + a_1\phi(\xi) + a_2\phi^2(\xi), \\
(\phi')^2 &= c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3 + c_4\phi^4.
\end{aligned} \tag{21}
$$

Then solve Eq. 21 and substitute the solutions for $u(\xi)$ into Eq. 14. The following solutions will be found:

If $c_0 = c_1 = c_2 = c_3 = 0, c_4 > 0$, then $\phi = -\frac{1}{\sqrt{c_4}\xi}$, and
$u_1 = -2 - \sqrt{3} + \frac{30}{(x-ct)^2}$, where $c = 4(2 \mp \sqrt{3})$.
If $c_0 = c_1 = c_3 = 0, c_2 < 0, c_4 > 0$, then $\phi = \sqrt{\frac{-c_2}{c_4}} Sec(\sqrt{-c_2}\xi)$, and
$u_2 = \frac{1}{4}(40c_2 - c) - 30c_2 Sec^2\left(\sqrt{-c_2}(x-ct)\right)$, where $c = 4(2 \mp \sqrt{3}\sqrt{1-20c_2^2})$.

If $c_1 = c_3 = 0, c_0 = \frac{c_2^2}{4c_4}, c_2 > 0, c_4 > 0$, then $\phi = \sqrt{\frac{c_2}{2c_4}} Tan(\sqrt{\frac{c_2}{2}}\xi)$, and
$u_3 = \frac{1}{4}(40c_2 - c) + 15c_2 Tan^2\left(\frac{\sqrt{c_2}(x-ct)}{\sqrt{2}}\right)$, where $c = 4(2 \mp \sqrt{3}\sqrt{1-5c_2^2})$.

If $c_0 = c_1 = c_3 = 0, c_2 > 0, c_4 < 0$, then $\phi = \sqrt{\frac{-c_2}{c_4}} Sech(\sqrt{c_2}\xi)$and
$u_4 = \frac{1}{4}(40c_2 - c) - 30c_2 Sech^2\left(\sqrt{c_2}(x-ct)\right)$, where $c = 4(2 \mp \sqrt{3}\sqrt{1-20c_2^2})$.

If $c_1 = c_3 = 0, c_0 = \frac{c_2^2}{4c_4}, c_2 < 0, c_4 > 0$, then $\phi = \sqrt{\frac{-c_2}{2c_4}} Tanh\left(\sqrt{\frac{-c_2}{2}}\xi\right)$, and
$u_5 = \frac{1}{4}(40c_2 - c) - 15c_2 Tanh^2\left(\frac{\sqrt{-c_2}(x-ct)}{\sqrt{2}}\right)$, where $c = 4(2 \mp \sqrt{3}\sqrt{1-5c_2^2})$.

If $c_0 = c_1 = 0, c_2 < 0$, then $\phi = -\frac{c_2 Sec^2[\frac{\sqrt{-c_2}}{2}\xi]}{2\sqrt{-c_2c_4}Tan[\frac{\sqrt{-c_2}}{2}\xi] + c_3}$, and

$$
u_6 = \left[\begin{array}{l}
\dfrac{20c_2c_4 - 2cc_4}{8c_4} + \dfrac{30c_2^2c_4Sec^4[\frac{\sqrt{-c_2}(x-ct)}{2}]}{(2\sqrt{c_2c_4} + 2\sqrt{-c_2c_4}Tan[\frac{\sqrt{-c_2}(x-ct)}{2}])^2} \\
-\dfrac{30c_2^{\frac{3}{2}}\sqrt{c_4}Sec^2[\frac{\sqrt{-c_2}(x-ct)}{2}]}{2\sqrt{c_2c_4} + 2\sqrt{-c_2c_4}Tan[\frac{\sqrt{-c_2}(x-ct)}{2}]}
\end{array}\right],
$$

where, $c = 2(4 \mp \sqrt{3}\sqrt{4-5c_2^2})$.

If $c_0 = c_1 = 0, c_2 > 0$, then $\phi = \frac{c_2 Sech^2[\frac{\sqrt{c_2}}{2}\xi]}{2\sqrt{c_2c_4}Tanh[\frac{\sqrt{c_2}}{2}\xi] - c_3}$, and

$$
u_7 = \left[\begin{array}{l}
\dfrac{20c_2c_4 - 2cc_4}{8c_4} + \dfrac{30c_2^2c_4Sech^4[\frac{\sqrt{c_2}(x-ct)}{2}]}{(-2\sqrt{c_2c_4} + 2\sqrt{c_2c_4}Tanh[\frac{\sqrt{c_2}(x-ct)}{2}])^2} \\
+\dfrac{30c_2^{\frac{3}{2}}\sqrt{c_4}Sech^2[\frac{\sqrt{c_2}(x-ct)}{2}]}{-2\sqrt{c_2c_4} + 2\sqrt{c_2c_4}Tanh[\frac{\sqrt{c_2}(x-ct)}{2}]}
\end{array}\right],
$$

where $c = 2(4 \mp \sqrt{3}\sqrt{4 - 5c_2^2})$.
The following figures display different type of solutions.

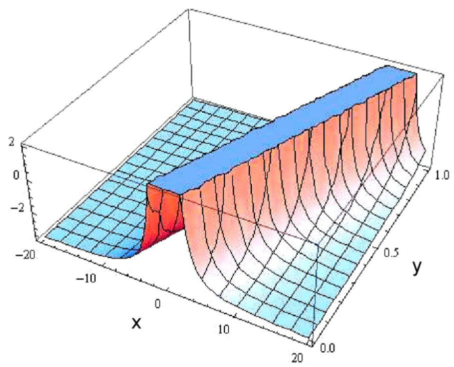

Fig. 1. The peakon solution $u_1 = -2 - \sqrt{3} + \frac{30}{(x-ct)^2}$.

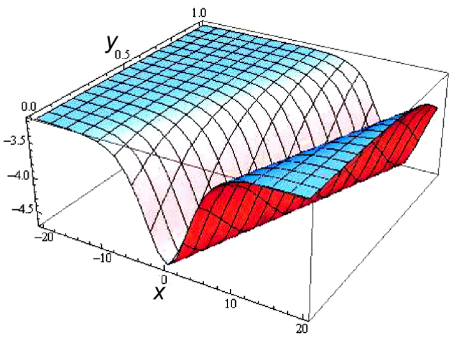

Fig. 2. The soliton solution u_7 when c_2=1/5, and c_4=1.

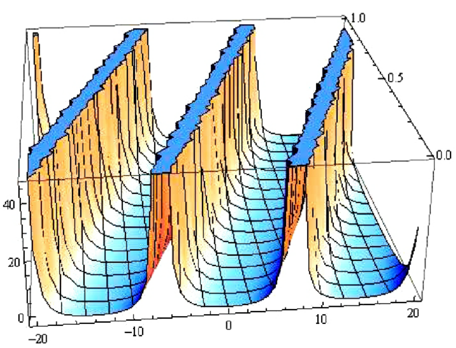

Fig. 3. The periodic solution u_6 when c_2= -1/5, and c_4=1.

Conclusion

As seen from above, applying this method we've got peakon, periodic and solitary wave solutions to the modified Fornberg-Whitham equation that are not found by the bifurcation theory and the method of phase portraits analysis. So this method is very useful to construct several kinds of exact solutions for nonlinear partial differential equations.

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Ameina S. Nuseir
Department of Mathematics and Statistics
Jordan University of Science and Technology
P. O. Box 3030
Irbid 22110
Jordan
E-mail: anuseir@just.edu.jo