

FEEDBACK VERTEX SET ON PLANAR GRAPHS

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Abstract. A feedback vertex set of a graph is a set of vertices whose removal results an acyclic graph. This paper shows that for every planar graph the minimum cardinality of a feedback vertex set is at most three times the maximum number of vertex disjoint cycles in the graph.

1. INTRODUCTION

A feedback vertex set of a graph is a set of vertices whose removal results an acyclic graph. In other words, each feedback vertex set contains at least one vertex of any cycle in the graph. The feedback vertex set problem plays an important role in the study of deadlock recovery in operating systems [16] where each directed cycle corresponds to a deadlock situation. Indeed, a deadlock is a situation wherein two or more competing processes are each waiting for the other to finish, and thus neither ever does. To resolve all deadlocks, some blocked processes have to be aborted. Therefore, a minimum feedback vertex set corresponds to the least processes needed to be aborted to resolve all deadlocks.

The feedback vertex set problem, as well as its undirected or weighted version, has been extensively studied. In fact, it is one of the first NP-complete problems shown by Richard Karp in 1972 [11]. In contrast, the problem of finding a minimum edge set containing at least one edge of any cycle is equivalent to finding a spanning tree, which has been shown solvable in polynomial time. A vast amount of algorithmic results on the feedback vertex set problem have been proposed, including approximation algorithms [1, 2, 4, 10], APX-completeness [5], exact algorithms [8, 9, 13, 15] and enumeration algorithms [8]. Readers are referred to the survey [7] for an overview.

Throughout this paper, we consider finite and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$ without loops or multiple edges. The *feedback number* of a graph G , denoted as $\tau_c(G)$, is the minimum size of a feedback vertex set of G . Clearly, $\tau_c(G) \geq \nu_c(G)$ for every graph G , where $\nu_c(G)$ is the *cycle packing*

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number of G , i.e., the maximum size of a set of vertex disjoint cycles in G . Dirac and Gallai wondered if there is any inverse relation between $\tau_c(G)$ and $\nu_c(G)$. Define $\tau_c(k) = \max\{\tau_c(G) \mid \nu_c(G) \leq k\}$. Bollobás [3] proved that $\tau_c(1) \leq 3$ and the complete graph of five vertices shows that this bound is sharp. Later, Voss [17] showed that $\tau_c(2) = 6$ and $9 \leq \tau_c(3) \leq 12$. Erdős and Pósa [6] proved that $c_1\nu_c(k) \log \nu_c(k) \leq \tau_c(k) \leq c_2\nu_c(k) \log \nu_c(k)$ by a probabilistic argument, where c_1 and c_2 are constants.

This paper mainly focuses on such a relation between $\tau_c(G)$ and $\nu_c(G)$ for planar graphs. Kloks, Lee and Liu [12] noticed that $\tau_c(W) = 2\nu_c(W)$ for every wheel W , which is a special case of planar graphs, and conjectured the following.

Conjecture 1. (Kloks-Lee-Liu [12]). *For every planar graph G , $\tau_c(G) \leq 2\nu_c(G)$.*

They also verified the conjecture for the special case of outerplanar graphs and proved a general result that $\tau_c(G) \leq 5\nu_c(G)$ for every planar graph G by using a constructive algorithm. The present paper modifies Kloks-Lee-Liu's constructive algorithm and improves upon their result. More precisely, we prove that $\tau_c(G) \leq 3\nu_c(G)$ for every planar graph G . Instrumental to the main results is a theoretical result on planar graphs proved by using the discharging method.

2. MAIN RESULTS

This section starts with some definitions and notations used later. Readers are referred to the book [18] for graph terminologies. The *degree* of a vertex v in G , denoted as d_v , is the number of edges incident to v . For the sake of brevity, a vertex of degree d is denoted by a *d-vertex*. A *cycle* in G is a connected subgraph of G in which all vertices have degree 2. Let C_k denote a cycle of length k , i.e., a cycle with k edges. A graph is called *triangle-free* if it contains no C_3 as its subgraph. A *plane graph* is a planar graph embedded in the plane without crossing edges. A *face* f of a plane graph is a circuit that surrounds a region bounded by edges; let ℓ_f denote the length of f , i.e., the number of surrounding edges. For a plane graph G , let $F(G)$ be the set of faces of the embedding. *Euler's formula* states that for every plane graph G ,

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

We now prove a theoretical result of graph theory, which is an essential instrument of our algorithm.

Lemma 2. *Every 2-edge-connected triangle-free planar graph G with minimum degree at least three has either a C_4 containing a 3-vertex or a C_5 containing at least four 3-vertices.*

Proof. Let G be embedded in the plane. We charge each vertex v of degree d_v by $4 - d_v$ and each face f of length ℓ_f by $4 - \ell_f$. The total charge, by Euler's formula, is

$$\sum_{v \in V(G)} (4 - d_v) + \sum_{f \in F(G)} (4 - \ell_f) = 4|V(G)| - 2|E(G)| + 4|F(G)| - 2|E(G)| = 8.$$

Discharge every 3-vertex equally to its incident faces. Note that each 3-vertex in a 2-edge-connected plane graph must be exactly incident to 3 faces. According to the discharging rule, each 3-vertex gives a charge of $\frac{1}{3}$ to each of three faces incident to it. Thus, each 3-vertex has a final charge 0. As a result, every vertex has a non-positive charge after discharging. This implies some face f must have a positive charge after discharging since the initial total charge is positive. Thus, we have

$$4 - \ell_f + \frac{1}{3}k > 0,$$

where k is the number of 3-vertices in the face f . The above inequality holds only when $\ell_f \leq 5$ since $k \leq \ell_f$. It is easily verified that satisfaction of the inequality when $\ell_f = 4, 5$ would lead to the conclusion of this lemma. ■

Everything now is prepared to find a feedback vertex set on a given planar graph. Our algorithm starts with an empty set \mathcal{F} and goes step by step as follows.

- A. Remove all vertices and edges not lying on any cycle. Notice that the resulting graph will be 2-edge-connected. Once no vertex exists, then the process stops and outputs \mathcal{F} .
- B. Repeatedly remove from the resulting graph 2-vertices (vertices of degree 2) that have nonadjacent neighbors and connect an edge between these two neighbors. Go to the next step.
- C. If there is a C_3 , then take these three vertices into \mathcal{F} and remove them from the remaining graph, and go back step A. Otherwise, do the next step.
- D. Remark that the process enters this step only when all vertices are of degree at least 3 and no C_3 exists. By Lemma 2, there must be either a C_4 containing a 3-vertex or a C_5 containing at least four 3-vertices. In the former case, take the three vertices other than the 3-vertex into \mathcal{F} and remove them, then go back step A. In the later case, there must be at least two 3-vertices that are nonadjacent in the C_5 . Take the other three vertices into \mathcal{F} and remove them, then go back step A.

Theorem 3. *For every planar graph G , $\tau_c(G) \leq 3\nu_c(G)$.*

Proof. The proof is based on the above algorithm. We first show that the output \mathcal{F} is a feedback vertex set of G . Note that steps A and B do not affect the result since any feedback vertex set of the remaining graph is a feedback vertex set of G . Hence, once the algorithm terminates, \mathcal{F} is a feedback vertex set since all cycles are broken when \mathcal{F} is removed from G . Consequently, $\tau_c(G) \leq |\mathcal{F}|$.

It then suffices to show that $|\mathcal{F}| \leq 3\nu_c(G)$. Notice that the algorithm collects at most 3 vertices a time as long as it finds a cycle satisfying the requirement in steps C or D. Moreover, these cycles are mutually vertex disjoint because all vertices on such a cycle are removed from G once the cycle is found. Though step D leaves some 3-vertices behind, they do not lie on any other cycles discovered later because two neighbors of each of them are removed. According to the discussion above, \mathcal{F} contains at most 3 times the maximum number of vertex disjoint cycles of G . ■

3. CONCLUSION

The algorithm presented in this paper strengthens and improves the algorithm in [12]. Instead of removing all vertices on each discovered small cycle, our algorithm collects only part of vertices on each discovered cycle according to the observation that every 3-vertex which has two neighbors collected in a feedback vertex set is redundant and unnecessary for the feedback vertex set. However, this algorithmic approach seems difficult to be further improved to match the conclusion of the Kloks-Lee-Liu's Conjecture, which we believe to be affirmative. The main reason is that little is known about whether vertices on a discovered cycle left behind after removing at most 2 vertices from it are no longer lying on any cycle.

The most interesting idea in this paper is to exploit the discharging method, which was most well-known for its central role in the proof of the four color theorem [14]. We believe that the result presented in this paper does not leverage full potential of the method due to the limitation of the proposed algorithmic approach. A possible research direction for further improvement is to design creative discharging rules to prove the Kloks-Lee-Liu's Conjecture directly. For example, one may derive a contradiction to minimal counterexamples in terms of reducible configurations which are formulated by the discharging method. ■

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