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ON CIRCULAR-$L(2,1)$-EDGE-LABELING OF GRAPHS

Wensong Lin and Jianzhuan Wu

Abstract. Let m, j and k be positive integers with $j \geq k$. An m-circular-$L(j,k)$-edge-labeling of a graph G is an assignment f from $\{0,1,\ldots,m-1\}$ to the edges of G such that, for any two edges e_1 and e_2, $|f(e_1)-f(e_2)|_m \geq j$ if e_1 and e_2 are adjacent, and $|f(e_1)-f(e_2)|_m \geq k$ if e_1 and e_2 are at distance 2, where $|a|_m = \min\{a, m-a\}$. The minimum m such that G has an m-circular-$L(j,k)$-edge-labeling is defined as the circular-$L(j,k)$-edge-labeling number of G, denoted by $\sigma'_{j,k}(G)$. This paper determines the circular-$L(2,1)$-edge-labeling numbers of the infinite Δ-regular tree for $\Delta \geq 2$ and the n-dimensional cube for $n \in \{2,3,4,5\}$.

1. INTRODUCTION

Let j and k be two positive integers with $j \geq k$. An $L(j,k)$*-labeling* of a graph G is an assignment of nonnegative integers, called labels, to the vertices of G such that the difference between labels of any two vertices at distance one is at least j, and the difference between labels of any two vertices that are distance two apart is at least k. Given a graph G, for an $L(j,k)$-labeling f of G, we define the *span* of f, $span(f)$, to be the absolute difference between the maximum and minimum vertex labels of f. The $L(j,k)$*-labeling number* of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelings of G.

Motivated from the channel assignment problem introduced by Hale [6], Griggs and Yeh [5] first proposed and studied the $L(2,1)$-labeling of a graph. Since then the $L(2,1)$-labelings and $L(j,k)$-labelings of graphs have been studied extensively, please refer to the surveys [1, 4, 14].

One interesting variation of $L(j,k)$-labeling number is the so called circular-$L(j,k)$-labeling number, which was introduced by Heuvel, Leese, and Shepherd in [7].

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Let m, j and k be positive integers with $j \geq k$. An *m-circular-$L(j,k)$-labeling* of a graph G is an assignment f from $\{0, 1, \ldots, m-1\}$ to the vertices of G such that, for any two vertices u and v, $|f(u)-f(v)|_m \geq j$ if $uv \in E(G)$, and $|f(u)-f(v)|_m \geq k$ if $d_G(u,v) = 2$, where $|a|_m = \min\{a, m-a\}$. The minimum m such that G has an m-circular-$L(j,k)$-labeling is called the *circular-$L(j,k)$-labeling number* of G, denoted by $\sigma_{j,k}(G)$.

Heuvel, Leese and Shepherd in [7] determined the circular-$L(j,k)$-labeling numbers of triangular lattice and square lattice for any two positive integers j and k with $j \geq k$. The relationship between the circular-$L(2,1)$-labeling number of a graph G and the path covering number of its complement was revealed by Liu in [9]. Wu and Yeh [13] showed that $\sigma_{j,1}(T) = 2j + \Delta - 1$ for any tree T with maximum degree Δ. In [12, 11], it was proved that, for $j \geq k$, $\sigma_{j,k}(T) = 2j + (\Delta - 1)k$ for any tree T with maximum degree Δ. The circular-$L(j,k)$-labeling numbers of cycles for $j \geq k$ were completely determined in [11]. In [8], the circular-$L(j,k)$-labeling numbers of the Cartesian product of two complete graphs, the direct product of two complete graphs for $j \geq k$ were determined. Recently, the circular-$L(2,1)$-labeling numbers of the Cartesian products of three complete graphs were obtained in [10].

Let $G = (V(G), E(G))$ be a graph. Denote by $L(G)$ the line graph of G. Let $\Delta(G)$ denote the maximum degree of G and $\Delta_L(G)$ the maximum edge degree of G (or equivalently the maximum degree of $L(G)$). Let e_1 and e_2 be any two edges of G. The distance between e_1 and e_2, denoted by $d(e_1, e_2)$, is defined as the distance between the corresponding two vertices in the line graph of G.

The edge version of $L(j,k)$-labeling and circular-$L(j,k)$-labeling of a graph G are defined as the $L(j,k)$-labeling and the circular-$L(j,k)$-labeling of $L(G)$, respectively. The *$L(j,k)$-edge-labeling number* of G is denoted by $\lambda'_{j,k}(G)$ and the *circular-$L(j,k)$-edge-labeling number* of G is denoted by $\sigma'_{j,k}(G)$.

The edge version of distance two labeling was first investigated by Georges and Mauro in [3]. Several classes of graphs were studied by Georges and Mauro. Among them, they determined the $L(2,1)$-edge-labeling numbers of Δ-regular tree for $\Delta \geq 2$ and the n-dimensional cube for small n.

The following theorem was proved by Chen and Lin in [2].

Theorem 1.1. *Let G be a simple graph and let Δ be the maximum degree of G. Suppose $\Delta \geq 2$. If G is $K_{1,3}$-free then, except the case that G is a 5-cycle and $j = k$, we have $\lambda_{j,k}(G) \leq k\lfloor \Delta^2/2 \rfloor + j\Delta - 1$.*

Since a line graph is $K_{1,3}$-free, the upper bound for $\lambda_{j,k}(G)$ in this theorem obviously holds for all line graphs, and hence $\lambda'_{j,k}(G) \leq k\lfloor \Delta_L^2/2 \rfloor + j\Delta_L - 1$ holds for any graph G.

With this result, Chen and Lin in [2] proved that the conjecture "$\lambda_{2,1}(G) \leq \Delta^2(G)$" (Griggs and Yeh [5]) holds for all $K_{1,3}$-free graphs and hence for all line graphs.

[3] and [2] are the only references we have found in the literature concerning the

$L(j,k)$-edge-labeling of graphs.

The following lemma was mentioned by Heuvel, Leese, and Shepherd in [7].

Lemma 1.1. *For any graph G, we have* $\lambda_{j,k}(G)+1 \le \sigma_{j,k}(G) \le \lambda_{j,k}(G)+j$.

We would like to point out that even in the case when $j=2$ and $k=1$ it is not easy to determine whether $\sigma_{2,1}(G)$ equals $\lambda_{2,1}(G)+1$ or $\lambda_{2,1}(G)+2$ provided that $\lambda_{2,1}(G)$ is known. We obviously have the edge version of Lemma 1.1.

Lemma 1.2. *For any graph G, we have* $\lambda'_{j,k}(G)+1 \le \sigma'_{j,k}(G) \le \lambda'_{j,k}(G)+j$.

In this paper, we determine the circular-$L(2,1)$-edge-labeling numbers of the infinite Δ-regular tree for any $\Delta \ge 2$ and the n-dimensional cube for $n \in \{2,3,4,5\}$, and as a consequence, the $L(2,1)$-edge-labeling number of the 5-dimensional cube.

For a positive real number r, let $S(r)$ denote the circle obtained from the interval $[0,r]$ by identifying 0 and r into a single point. For any $x \in \mathbb{R}$, $[x]_r \in [0,r)$ denotes the remainder of x upon division of r. For $a,b \in S(r)$, the interval $[a,b]_r$ is defined as $[a,b]_r = \{x \in S(r) : 0 \le [x-a]_r \le [b-a]_r\}$. And similarly, the open interval $(a,b)_r$ is defined as $(a,b)_r = \{x \in S(r) : 0 < [x-a]_r < [b-a]_r\}$. The length of the interval $[a,b]_r$ is equal to $[b-a]_r$. Two points $a,b \in S(r)$ partition $S(r)$ into two arcs: $[a,b]_r$ and $[b,a]_r$. The *circular distance* between a and b, denoted by $|a-b|_r$, is the length of the shorter arc. In other words, $|a-b|_r = \min\{[a-b]_r, [b-a]_r\} = \min\{|a-b|, r-|a-b|\}$.

A set of points on $S(r)$ is said to be $(r,2)$*-circular separated* if any two elements from the set are at circular distance at least 2 on $S(r)$. A sequence of points $a_1, a_2, \ldots, a_k$ on $S(r)$ are said to be in *cyclic order* if $(a_1,a_2)_r, (a_2,a_3)_r, \ldots, (a_{k-1},a_k)_r, (a_k,a_1)_r$ are pairwise disjoint open intervals on $S(r)$.

2. Circular-$L(2,1)$-Edge-labeling Numbers of Δ-Regular Trees

Let $\Delta(\ge 2)$ be any integer. A Δ-regular tree is an infinite tree with each vertex having degree Δ. Denote by $T_\infty(\Delta)$ the infinite Δ-regular tree. If $\Delta = 2$, then $T_\infty(\Delta)$ is an infinite path and $\lambda'_{2,1}(T_\infty(\Delta)) = 4$, $\sigma'_{2,1}(T_\infty(\Delta)) = 5$. For $\Delta > 2$, Georges and Mauro proved the following.

Theorem 2.1. ([3]). *Let Δ be a positive integer greater than 2. We have*

$$\lambda'_{2,1}(T_\infty(\Delta)) = \begin{cases} 2\Delta+1, & \text{if } \Delta = 3,4, \\ 2\Delta+2, & \text{if } \Delta = 5, \\ 2\Delta+3, & \text{if } \Delta \ge 6. \end{cases}$$

Hereafter in this section, we assume $\Delta > 2$. Let $e = xy$ be an edge and v a vertex. The distance between e and v, $d(e,v)$, is defined as $\min\{d(x,v), d(y,v)\}$.

Suppose we set a vertex v_0 as the center of the tree $T_\infty(\Delta)$. If $e = xy$ is an edge with $d(x, v_0) + 1 = d(y, v_0)$, then we call x the *father* of y and y a *son* of x. Our main theorem in this section is the following.

Theorem 2.2. *Let* Δ *be a positive integer greater than* 2. *We have*

$$\sigma'_{2,1}(T_\infty(\Delta)) = \begin{cases} 2\Delta + 2, & \text{if } \Delta = 3, \\ 2\Delta + 3, & \text{if } \Delta = 4, 5, \\ 2\Delta + 4, & \text{if } \Delta \geq 6. \end{cases}$$

Proof. By Theorems 2.1 and Lemma 1.2, we have $2\Delta + 2 \leq \sigma'_{2,1}(T_\infty(3)) \leq 2\Delta + 3$, $2\Delta + 2 \leq \sigma'_{2,1}(T_\infty(4)) \leq 2\Delta + 3$, $2\Delta + 3 \leq \sigma'_{2,1}(T_\infty(5)) \leq 2\Delta + 4$, and $2\Delta + 4 \leq \sigma'_{2,1}(T_\infty(\Delta)) \leq 2\Delta + 5$ if $\Delta \geq 6$. The proof is split into the following four cases. ■

Case 1. $\Delta = 3$.

We show $\sigma'_{2,1}(T_\infty(3)) = 2\Delta + 2 = 8$ by giving an 8-circular-$L(2,1)$-edge-labeling of $T_\infty(3)$. Let v_0 be a vertex of $T_\infty(3)$. We shall label its edges in the order according to the distance from v_0 to the edges. We first label the 3 edges at distance 0 from v_0 (i.e. the edges incident to v_0), and then label the 6 edges at distance 1 from v_0, and so on. Clearly the first three edges can be labeled properly. Suppose all edges at distance less than i from v_0 have been labeled. We then label the edges at distance i from v_0 in a greedy way. For any two adjacent edges e and e' at distance i from v_0, notice that there are only three labeled edges, one at distance 1 from them and two at distance 2 from them, thus the number of labels that are forbidden for these two edges is at most 5. It follows that there are at least three labels available for e and e' and therefore we can label them properly. In this way, one can construct an 8-circular-$L(2,1)$-edge-labeling of $T_\infty(3)$. Therefore $\sigma'_{2,1}(T_\infty(3)) = 2\Delta + 2 = 8$.

Case 2. $\Delta = 4$.

We show $\sigma'_{2,1}(T_\infty(4)) = 2\Delta + 3 = 11$ by proving that there is no 10-circular-$L(2,1)$-edge-labeling of $T_\infty(4)$. Suppose to the contrary that f is a 10-circular-$L(2,1)$-edge-labeling of $T_\infty(4)$. We shall reach a contradiction. Suppose the labels used by f are $0, 1, \ldots, 9$.

Let v be any vertex and let e_0, e_1, e_2, e_3 be the four edges incident to v. Then the set of labels assigned to them should be $(10, 2)$-circular separated. Without loss of generality, assume that $f(e_0), f(e_1), f(e_2), f(e_3)$ occur in $S(10)$ in this cyclic order. For $i = 0, 1, 2, 3$, let x_i denote the number of integer points in the open interval $(f(e_i), f(e_{i+1}))_{10}$, where "+"s in the subscripts are taken modulo 4. Then $1 \leq x_i \leq 3$ for all $i = 0, 1, 2, 3$ and $x_0 + x_1 + x_2 + x_3 = 10 - 4 = 6$. Each solution of this equation corresponds to an ordered 4-tuple (x_0, x_1, x_2, x_3). Two ordered 4-tuples (and so the corresponding two solutions) are said to be equivalent if one can be obtained from the other by shifting each of its elements cyclically. For example, $(2, 2, 1, 1)$ is equivalent to each of $(1,2,2,1)$, $(1,1,2,2)$, and $(2,1,1,2)$. Therefore, it is easy to see that the above system has only three non-equivalent integer solutions: $S_1 = (2,2,1,1)$, $S_2 = (3,1,1,1)$, and $S_3 = (2,1,2,1)$.

We say that a vertex w is of type S_i ($i = 1,2,3$) if the corresponding system described in the previous paragraph has solution S_i. Let w be any vertex and $e_i = wu_i$ ($i = 0,1,2,3$) are the four edges incident to w. We shall get contradictions no matter of what type w is, thus complete the proof for the case $\Delta = 4$.

If $(x_0, x_1, x_2, x_3) = S_1 = (2,2,1,1)$, since the labels on $S(10)$ are cyclic, we may assume that the labels assigned to e_0, e_1, e_2, e_3 are $0,3,6,8$, respectively. It follows that there are only four labels $1,2,4,5$ which are legal for the three edges incident with u_3 other than wu_3. It is clear that we can not label them properly. Therefore, there is no vertex of type S_1 in any 10-circular-$L(2,1)$-edge-labeling of $T_\infty(4)$.

If $(x_0, x_1, x_2, x_3) = S_2 = (3,1,1,1)$, then we may assume that the labels assigned to e_0, e_1, e_2, e_3 are $0,4,6,8$, respectively. This implies that the four labels assigned to the four edges incident to u_3 should be $1,3,5,8$, which is of type S_1, contradicting the previous case.

If $(x_0, x_1, x_2, x_3) = S_3 = (2,1,2,1)$, then we may assume that the labels assigned to e_0, e_1, e_2, e_3 are $0,3,5,8$, respectively. It is easy to check that the four labels assigned to the four edges incident to u_3 should be $1,4,6,8$, or $2,4,6,8$, which are of types S_1 and S_2 respectively. This is a contradiction. Hence $T_\infty(4)$ has no 10-circular-$L(2,1)$-edge-labeling. The proof of Case 2 is completed.

Case 3. $\Delta = 5$.

We show $\sigma'_{2,1}(T_\infty(5)) = 2\Delta + 3 = 13$ by constructing a 13-circular-$L(2,1)$-edge-labeling of $T_\infty(5)$.

Let v be any vertex and let e_0, e_1, e_2, e_3, e_4 be five edges incident to v. Then the set of labels assigned to them should be $(13,2)$-circular separated. Without loss of generality, assume that $f(e_0), f(e_1), f(e_2), f(e_3), f(e_4)$ occur in $S(13)$ in this cyclic order. For $i = 0,1,2,3,4$, let x_i denote the number of integer points in the open interval $(f(e_i), f(e_{i+1}))_{13}$, where "+"s in the subscripts are taken modulo 5. Then $1 \le x_i \le 4$ for all $i = 0,1,2,3,4$ and $x_0 + x_1 + x_2 + x_3 + x_4 = 13 - 5 = 8$. Each solution of this equation corresponds to an ordered 5-tuple $(x_0, x_1, x_2, x_3, x_4)$. Two ordered 5-tuples (and so the corresponding two solutions) are said to be equivalent if one can be obtained from the other by shifting each of its elements cyclically or by reversing the order of its elements. For example, $(3,2,1,1,1)$ is equivalent to $(1,3,2,1,1)$ and $(1,1,1,2,3)$. The type of a vertex is defined similarly as in Case 2.

In the following, we shall label all vertices of $T_\infty(5)$ such that each vertex is of one of the three types: $S_1 = (3,2,1,1,1)$, $S_2 = (3,1,2,1,1)$, and $S_3 = (2,2,1,2,1)$. We first choose a vertex v_0 and label the five edges incident to it such that v_0 becomes one of the above three types. We then label edges at distance 1 from v_0, and so on. Each time when we are at a vertex v (other than v_0) with exactly one incident edge labeled, we try to label the other four edges and make that vertex of one of the above three types. The only thing we need to prove is that, no matter what the type of its father is, we can always properly label the other four unlabeled edges incident to the vertex v and make it to be one of the three types. Let w be the father of v. We split the proof into three cases according to the type of w.

Case 3.1. w is of type $S_1 = (3, 2, 1, 1, 1)$.
With no loss of generality, we may assume that the five labels assigned to the five edges incident to w are $0, 4, 7, 9, 11$. If wv is labeled by 0, then we can label the four other edges by $2, 5, 8, 10$ and make v be of type S_3. If the label of wv is 4, then we label the remaining four edges by $2, 6, 8, 12$ and thus make v be of type S_1. If the label of wv is 7, then we label the remaining four edges by $1, 5, 10, 12$ and make v be of type S_2. If the label of wv is 9, then we label the remaining four edges by $1, 3, 5, 12$ and make v be of type S_1. If the label of wv is 11, then we label the remaining four edges by $1, 3, 6, 8$ and make v be of type S_3.

Case 3.2. w is of type $S_2 = (3, 1, 2, 1, 1)$.
With no loss of generality, we may assume that the five labels assigned to the five edges incident to w are $0, 4, 6, 9, 11$. If wv is labeled by 0, then we can label the four other edges by $2, 5, 7, 10$ and make v be of type S_3. If the label of wv is 4, then we label the remaining four edges by $2, 7, 10, 12$ and thus make v be of type S_3. If the label of wv is 6, then we label the remaining four edges by $1, 3, 8, 10$ and make v be of type S_2. If the label of wv is 9, then we label the remaining four edges by $3, 5, 7, 12$ and make v be of type S_1. If the label of wv is 11, then we label the remaining four edges by $1, 3, 5, 7$ and make v be of type S_1.

Case 3.3. w is of type $S_3 = (2, 2, 1, 2, 1)$.
With no loss of generality, we may assume that the five labels assigned to the five edges incident to w are $0, 3, 6, 8, 11$. If wv is labeled by 0, then we can label the four other edges by $2, 5, 7, 10$ and make v be of type S_3. If the label of wv is 3, then we label the remaining four edges by $1, 5, 9, 12$ and thus make v be of type S_1. If the label of wv is 6, then we label the remaining four edges by $1, 4, 9, 12$ and make v be of type S_3. If the label of wv is 8, then we label the remaining four edges by $1, 5, 10, 12$ and make v be of type S_1. If the label of wv is 11, then we label the remaining four edges by $2, 5, 7, 9$ and make v be of type S_1.

Case 4. $\Delta \geq 6$.
We shall recursively define a $(2\Delta + 4)$-circular-$L(2, 1)$-edge-labeling of $T_\infty(\Delta)$ for $\Delta \geq 6$.

Choose any vertex v_0 of $T_\infty(\Delta)$. For all positive integers k, let $W_k = \{u|\, d(u, v_0) = k\}$. Let X_0 denote the set of even labels $0, 2, \ldots, 2\Delta + 2$ and X_1 denote the set of odd labels $1, 3, \ldots, 2\Delta + 3$. We first assign any Δ different labels from X_0 to the Δ edges incident to v_0. Suppose all edges at distance less than k from v_0 have been labeled.

The next step is to label all edges at distance k from v_0. We do it by considering vertices in W_k one by one. Select any vertex $u \in W_k$ that is not considered yet. Let w be the father of u and let t be the father of w if $k \geq 2$. Clearly, exactly one edge wu incident to u has been labeled at this moment. Let h be the label assigned to wu and r the label assigned to tw. We then label the remaining $\Delta - 1$ edges incident to u with $\Delta - 1$ distinct labels from $X_i - \{h-1, h+1, r\}$, where $i = 0$ if k is even and $i = 1$ if k is odd. We can always do this since $|X_i - \{h-1, h+1, r\}| \geq \Delta - 1$. Since X_i is $(2\Delta+4, 2)$-circular separated and since the $\Delta - 1$ edges incident to w other than tw are labeled with labels from X_{1-i}, the labeling constructed in this way is proper.

3. Circular-$L(2,1)$-Edge-labeling Numbers of n-Cubes for $n \leq 5$

For an integer $n \geq 2$, the *n-dimensional cube*, denoted by Q_n, is the simple graph whose vertices are the n-tuples with entries in $\{0,1\}$ and whose edges are the pairs of n-tuples that differ in exactly one position. The vertices of Q_n will be denoted by binary bit strings of length n. Let E' be a set of edges. Denote by $N(E')$ the set of edges that are adjacent to at least one edge in E'. By $\overline{N}(E')$ we denote the set $N(E') \cup E'$. In case $E' = \{e\}$, we simply write as $N(e)$ and $\overline{N}(e)$.

For $1 \leq i \leq n$, let E_i denote the set of edges whose two endvertices differ only in the ith coordinate. Let uv be an edge in E_i. Denote by $\xi_i(uv)$ the sum of coordinates of u except the ith one. For $h = 0, 1$, let E_i^h denote the set of edges uv in E_i with $\xi_i(uv) \equiv h \pmod 2$. The following six observations were made by Georges and Mauro in [3].

(A1) Each E_i is a perfect matching in Q_n; hence $|E_i| = 2^{n-1}$ and no two edges in E_i are adjacent.

(A2) The set $\{E_1, E_2, \ldots, E_n\}$ is a partition of $E(Q_n)$.

(A3) For $i \in \{1, 2 \ldots, n\}$, The set $\{E_i^0, E_i^1\}$ is a partition of E_i, and for $h \in \{0,1\}$, $|E_i^h| = 2^{n-2}$ and the edges in E_i^h are pairwise at distance at least three.

(A4) For $n \geq 2$ and $i \in \{1, 2 \ldots, n\}$, $Q_n - E_i$ is isomorphic to the disjoint union of two copies of Q_{n-1}.

(A5) For $h \in \{0,1\}$ and $i \in \{1, 2 \ldots, n\}$, every edge in $Q_n - E_i$ is adjacent to some edge in E_i^h.

(A6) For $n \geq 2$, if $X \subseteq E(Q_n)$ with $|X| = 2^{n-2}$ such that elements of X are pairwise at distance at least three, then $X = E_i^h$ for some $i \in \{1, 2 \ldots, n\}$ and $h \in \{0,1\}$.

Georges and Mauro [3] proved that $\lambda'_{2,1}(Q_n) \leq 3n - 2$ for $n \geq 2$. In addition, they proved the following.

Theorem 3.1. ([3]).

(1) $\lambda'_{2,1}(Q_2) = 4$,

(2) $\lambda'_{2,1}(Q_3) = 7$,
(3) $\lambda'_{2,1}(Q_4) = 10$,
(4) $\lambda'_{2,1}(Q_5) = 12 or 13$,
(5) $\lambda'_{2,1}(Q_6) = 15 or 16$.

By Lemma 1.2, $\sigma'_{2,1}(Q_n) \leq 3n$ for $n \geq 2$. Our purpose in this section is to prove the following theorem.

Theorem 3.2. *$\sigma'_{2,1}(Q_n) = 3n$ for $n \in \{2, 3, 4, 5\}$. $\lambda'_{2,1}(Q_5) = 13$.*

Proof. By Theorem 3.1 and Lemma 1.2, we have $5 \leq \sigma'_{2,1}(Q_2) \leq 6$, $8 \leq \sigma'_{2,1}(Q_3) \leq 9$, and $11 \leq \sigma'_{2,1}(Q_4) \leq 12$. We prove the theorem case by case. ■

Case 1. $\sigma'_{2,1}(Q_2) = 6$.
Q_2 is a 4-cycle. And the line graph of a 4-cycle is also a 4-cycle. It follows from Theorem 3.3 in [9] that $\sigma'_{2,1}(Q_2) = 6$.

Case 2. $\sigma'_{2,1}(Q_3) = 9$.
Suppose to the contrary that $\sigma'_{2,1}(Q_3) < 9$. Then $\sigma'_{2,1}(Q_3) = 8$. Let L be an 8-circular-$L(2,1)$-edge-labeling of Q_3. For $j = 0, 1, \ldots, 7$, denote by L_j the set of edges labeled by j and let $l_j = |L_j|$. Clearly $\sum_{j=0}^{7} l_j = |E(Q_3)| = 12$. From (A6), we know that $0 \leq l_j \leq 2$, for $j = 0, 1, \ldots, 7$.

If there is some j with $l_j = l_{j+1} = 2$, then by (A5) and (A6), $l_{j-1} = l_{j+2} = 0$, where "+" and "−" in the subscripts are taken modulo 8. It follows that $\sum_{i=0}^{7} l_j < 12$, a contradiction. Thus for each j, $l_j + l_{j+1} \leq 3$. Since $\sum_{j=0}^{7} l_j = 12$, we have $l_j + l_{j+1} = 3$ for each j. With no loss of generality, we assume $l_0 = 2$. Then $l_1 = l_7 = 1$ and $l_2 = 2$. By (A6), $L_0 = E_i^h$ for some i and h. Then by (A5), $L_7 \cup L_1 \subseteq E_i$. And so $L_7 \cup L_0 \cup L_1 = E_i$. But then it is easy to check that the only four edges outside E_i not adjacent to the edge with label 1 are pairwise at distance less than 3. This contradicts $l_2 = 2$. Hence $\sigma'_{2,1}(Q_3) = 9$.

Case 3. $\sigma'_{2,1}(Q_4) = 12$.
Suppose to the contrary that $\sigma'_{2,1}(Q_4) < 12$. Then $\sigma'_{2,1}(Q_4) = 11$. Let L be an 11-circular-$L(2,1)$-edge-labeling of Q_4. For $j = 0, 1, \ldots, 10$, denote by L_j the set of edges labeled by j and let $l_j = |L_j|$. Clearly $\sum_{j=0}^{10} l_j = |E(Q_4)| = 32$. From (A6), we know that $0 \leq l_j \leq 4$, for $j = 0, 1, \ldots, 10$. We first prove the following two properties of the sequence $(l_0, l_1, \ldots, l_{10})$.

Property 1. *For $0 \leq j \leq 10$, if $l_j = 4$ then $L_j = E_i^h$ for some i, h and $L_{j-1} \cup L_{j+1} \subseteq E_i^{1-h}$.*

Proof. If $l_j = 4$ then by (A6) $L_j = E_i^h$ for some i, h. And by (A5), $L_{j-1} \cup L_{j+1} \subseteq E_i^{1-h}$. ■

Property 2. *For $0 \le j \le 10$, if $l_j = 3$ then $L_j \subseteq E_i^h$ for some i, h. Let e be the only edge in $E_i^h \setminus L_j$. We have $L_{j-1} \cup L_{j+1} \subseteq E_i^{1-h} \cup \overline{N}(e)$.*

Proof. Let $L_j = \{e_1, e_2, e_3\}$. We first prove that $L_j \subseteq E_i$ for some i. Suppose to the contrary there are two integers p and q such that $e_1 \in E_p$ and $e_2 \in E_q$. Without loss of generality, let $e_1 = (0101, 0001)$. Denote the edge $(1110, 1010)$ by e_4. Then all edges outside E_p that are at distance greater than 2 from e_1 are in $N(e_4)$. Thus $e_2 \in N(e_4) \setminus E_p$. Note that all edges in $N(e_4)$ are pairwise at distance at most 2. It follows that e_3 should be in E_p. However, it is not difficult to see that any edge in E_p at distance greater than 2 from e_1 is at distance at most 2 from any edge in $N(e_4)$. (See Figure 1 for illustration.) This is a contradiction. Thus $L_j \subseteq E_i$ for some i. Note that for any edge in E_i^h, there is only one edge in E_i^{1-h} that is at distance greater than 2. Therefore $L_j \subseteq E_i^h$ for some i, h.

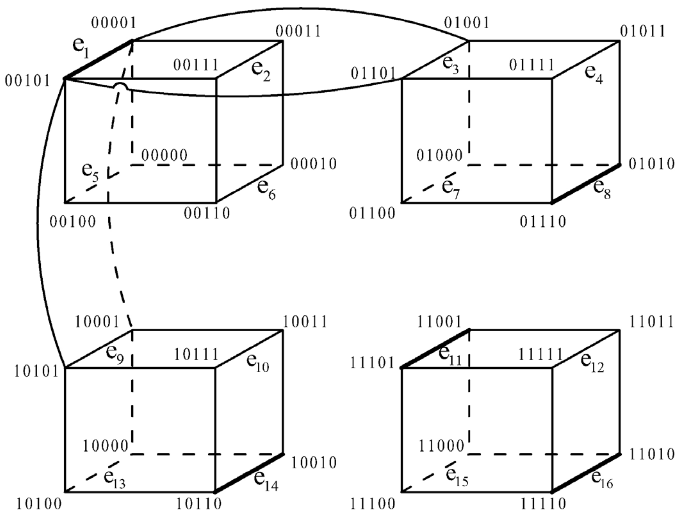

Fig. 1. Q_4 with $e_1 \in L_j$.

Let e be the only edge in $E_i^h \setminus L_j$. By (A5), the edges outside E_i that can be labeled by $j-1$ or $j+1$ are in $N(e)$. Therefore $L_{j-1} \cup L_{j+1} \subseteq E_i^{1-h} \cup N(e) \cup \{e\}$. This proves Property 2. ∎

We next prove that $l_{j-1} + l_j + l_{j+1} \le 8$ for each $j = 0, 1, \ldots, 10$.

If $l_j = 0$ then clearly $l_{j-1} + l_j + l_{j+1} \le 8$. Suppose $l_j = 1$. If $l_{j-1} + l_{j+1} = 8$ then by Property 1, $L_{j-1} = E_i^h$ for some i, h and $L_{j+1} = E_{i'}^{h'}$ for some i', h'. It follows that $L_j \subseteq E_i^{1-h} \cap E_{i'}^{1-h'} = \varnothing$, a contradiction. Thus $l_{j-1} + l_{j+1} \le 7$ and $l_{j-1} + l_j + l_{j+1} \le 8$. If $l_j = 4$ then, by Property 1, $l_{j-1} + l_j + l_{j+1} \le 8$.

If $l_j = 3$ then by Property 2, $L_j \subseteq E_i^h$ for some i, h. Let e be the only edge in

$E_i^h \setminus L_j$. We have $L_{j-1} \cup L_{j+1} \subseteq E_i^{1-h} \cup \overline{N}(e)$. If $l_{j-1} = 4$ then $L_{j-1} = E_i^{1-h}$. Thus $L_{j+1} \subseteq \overline{N}(e)$. Note that any two edges in $\overline{N}(e)$ are at distance at most 2. We have $l_{j+1} \leq 1$. Similarly, if $l_{j+1} = 4$ then $l_{j-1} \leq 1$. In both cases we have $l_{j-1} + l_j + l_{j+1} \leq 8$. Thus we now assume $l_{j-1}, l_{j+1} \leq 3$. If one of l_{j-1} and l_{j+1} is less than 3, then we are done. If $l_{j-1} = l_{j+1} = 3$, then by Property 2, $L_{j-1} \cup L_{j+1} \subseteq E_i^{1-h}$. This is a contradiction since $|E_i^{1-h}| = 4 < 6 = l_{j-1} + l_{j+1}$. Therefore we conclude that if $l_j = 3$ then $l_{j-1} + l_j + l_{j+1} \leq 8$.

Now suppose $l_j = 2$. If one of l_{j-1} and l_{j+1} is less than 3, then we clearly have $l_{j-1} + l_j + l_{j+1} \leq 8$. If $l_{j-1} = l_{j+1} = 3$, then we are done. Thus without loss of generality we assume $l_{j-1} = 4$ and $l_{j+1} \geq 3$. Then by Property 1, $L_{j-1} = E_i^h$ for some i, h and $L_j \subseteq E_i^{1-h}$. If l_{j+1} is also equal to 4, then $L_{j+1} = E_{i'}^{h'}$ for some i', h' and $L_j \subseteq E_{i'}^{1-h'}$. It follows that $i = i'$ and $h = h'$. This is a contradiction. Therefore we assume $l_{j+1} = 3$. Let e' be the only edge in $E_{i'}^{h'} \setminus L_{j+1}$. By Property 2, $L_{j+1} \subseteq E_{i'}^{h'}$ for some i', h' and $L_j \subseteq E_{i'}^{1-h'} \cup N(e') \cup \{e'\}$. Now we have $L_j \subseteq E_i^{1-h}$ and $L_j \cap E_{i'}^{1-h'} \neq \varnothing$. It follows that $i = i'$ and $h = h'$. This is a contradiction.

Therefore $96 = 3 \times 32 = 3 \times \sum_{j=0}^{10} l_j = \sum_{j=0}^{10} (l_{j-1} + l_j + l_{j+1}) \leq 8 \times 11 = 88$. This is contradiction. Case 3 holds.

Case 4. $\sigma'_{2,1}(Q_5) = 15$ and $\lambda'_{2,1}(Q_5) = 13$.

Suppose to the contrary that $\sigma'_{2,1}(Q_5) < 15$. Let L be a 14-circular-$L(2,1)$-edge-labeling of Q_5. Clearly $\sum_{j=0}^{13} l_j = |E(Q_5)| = 80$. From (A6), we know that $0 \leq l_j \leq 8$, for $j = 0, 1, \ldots, 13$. We first prove the following property of the sequence $(l_0, l_1, \ldots, l_{13})$.

For convenience, we use $[0, 13]$ to denote the set of integers $0, 1, \ldots, 13$.

Property 3. *Let* $j \in [0, 13]$. *If* $l_j \geq 6$ *then* $L_j \subseteq E_i^h$ *for some* i, h.

Proof. Let j be any integer in $[0, 13]$. Suppose $l_j \geq 6$. For convenience, we name the 16 edges in E_3 as $e_1, e_2, \ldots, e_{16}$ (See Figure 2). Then $E_3^1 = \{e_1, e_4, e_6, e_7, e_{10}, e_{11}, e_{13}, e_{16}\}$ and $E_3^0 = \{e_2, e_3, e_5, e_8, e_9, e_{12}, e_{14}, e_{15}\}$.

Suppose without loss of generality that $e_1 \in L_j$ ($e_1 \in E_3^1$). Then after carefully checking all edges outside E_3^1 that are at distance at least 3 from e_1, we may find that $L_j \setminus E_3^1$ is contained in $\overline{N}(e_8) \cup \overline{N}(e_{12}) \cup \overline{N}(e_{14}) \cup \overline{N}(e_{15})$. (Please see Figure 2 for illustration.) Since edges in E_3 that are at distance at least 3 from e_1 are contained in $M = E_3 \setminus \{e_2, e_3, e_5, e_9\}$, we have $L_j \cap E_3 \subseteq M$.

In the following we want to show that if $L_j \setminus E_3^1 \neq \emptyset$ then $l_j \leq 5$ and thus prove that $L_j \subseteq E_3^1$. We first construct a bipartite graph $H(X, Y)$ as follows. The partite sets $X = \{\overline{N}(e_8), \overline{N}(e_{12}), \overline{N}(e_{14}), \overline{N}(e_{15})\}$ and $Y = M \setminus \{e_1, e_8, e_{12}, e_{14}, e_{15}\}$. If some edge e in $\overline{N}(e_8)$ (or $\overline{N}(e_{12})$, or $\overline{N}(e_{14})$, or $\overline{N}(e_{15})$) is in L_j, then the four edges e_4, e_6, e_7, e_{16} (or $e_4, e_{10}, e_{11}, e_{16}$, or $e_6, e_{10}, e_{13}, e_{16}$, or $e_7, e_{11}, e_{13}, e_{16}$) from Y can not be in L_j since they are at distance less than 3 from e_8 (or e_{12}, or e_{14}, or e_{15}). In this case, we draw edges in H between $\overline{N}(e_8)$ (or $\overline{N}(e_{12})$, or $\overline{N}(e_{14})$, or $\overline{N}(e_{15})$) and the vertices in Y corresponding to those edges. The graph H is presented in Figure 3.

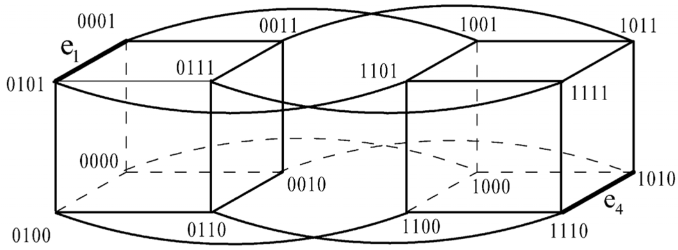

Fig. 2. Q_5 with $e_1 \in L_j$.

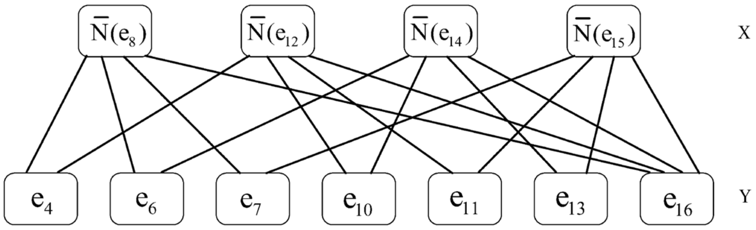

Fig. 3. The graph H.

If $L_j \setminus E_3^1 \neq \emptyset$, then $L_j \cap (\overline{N}(e_8) \cup \overline{N}(e_{12}) \cup \overline{N}(e_{14}) \cup \overline{N}(e_{15})) \neq \emptyset$. Suppose some edge in $\overline{N}(e_8)$ is in L_j, then e_4, e_6, e_7, e_{16} are not in L_j. The subgraph of H induced by $V(H) \setminus \{\overline{N}(e_8), e_4, e_6, e_7, e_{16}\}$ is a cycle of order 6. It is easy to see from this subgraph that at most three edges corresponding to the vertices in this subgraph can be in L_j. It follows that $l_j \leq 5$. It is not difficult to check that the same is true for $\overline{N}(e_{12})$, $\overline{N}(e_{14})$, and $\overline{N}(e_{15})$. Thus we conclude that if $L_j \setminus E_3^1 \neq \emptyset$ then $l_j \leq 5$. This implies that if $l_j \geq 6$ then $L_j \subseteq E_i^h$ for some i, h. Property 3 holds. ■

For $j \in [0, 13]$, denote by h_j the sum $l_{j-1} + l_j + l_{j+1}$.

Let $j \in [0, 13]$. Suppose $l_j \geq 6$. Then, by Property 3, $L_j \subseteq E_i^h$ for some i, h. Let $E' = E_i^h \setminus L_j$. Then, by (A5), $L_{j-1} \cup L_{j+1} \subseteq (E_i \setminus L_j) \cup N(E')$. Let e' be an edge in E'. It is clear that $|(L_{j-1} \cup L_{j+1}) \cap \overline{N}(e')| \leq 2$. Furthermore, if the equality holds then at least two edges in E_i^{1-h} cannot be in $L_{j-1} \cup L_{j+1}$. It follows that if $l_j \geq 6$ then $h_j \leq 16$.

Now suppose $l_{j-1} \geq 6$ and $l_{j+1} \geq 6$. Then $L_{j-1} \subseteq E_i^h$ for some i, h and $L_{j+1} \subseteq E_{i'}^{h'}$ for some i', h'. Let $E' = E_i^h \setminus L_{j-1}$ and $E'' = E_{i'}^{h'} \setminus L_{j+1}$. For any edge $e' \in E' \cup E''$, it is clear that $|L_j \cap \overline{N}(e')| \leq 1$. If $i \neq i'$ then $L_j \subseteq \overline{N}(E') \cup \overline{N}(E'')$ and so $h_j \leq 16$. If $i \neq i'$ then $h_j \leq 16$ since $l_{j-1} + |E'| + l_{j+1} + |E''| = 16$.

We conclude that if $l_j \geq 6$ or both $l_{j-1} \geq 6$ and $l_{j+1} \geq 6$ then $h_j \leq 16$.

Therefore, if $h_j \geq 18$ then $l_j \leq 5$ and l_{j-1} or $l_{j+1} \leq 5$. This implies that if $h_j \geq 18$ then $(l_{j-1}, l_j, l_{j+1}) = (8, 5, 5)$ or $(5, 5, 8)$. And so, $h_j \leq 18$ for all $j \in [0, 13]$; furthermore, if $h_j = 18$ then $h_{j-1} \leq 16$ or $h_{j+1} \leq 16$. If $h_j = h_{j+1} = 18$, then $(l_{j-1}, l_j, l_{j+1}, l_{j+2})$ should be of the form $(8, 5, 5, 8)$. In this case, we have $h_{j-1} \leq 16$ and $h_{j+2} \leq 16$. It is easy to see that the case $h_j = h_{j+1} + 2 = h_{j+2} = 18$ will never happen. From these discussions, we have

$$240 = 3 \times \sum_{j=0}^{13} l_j = \sum_{j=0}^{13} h_j \leq 14 \times 17 = 238.$$

This contradiction proves that $\sigma'_{2,1}(Q_5) = 15$.

Since $\lambda'_{2,1}(Q_5) \geq \sigma'_{2,1}(Q_5) - 2 = 13$, by Theorem 3.1, $\lambda'_{2,1}(Q_5) = 13$. ■

It seems difficult to extend the method in this section to the case Q_n for $n \geq 6$.

We conclude this paper by proposing the following three questions.

Question 1. Notice that $\lambda'_{2,1}(K_{1,\Delta}) = 2\Delta - 2$, Georges and Mauro in [3] asked the question: for each integer from $2\Delta - 2$ to $\lambda'_{2,1}(T_\infty(\Delta))$, is there a tree with maximum degree Δ such that its $L(2,1)$-edge-labeling number is that integer? Since $\sigma'_{2,1}(K_{1,\Delta}) = 2\Delta$, the similar question as above is: for each integer from 2Δ to $\sigma'_{2,1}(T_\infty(\Delta))$, is there a tree with maximum degree Δ such that its circular-$L(2,1)$-edge-labeling number is that integer?

Question 2. Is there a polynomial time algorithm to compute $\lambda'_{2,1}(T)$ (or $\sigma'_{2,1}(T)$) for any tree T?

Question 3. From Theorems 3.2 and 3.1, $\lambda'_{2,1}(Q_n) + 2 = \sigma'_{2,1}(Q_n) = 3n$ for $n \in \{2, 3, 4, 5\}$. That is, the upper bounds $3n - 2$ and $3n$ for $\lambda'_{2,1}(Q_n)$ and $\sigma'_{2,1}(Q_n)$ respectively are attained for $n \in \{2, 3, 4, 5\}$. This is an interesting phenomenon. Is it true that $\lambda'_{2,1}(Q_n) + 2 = \sigma'_{2,1}(Q_n) = 3n$ for all $n \geq 2$?

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Wensong Lin and Jianzhuan Wu
Department of Mathematics
Southeast University
Nanjing 210096
P. R. China
E-mail: wslin@seu.edu.cn