

SEMI-INVARIANT ξ^\perp -SUBMANIFOLDS OF GENERALIZED QUASI-SASAKIAN MANIFOLDS

Constantin Călin*, Mircea Crasmareanu, Marian Ioan Munteanu
and Vincenzo Saltarelli

Dedicated to the memory of Prof. Stere Ianuș (1939-2010).

Abstract. A structure on an almost contact metric manifold is defined as a generalization of well-known cases: Sasakian, quasi-Sasakian, Kenmotsu and cosymplectic. This was suggested by a local formula of Eum [9]. Then we consider a semi-invariant ξ^\perp -submanifold of a manifold endowed with such a structure and two topics are studied: the integrability of distributions defined by this submanifold and characterizations for the totally umbilical case. In particular we recover results of Kenmotsu [11], Eum [9, 10] and Papaghiuc [16].

1. PRELIMINARIES AND BASIC FORMULAE

An interesting topic in the differential geometry is the theory of submanifolds in spaces endowed with additional structures, see [7]. In 1978, A. Bejancu (in [2]) studied CR-submanifolds in Kähler manifolds. Starting from it, several papers have been appeared in this field. Let us mention only few of them: a series of papers of B.Y. Chen (e.g. [6]), of A. Bejancu and N. Papaghiuc (e.g. [3] in which the authors studied semi-invariant submanifolds in Sasakian manifolds). See also [14]. The study was extended also to other ambient spaces, for example A. Bejancu in [4] also studied QR-submanifolds in quaternionic manifolds and M. Barros, B.Y. Chen, F. Urbano in [1] investigated CR-submanifolds in quaternionic manifolds. Several important results on CR-submanifolds are being brought together in [4, 6, 12, 14, 15] and the corresponding references. The purpose of the present paper is to investigate the semi-invariant ξ^\perp -submanifolds in a generalized Quasi-Sasakian manifold.

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*Corresponding author.

Let \widetilde{M} be a real $(2n + 1)$ -dimensional smooth manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$:

$$\begin{cases} \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0 \\ \eta(X) = \tilde{g}(X, \xi), \quad \tilde{g}(\phi X, Y) + \tilde{g}(X, \phi Y) = 0 \end{cases}$$

for any vector fields X, Y tangent to \widetilde{M} where I is the identity on sections of the tangent bundle $T\widetilde{M}$, ϕ is a tensor field of type $(1, 1)$, η is a 1-form, ξ is a vector field and \tilde{g} is a Riemannian metric on \widetilde{M} . Throughout the paper all manifolds and maps are smooth. We denote by $\mathcal{F}(\widetilde{M})$ the algebra of the smooth functions on \widetilde{M} and by $\Gamma(E)$ the $\mathcal{F}(\widetilde{M})$ -module of the sections of a vector bundle E over \widetilde{M} .

The almost contact manifold $\widetilde{M}(\phi, \xi, \eta)$ is said to be *normal* if

$$N_\phi(X, Y) + 2d\eta(X, Y)\xi = 0$$

where

$$N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \quad X, Y \in \Gamma(T\widetilde{M})$$

is the Nijenhuis tensor field corresponding to the tensor field ϕ .

The *fundamental 2-form* Φ on \widetilde{M} is defined by $\Phi(X, Y) = \tilde{g}(X, \phi Y)$.

In [9], the author studied hypersurfaces of an almost contact metric manifold \widetilde{M} whose structure tensor fields satisfy the following relation (expressed only in coordinates)

$$(1) \quad (\tilde{\nabla}_X \phi)Y = \tilde{g}(\tilde{\nabla}_{\phi X} \xi, Y)\xi - \eta(Y)\tilde{\nabla}_{\phi X} \xi$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the metric tensor \tilde{g} . See also [10]. For the sake of simplicity we say that a manifold \widetilde{M} endowed with an almost contact metric structure satisfying (1) is a *generalized Quasi-Sasakian manifold*, in short G.Q.S. An example of such a manifold is furnished in [5]. Define a $(1, 1)$ type tensor field F by

$$(2) \quad FX = -\tilde{\nabla}_X \xi.$$

Proposition 1. *If \widetilde{M} is a G.Q.S manifold then any integral curve of the structure vector field ξ is a geodesic i.e. $\tilde{\nabla}_\xi \xi = 0$. Moreover $d\Phi = 0$ if and only if ξ is a Killing vector field.*

Proof. The first assertion follows immediately from (1) with $X = Y = \xi$, and taking into account that $\eta(\tilde{\nabla}_\xi \xi) = 0$. Next, we deduce

$$\begin{aligned} & 3d\Phi(X, Y, Z) \\ &= \tilde{g}((\tilde{\nabla}_X \phi)Z, Y) + \tilde{g}((\tilde{\nabla}_Z \phi)Y, X) + \tilde{g}((\tilde{\nabla}_Y \phi)X, Z) = \\ &+ \eta(X) \left(\tilde{g}(Y, \tilde{\nabla}_{\phi Z} \xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y \xi) \right) + \eta(Y) \left(\tilde{g}(Z, \tilde{\nabla}_{\phi X} \xi) + \tilde{g}(\phi X, \tilde{\nabla}_Z \xi) \right) \\ &+ \eta(Z) \left(\tilde{g}(X, \tilde{\nabla}_{\phi Y} \xi) + \tilde{g}(\phi Y, \tilde{\nabla}_X \xi) \right). \end{aligned}$$

If we suppose that ξ is Killing then, from the last equation, we obtain $d\Phi = 0$.

Conversely, suppose that $d\Phi = 0$. Taking into account the first part of the statement, for $X = \xi$, $\eta(Y) = \eta(Z) = 0$, the last relation implies

$$\tilde{g}(Y, \tilde{\nabla}_{\phi Z}\xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y\xi) = 0.$$

Finally, by replacing Z with ϕZ and Y by $Y - \eta(Y)\xi$ we deduce that ξ is a Killing vector field. ■

The next result can be obtained by direct computation:

Proposition 2. *A G.Q.S manifold \tilde{M} is normal and*

$$(3) \quad \phi \circ F = F \circ \phi, \quad F\xi = 0, \quad \eta \circ F = 0, \quad \tilde{\nabla}_\xi\phi = 0.$$

Remark 1. (a) It is easy to see that on such manifold \tilde{M} the structure vector field ξ is not necessarily a Killing vector field i.e. \tilde{M} is not necessarily a K-contact manifold. Note that any submanifold of a K-contact manifold, normal to the structure vector field, is anti-invariant (see [19]).

(b) It is also interesting to point out that the following particular situations hold

- (1) $FX = -\phi X$ then \tilde{M} is Sasakian,
- (2) $FX = -X + \eta(X)\xi$ then \tilde{M} is Kenmotsu; this is an example of non K-contact structure,
- (3) $FX = 0$ then \tilde{M} is cosymplectic,
- (4) if ξ is a Killing vector field then \tilde{M} is a quasi-Sasakian manifold,
- (5) another example of non K-contact manifold arises when F has not a maximal rank,
- (6) finally, trans-Sasakian manifolds are not K-contact (see e.g. [8, 13]).

Now, let \tilde{M} be a G.Q.S manifold and consider an m -dimensional submanifold M , isometrically immersed in \tilde{M} . Denote by g the induced metric on M and by ∇ its Levi-Civita connection. Let ∇^\perp and h be the normal connection induced by $\tilde{\nabla}$ on the normal bundle TM^\perp and the second fundamental form of M , respectively. Then one has the direct sum decomposition $T\tilde{M} = TM \oplus TM^\perp$. Recall the Gauss and Weingarten formulae

$$(G) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

$$(W) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad X, Y \in \Gamma(TM)$$

where A_N is the shape operator with respect to the normal section N and satisfies

$$\tilde{g}(h(X, Y), N) = g(A_N X, Y) \quad X, Y \in \Gamma(TM), \quad N \in \Gamma(TM^\perp).$$

The aim of the present paper is to investigate the semi-invariant ξ^\perp -submanifolds in a G.Q.S manifold. More precisely, we suppose that the structure vector field ξ is orthogonal to the submanifold M . According to Bejancu [4] we say that M is a *semi-invariant ξ^\perp -submanifold* if there exist two orthogonal distributions, \mathcal{D} and \mathcal{D}^\perp , in TM such that:

$$(4) \quad TM = \mathcal{D} \oplus \mathcal{D}^\perp, \quad \phi\mathcal{D} = \mathcal{D}, \quad \phi\mathcal{D}^\perp \subseteq TM^\perp$$

where \oplus denotes the orthogonal sum. If $\mathcal{D}^\perp = \{0\}$ then M is an *invariant ξ^\perp -submanifold*. The normal bundle can also be decomposed as $TM^\perp = \phi\mathcal{D}^\perp \oplus \mu$, where $\phi\mu \subseteq \mu$. Hence μ contains ξ .

2. INTEGRABILITY OF DISTRIBUTIONS ON A SEMI-INVARIANT ξ^\perp -SUBMANIFOLD

Let M be a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} . Denote by P and Q the projections of TM on \mathcal{D} and \mathcal{D}^\perp respectively, namely for any $X \in \Gamma(TM)$

$$(5) \quad X = PX + QX.$$

Moreover, for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$ we put

$$(6) \quad \phi X = tX + \omega X$$

$$(7) \quad \phi N = BN + CN$$

with $tX \in \Gamma(\mathcal{D})$, $BN \in \Gamma(TM)$ and $\omega X, CN \in \Gamma(TM^\perp)$. We also consider, for $X \in \Gamma(TM)$, the decomposition

$$(8) \quad FX = \alpha X + \beta X, \quad \alpha X \in \Gamma(\mathcal{D}), \quad \beta X \in \Gamma(TM^\perp).$$

The purpose of this section is to study the integrability of both distributions \mathcal{D} and \mathcal{D}^\perp . With this scope in mind, we state first the following result.

Proposition 3. *Let M be a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} . Then we have*

$$(9) \quad \begin{aligned} (a) \quad & (\nabla_X t)Y = A_{\omega Y}X + Bh(X, Y), \\ (b) \quad & (\nabla_X \omega)Y = Ch(X, Y) - h(X, tY) + g(FX, \phi Y)\xi, \quad X, Y \in \Gamma(TM). \end{aligned}$$

Proof.

The statement follows immediately from (6)-(8). ■

Taking into consideration the decomposition of TM^\perp , it can be easily proved:

Proposition 4. *Let M be a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} . Then for any $N \in \Gamma(TM^\perp)$ one has:*

- (a) $BN \in \mathcal{D}^\perp$,
- (b) $CN \in \mu$.

Proposition 5. *If M is a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} then*

$$(10) \quad A_{\omega Z}W = A_{\omega W}Z$$

for any $Z, W \in \Gamma(\mathcal{D}^\perp)$.

The following two results give necessary and sufficient conditions for the integrability of the two distributions.

Theorem 1. *Let M be a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} . Then the distribution \mathcal{D}^\perp is integrable.*

Proof. Let $Z, W \in \Gamma(\mathcal{D}^\perp)$. Then from (6), (9) and (10) we deduce that

$$t[Z, W] = A_{\omega Z}W - A_{\omega W}Z = 0.$$

Hence the conclusion. ■

Theorem 2. *If M is a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} then the distribution \mathcal{D} is integrable if and only if*

$$(11) \quad h(tX, Y) - h(X, tY) = (\mathcal{L}_\xi \tilde{g})(X, \phi Y) \xi, \quad X, Y \in \Gamma(\mathcal{D}).$$

Proof.

The statement yields directly from (3) and (9)

$$\omega([X, Y]) = h(X, tY) - h(tX, Y) + (\mathcal{L}_\xi \tilde{g})(X, \phi Y) \xi. \quad \blacksquare$$

Notice that the two results above are analogue to those obtained in the Kenmotsu case in [16] and for the cosymplectic case in [18]. See also [14] when the submanifold is tangent to the structure vector field of the Sasakian manifold.

Moreover, from (8) we deduce

Proposition 6. *Let M be a ξ^\perp -semi-invariant submanifold of a G.Q.S manifold \widetilde{M} . Then*

$$(12) \quad A_\xi X = \alpha X, \quad \nabla_X^\perp \xi = -\beta X, \quad X \in \Gamma(TM).$$

Let now $\{e_i, \phi e_i, e_{2p+j}\}$, $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$ be an adapted orthonormal local frame on M , where $q = \dim \mathcal{D}^\perp$ and $2p = \dim \mathcal{D}$. One can state the following

Theorem 3. *If M is a ξ^\perp -semi-invariant submanifold of a G.Q.S manifold \widetilde{M} one has*

$$\eta(H) = \frac{1}{m} \text{trace}(A_\xi), \quad m = 2p + q.$$

Proof.

Using a general formula for the mean curvature, e.g. $H = \frac{1}{m} \sum_{a=1}^s \text{trace}(A_{\xi_a}) \xi_a$, where $\{\xi_1, \dots, \xi_s\}$ is an orthonormal basis in TM^\perp , the conclusion holds by straightforward computations. ■

In the case when the ambient space is a Kenmotsu manifold we retrieve the known result from [16, p. 614].

Corollary 1. *There does not exist a minimal semi-invariant ξ^\perp -submanifold of a Kenmotsu manifold.*

Also it is not difficult to prove:

Theorem 4. *Let M be a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} . Then*

- (1) *the distribution \mathcal{D} is integrable and its leaves are totally geodesic in M if and only if $h(X, Y) \in \Gamma(\mu)$, where $X, Y \in \Gamma(\mathcal{D})$;*
- (2) *any leaf of the integrable distribution \mathcal{D}^\perp is totally geodesic in M if and only if $h(X, Z) \in \Gamma(\mu)$ if $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$.*

Proof. Let us prove only the first statement. For any $Z \in \Gamma(\mathcal{D})^\perp$ we have

$$\begin{aligned} \tilde{g}(h(X, Y), \phi Z) &= \tilde{g}(\tilde{\nabla}_X Y, \phi Z) = -\tilde{g}(Y, \tilde{\nabla}_X(\phi Z)) \\ &= -\tilde{g}(Y, (\tilde{\nabla}_X \phi)Z) - \tilde{g}(\phi Y, \tilde{\nabla}_X Z) = g(\nabla_X(\phi Y), Z). \end{aligned}$$

Let M^* be a leaf of the integrable distribution \mathcal{D} and h^* the second fundamental form of M^* in M .

For any $Z \in \Gamma(\mathcal{D}^\perp)$ we have:

$$g(h^*(X, Y), Z) = \tilde{g}(\tilde{\nabla}_X tY, Z) = \tilde{g}((\tilde{\nabla}_X \varphi)Y + \varphi(\tilde{\nabla}_X Y), Z) = -\tilde{g}(h(X, Y), \varphi Z)$$

which proves that the leaf M^* of the integrable distribution \mathcal{D} is totally geodesic in M if and only if $h(X, Y) \in \Gamma(\mu)$. ■

Notice that the part (2) of the previous Theorem was obtained in the Kenmotsu case by Papaghiuc in [17, p. 115].

We conclude this section with the following

Corollary 2. *If the leaves of the integrable distribution \mathcal{D} are totally geodesic in M then the structure vector field ξ is \mathcal{D} -Killing, that is $(\mathcal{L}_\xi g)(X, Y) = 0$, $X, Y \in \Gamma(\mathcal{D})$.*

3. TOTALLY UMBILICAL SEMI-INVARIANT ξ^\perp -SUBMANIFOLDS

The main purpose of this section is to obtain a complete characterization of a totally umbilical semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} . Recall that for a totally umbilical submanifold we have

$$h(X, Y) = g(X, Y)H, \quad X, Y \in \Gamma(TM).$$

First we state:

Theorem 5. *An invariant ξ^\perp -submanifold M of a G.Q.S manifold is totally umbilical if and only if*

$$(13) \quad h(X, Y) = \frac{1}{m} g(X, Y) \text{trace}(A_\xi)\xi.$$

Proof. If M is an invariant ξ^\perp -submanifold then for any $X, Y \in \Gamma(TM)$ we have $h(X, \phi Y) = \phi h(X, Y) - g(A_\xi \phi X, Y)\xi$. Let us consider an orthonormal frame $\{e_i, e_{p+i}\}$, $i = 1, \dots, p$ on M ; from the above relation one obtains that $\phi H = 0$. Again, since M is an invariant submanifold:

$$(14) \quad H = g(H, \xi)\xi = \frac{1}{m} \sum_{i=1}^m g(h(e_i, e_i), \xi)\xi = \frac{1}{m} \text{trace}(A_\xi)\xi$$

and the proof is complete. ■

Corollary 3. *An invariant ξ^\perp -submanifold of a quasi-Sasakian manifold is minimal.*

The case of a semi-invariant ξ^\perp -submanifold in a G.Q.S manifold \widetilde{M} is solved in the next Theorem.

Theorem 6. *Let M be a semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} with $\dim \mathcal{D}^\perp > 1$. Then M is totally umbilical if and only if (13) holds.*

Proof. Suppose that M is totally umbilical. Let $X \in \Gamma(\mathcal{D})$ be a unit vector field and $N \in \Gamma(\mu) \setminus \text{span}\{\xi\}$. By direct calculation it follows that:

$$g(H, N) = g(h(X, X), N) = g(\widetilde{\nabla}_X \phi X - (\widetilde{\nabla}_X \phi)X, \phi N) = g(h(X, \phi X), \phi N) = 0$$

which proves that $H \in \phi \mathcal{D}^\perp \oplus \text{span}\{\xi\}$.

For $Z, W \in \Gamma(\mathcal{D}^\perp)$, from (9) we derive $QA_{\phi Z}W = -g(Z, W)\phi H$ i.e.

$$(15) \quad g(Z, \phi H)g(W, \phi H) = g(Z, W)g(\phi H, \phi H).$$

If we take $Z = W$ orthogonal to ϕH , since $\dim \mathcal{D}^\perp > 1$, from the above relation we infer $\phi H = 0 \Rightarrow H \in \text{span}\{\xi\}$. At this point the conclusion is straightforward.

Conversely, if (13) is supposed to be true, then we get (14) which together with (13) we deduce that M is totally umbilical. ■

Let us remark that when \widetilde{M} is a Kenmotsu manifold the result of the Theorem 6 was proved in [16].

Corollary 4. *Every ξ^\perp -hypersurface of a G.Q.S manifold \widetilde{M} is totally umbilical.*

Proof. If M is a hypersurface then $TM^\perp = \text{span}\{\xi\}$ that is $h(X, Y) \in \text{span}\{\xi\}$. Next, from (14) it follows (13). ■

In the particular case of a Kenmotsu manifold this result was obtained by Papaghiuc in [16, p. 617].

As a consequence of Theorem 6, we obtain

Theorem 7. *If M is a totally umbilical semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} with $\dim \mathcal{D}^\perp > 1$, then M is a semi-invariant product.*

Here, by a semi-invariant product we mean a semi-invariant ξ^\perp -submanifold of \widetilde{M} which can be locally written as a Riemannian product of a ϕ -invariant submanifold and a ϕ -anti-invariant submanifold of \widetilde{M} , both of them orthogonal to ξ .

Proof. From the definition of a totally umbilical submanifold we have $h(X, Z) = 0$ for any $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, so that, by (b) of Theorem 4, the leaves of \mathcal{D}^\perp are totally geodesic submanifolds of M . By Theorem 6, we know that $h(X, Y) \in \text{span}\{\xi\} \subset \mu$ for any $X, Y \in \Gamma(\mathcal{D})$. By virtue of (a) of Theorem 1, this implies that the invariant distribution \mathcal{D} is integrable and its integral manifolds are totally geodesic submanifolds of M . Therefore, we conclude that M is a semi-invariant product. ■

Without any restriction on the dimension of \mathcal{D}^\perp , we have the following

Theorem 8. *Let M be a totally umbilical semi-invariant ξ^\perp -submanifold of a G.Q.S manifold \widetilde{M} . If \mathcal{D} is integrable, then each leaf of \mathcal{D} is a totally geodesic submanifold of M .*

Proof. By using (b) of Proposition 3, for any $X \in \Gamma(\mathcal{D})$, we have

$$\omega(\nabla_X X) = -g(X, X)CH - g(FX, \phi Y)\xi.$$

Since $CH \in \mu$ by b) of Lemma 4 and $\omega U \in \phi \mathcal{D}^\perp$ for any $U \in \Gamma(TM)$, from the above equation we deduce that $\omega(\nabla_X X) = 0$, or equivalently

$$\nabla_X X \in \mathcal{D}, \quad \forall X \in \Gamma(\mathcal{D}).$$

Replacing X by $X + Y$, we get $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$ for all $X, Y \in \Gamma(\mathcal{D})$. This condition, together with the integrability of \mathcal{D} , implies

$$(16) \quad \nabla_X Y \in \Gamma(\mathcal{D}), \quad \forall X, Y \in \Gamma(\mathcal{D}).$$

As \mathcal{D} is integrable, Frobenius theorem ensures that M is foliated by leaves of \mathcal{D} . Combining this fact with (16), we conclude that the leaves of \mathcal{D} are totally geodesic submanifolds of M . ■

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Constantin Călin
Technical University Gh. Asachi
Department of Mathematics
Iasi 700049
Romania
E-mail: c0nstc@yahoo.com

Mircea Crasmareanu and Marian Ioan Munteanu
'Al.I.Cuza' University of Iasi
Bd. Carol I, nr. 11
Iasi 700506
Romania
E-mail: mcrasm@uaic.ro
marian.ioan.munteanu@gmail.com

Vincenzo Saltarelli
Department of Mathematics, University of Study of Bari
Via E. Orabona 4, 70125-Bari
Italy
E-mail: saltarelli@dm.uniba.it