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NONSELFADJOINT SINGULAR STURM-LIOUVILLE OPERATORS IN LIMIT-CIRCLE CASE

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Abstract. In this paper, we study the maximal dissipative singular Sturm-Liouville operators (in Weyl's limit-circle case at singular point b) acting in the Hilbert space $L_w^2[a,b)$ $(-\infty < a < b \leq \infty$). In fact, we consider all extensions of a minimal symmetric operator and we investigate two classes of maximal dissipative operators with separated boundary conditions, called 'dissipative at a' and 'dissipative at b'. In both cases, we construct a selfadjoint dilation of the maximal dissipative operator and determine its incoming and outgoing spectral representations. This representations make it possible to determine the scattering matrix of the dilation in terms of the Titchmarsh-Weyl function of a selfadjoint Sturm-Liouville operator. We also construct a functional model of the maximal dissipative operator and determine its characteristic function in terms of the scattering matrix of the dilation (or of the Titchmarsh-Weyl function). Finally we prove theorems on the completeness of the eigenfunctions and associated functions of the maximal dissipative Sturm-Liouville operators.

1. INTRODUCTION

The contour integration method is one of the general methods of the spectral analysis of nonselfadjoint operators. This method is about separating the spectrum with expanding contours and may be applied to weak perturbations of selfadjoint operators and also to operators with sparse discrete spectrum. However this method has doesn't have application because there are not asymptotics of solutions of some class of singular differential equations.

In the spectral analysis of nonselfadjoint (dissipative) operators, the most adequate way is the functional model theory which shows that the characteristic function is unitary equivalent to the scattering function. In fact, in the spectral representation of

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dilation, the dissipative operator becomes the model. In the centre of this method, there is an information on the spectral properties of the dissipative operator. For example, the factorization of the characteristic function may help us about learning that whether the system of all eigenvectors and associated vectors is complete or not. To construct the characteristic function directly is quite hard. However, according to the results of Lax-Phillips, this construction can be done with the selfadjoint dilation and scattering matrix (see [11]). This approach for dissipative Schrödinger and Sturm-Liouville operators has been done, for example, in [1-4,14,15].

In this paper, we investigate the spectral analysis of singular dissipative Sturm-Liouville operators with the help of the extensions of a minimal symmetric operator with defect index $(2,2)$ (in Weyl's limit-circle case at singular end point b) acting in the Hilbert space $L_w^2\,[a,b)$ $(-\infty < a < b \leq \infty)$. In this investigation, we investigate two classes of maximal dissipative operators with separated boundary conditions, called 'dissipative at a' and 'dissipative at b'. In both cases, we construct a selfadjoint dilation of maximal dissipative operator and its incoming and outgoing spectral representations which will help us to determine the scattering matrix of the dilation according to the theory of Lax and Phillips [11]. With the help of the incoming spectral representation, we construct a functional model of maximal dissipative operator and define its characteristic function in terms of the Titchmarsh-Weyl function of the selfadjoint Sturm-Liouville operator. Finally, we prove the theorems on completeness of the system of eigenfunctions and associated functions of the maximal dissipative Sturm-Liouville operators with the help of the results obtained for the characteristic functions.

2. Preliminaries

Throughout this paper we consider the Sturm-Liouville differential expression with singular point b as

$$\ell(y): = \frac{1}{w(x)}[-(p(x)y'(x))' + q(x)y(x)] \; (x \in \mathbf{I} := [a,b)\,, \; -\infty < a < b \leq +\infty), \tag{2.1}$$

where p, q and w are real-valued, Lebesgue measurable functions on $\mathbf{I}$, and $p^{-1}, q, w \in L_{loc}^1(\mathbf{I}), w(x) > 0$ for almost all $x \in \mathbf{I}$.

Let us denote by D the linear set consisting of all vectors $y \in L_w^2(\mathbf{I})$ which consists of all complex-valued functions y such that $\int_a^b w(x) \mid y(x) \mid^2 \, dx < +\infty$ with the inner product $(y,z) = \int_a^b w(x)y(x)\overline{z(x)}dx$ such that y and py' are locally absolutely continuous functions on $\mathbf{I}$ and, $\ell(y) \in L_w^2\,(\mathbf{I})$ and define the operator L on D by the equality $Ly = \ell(y)$.

For $y, z \in D$ from Green's formula the equality

$$(Ly, z) - (y, Lz) = [y, z]_b - [y, z]_a\,, \tag{2.2}$$

holds, where $[y, z]_x := W_x[y, \overline{z}] := (yp\overline{z}' - py'\overline{z}x)(x)\,, x \in \mathbf{I}$ and $[y, z]_b := \lim_{x \to b^-} [y, z]_x\,.$

In $L^2_w(\mathbf{I})$, we consider the dense linear set D'_0 consisting of smooth, compactly supported functions on $\mathbf{I}$. Denote by L'_0 the restriction of the operator L to D'_0. It follows from (2.2) that L'_0 is symmetric. Consequently, it admits closure which we denote by L_0. The domain of L_0 consists of precisely those vectors $y \in D$ satisfying the conditions

$$y(a) = (py')(a) = 0, [y, z]_b = 0, \forall z \in D. \tag{2.3}$$

The operator L_0 is a symmetric operator with defect index $(1, 1)$ or $(2, 2)$, and $L = L^*_0$ [5-7,13,16,17]. The operators L_0 and L are called the *minimal* and *maximal operators*, respectively.

Let symmetric operator L_0 have defect index $(1, 1)$, so the Weyl's limit-point case occurs for ℓ or L_0. It is known that all selfadjoint extensions $\mathbf{L}_\alpha$ of the operator L_0 are described by the boundary conditions $y(a) \cos\alpha + (py')(a) \sin\alpha = 0, \alpha \in [0, \pi)$ $(y \in D)$ (see [5-7, 13, 16, 17]).

All maximal dissipative (maximal accretive) extensions $\mathbf{L}_h$ of the operator L_0 are described by the boundary conditions $(py')(a) - hy(a) = 0$ $(y \in D)$, where $Imh \geq 0$ or $h = \infty$ ($Imh \leq 0$ or $h = \infty$). For $h = \infty$, the corresponding boundary condition has the form $y(a) = 0$.

In this paper, we assume that L_0 has defect index $(2, 2)$, so that the Weyl's limit-circle case holds for the differential expression ℓ or the operator L_0 (see [5-7,9,10,13,16,17]).

Let $u(x)$ and $v(x)$ be the solutions of the equation

$$\ell(y) = 0 \ (x \in \mathbf{I}) \tag{2.4}$$

satisfying the initial conditions

$$u(a) = 1,\ (pu')(a) = 0,\ v(a) = 0,\ (pv')(a) = 1. \tag{2.5}$$

The Wronskian of the two solutions of (2.4) does not depend on x so it follows from (2.5) and the constancy of the Wronskian that the equality $W_x[u, v] = W_a[u, v] = 1$ $(x \in \mathbf{I})$ holds. Consequently, u and v form a fundamental system of solutions of (2.4). Since L_0 has defect index $(2, 2)$, $u, v \in L^2_w(\mathbf{I})$ and, moreover, $u, v \in D$.

It is clear that the set D_0 being the domain of the operator L_0 consists of all functions $y \in D$ satisfying the conditions

$$y(a) = (py')(a) = 0, [y, u]_b = [y, v]_b = 0. \tag{2.6}$$

A linear operator T (with dense domain $D(T)$) acting on some Hilbert space H is called *dissipative* (*accretive*) if $\mathrm{Im}(\mathrm{Tf},\mathrm{f}) \geq 0$ ($\mathrm{Im}(\mathrm{Tf},\mathrm{f}) \leq 0$) for all $f \in D(T)$ and *maximal dissipative (maximal accretive)* if it does not have a proper dissipative (accretive) extension.

To construct the nonselfadjoint operator we need linear mappings. So let Γ_1 and Γ_2 be linear mappings from D into $E := \mathbf{C}^2$ defined by

$$\Gamma_1 y = \begin{pmatrix} -y(a) \\ [y,u]_b \end{pmatrix}, \ \Gamma_2 y = \begin{pmatrix} (py')(a) \\ [y,v]_b \end{pmatrix}. \tag{2.7}$$

Then we have (see [1])

Theorem 2.1. *For any contraction K in E the restriction of the operator L to the set of functions $y \in D$ satisfying the boundary condition*

$$(K-I)\,\Gamma_1 y + i\,(K+I)\,\Gamma_2 y = 0 \tag{2.8}$$

or

$$(K-I)\,\Gamma_1 y - i\,(K+I)\,\Gamma_2 y = 0 \tag{2.9}$$

is, respectively, a maximal dissipative or a maximal accretive extension of the operator L_0. Conversely, every maximal dissipative (maximal accretive) extension of L_0 is the restriction of L to the set of vectors $y \in D$ satisfying (2.8) ((2.9)), and the contraction K is uniquely, determined by the extensions. These conditions define a selfadjoint extension if and only if K is unitary. In the latter case, (2.8) and (2.9) are equivalent to the condition $(\cos A)\,\Gamma_1 y - (\sin A)\,\Gamma_2 y = 0$, where A is a selfadjoint operator (Hermitian matrix) in E. The general form of the dissipative and accretive extensions of the operator L_0 is given by the conditions

$$K\left(\Gamma_1 y + i\Gamma_2 y\right) = \Gamma_1 y - i\Gamma_2 y, \Gamma_1 y + i\Gamma_2 y \in D\left(K\right) \tag{2.10}$$

$$K\left(\Gamma_1 y - i\Gamma_2 y\right) = \Gamma_1 y + i\Gamma_2 y, \Gamma_1 y - i\Gamma_2 y \in D\left(K\right) \tag{2.11}$$

respectively, where K is a linear operator with $\|Kf\| \leq \|f\|, f \in D(K)$. The general form of symmetric extensions is given by the formulae (2.10) and (2.11), where K is an isometric operator.

In particular, the boundary conditions ($y \in D$)

$$(py')(a) - h_1 y(a) = 0 \tag{2.12}$$

$$[y,u]_b - h_2\,[y,v]_b = 0 \tag{2.13}$$

with $\mathrm{Imh}_1 \geq 0$ *or* $h_1 = \infty$, *and* $\mathrm{Imh}_2 \geq 0$ *or* $h_2 = \infty$ ($\mathrm{Imh}_1 \leq 0$ or $h_1 = \infty$, and $\mathrm{Imh}_2 \leq 0$ or $h_2 = \infty$) *describe all the maximal dissipative (maximal accretive) extensions of* L_0 *with separated boundary conditions. The selfadjoint extensions of* L_0 are *obtained precisely when* $Imh_1 = 0$ *or* $h_1 = \infty$, *and* $Imh_2 = 0$ *or* $h_2 = \infty$. *Here for* $h_1 = \infty$ $(h_2 = \infty)$, *condition (2.12) ((2.13)) should be replaced by* $y(a) = 0$ $([y, v]_b = 0)$.

In the sequel we shall study the maximal dissipative operators $L_{h_1 h_2}$ generated by (2.1) and the boundary conditions (2.12) and (2.13) of two kinds: 'dissipative at a', i.e. when either $\mathrm{Imh}_1 > 0$ and $Imh_2 = 0$ *or* $h_2 = \infty$; and 'dissipative at b', when $Imh_1 = 0$ *or* $h_1 = \infty$ and $\mathrm{Imh}_2 > 0$.

3. Selfadjoint Dilation of the Maximal Dissipative Operator in the Case of Dissipative at a'

To investigate the spectral analysis of the maximal dissipative operators, the functional model theory of Sz.-Nagy-Foiaş may be helpful ([12]). In fact, to construct the characteristic function of a contraction directly is quite hard. On the other hand the abstract scattering function of Lax-Phillips is unitary equivalent to the characteristic function of Sz.-Nagy-Foiaş ([11,12]). This equivalence helps us to reach our main aim. So to pass to more easier, that is the scattering theory, we need to construct a selfadjoint dilation of the maximal dissipative operator $L_{h_1 h_2}$ ($\mathrm{Imh}_1 > 0$, Im $h_2 = 0$ or $h_2 = \infty$). So we associate with $H := L^2_w(\mathbf{I})$ the 'incoming' and 'outgoing' channels $D_- := L^2(\mathbf{R}_-)$ ($\mathbf{R}_- := (-\infty, 0]$) and $D_+ := L^2(\mathbf{R}_+)$ ($\mathbf{R}_+ := [0, \infty)$); we form the orthogonal sum $\mathcal{H} = D_- \oplus H \oplus D_+$.

Denote by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mappings acting according to the formulas $P : \langle \varphi_-, y, \varphi_+ \rangle \to y$ and $P_1 : y \to \langle 0, y, 0 \rangle$, respectively and we denote by $P^+ : \mathcal{H} \to L^2(\mathbf{R}_+)$ and $P_1^+ : L^2(\mathbf{R}_+) \to D_+$ the mappings acting according to the formulae $P^+ : \langle \varphi_-, u, \varphi_+ \rangle \to \varphi_+$ and $P_1^+ : \varphi \to \langle 0, 0, \varphi \rangle$, respectively.

In the space $\mathcal{H}$, we consider the operator $\mathcal{L}_{h_1 h_2}$ generated by the expression

$$\mathcal{L}\langle \varphi_-, y, \varphi_+ \rangle = \langle i\frac{d\varphi_-}{d\xi}, \ell(y), i\frac{d\varphi_+}{d\varsigma} \rangle \tag{3.1}$$

on the set $D(\mathcal{L}_{h_1 h_2})$ of elements $\langle \varphi_-, y, \varphi_+ \rangle$ satisfying the conditions $\varphi_\mp \in W_2^1(\mathbf{R}_\mp)$, $y \in D$ and

$$(py')(a) - h_1 y(a) = \alpha \varphi_-(0),$$

$$(py')(a) - \bar{h}_1 y(a) = \alpha \varphi_+(0), [y, u]_b - h_2 [y, v]_b = 0, \tag{3.2}$$

where $W_2^1(\mathbf{R}_\mp)$ is the Sobolev space, and $\alpha^2 := 2\mathrm{Imh}_1$, $\alpha > 0$. Then we have

Theorem 3.1. *The operator $\mathcal{L}_{h_1h_2}$ is selfadjoint in $\mathcal{H}$ and it is a selfadjoint dilation of the maximal dissipative operator $L_{h_1h_2}$.*

Proof. Let $f, g \in D(\mathcal{L}_{h_1h_2})$ where $f = \langle \varphi_-, y, \varphi_+ \rangle$ and $g = \langle \psi_-, z, \psi_+ \rangle$. Using integration by parts and (2.2), we get that

$$(\mathcal{L}_{h_1h_2} f, g)_{\mathcal{H}} = \int_{-\infty}^{0} i\varphi_-' \overline{\psi}_- d\xi + (Ly, z)_H + \int_{0}^{\infty} i\varphi_+' \overline{\psi}_+ d\varsigma = i\varphi_-(0)\overline{\psi}_-(0) \\ -i\varphi_+(0)\overline{\psi}_+(0) + [y, z]_b - [y, z]_a + (f, \mathcal{L}_{h_1h_2} g)_{\mathcal{H}}. \tag{3.3}$$

Now, using the boundary conditions (3.2) for the components of the vectors f and g and Lemma 2.1, a direct calculation shows that $i\varphi_-(0)\overline{\psi_-(0)} - i\varphi_+(0)\overline{\psi_+(0)} + [y, z]_b - [y, z]_a = 0$. So, $\mathcal{L}_{h_1h_2}$ is symmetric. Therefore, to prove that $\mathcal{L}_{h_1h_2}$ is selfadjoint, it suffices for us to show that $\mathcal{L}^*_{h_1h_2} \subseteq \mathcal{L}_{h_1h_2}$. Take $g = \langle \psi_-, z, \psi_+ \rangle \in D(\mathcal{L}^*_{h_1h_2})$. Let $\mathcal{L}^*_{h_1h_2} g = g^*$, $g^* = \langle \psi_-^*, z^*, \psi_+^* \rangle \in \mathcal{H}$, so that

$$(\mathcal{L}_{h_1h_2} f, g)_{\mathcal{H}} = (f, g^*)_{\mathcal{H}}, \forall f \in D(\mathcal{L}_{h_1h_2}). \tag{3.4}$$

By choosing suitable components for $f \in (\mathcal{L}_{h_1h_2})$ in (3.4), it is easy to show that $\psi_\mp \in W_2^1(\mathbf{R}_\mp)$, $z \in D$ and $g^* = \mathcal{L}g$, where the operator $\mathcal{L}$ is defined by (3.1). Consequently, (3.4) takes the form $(\mathcal{L}f, g)_{\mathcal{H}} = (f, \mathcal{L}g)_{\mathcal{H}}, \forall f \in D(\mathcal{L}_{h_1h_2})$. Therefore, the sum of the integral terms in the bilinear form $(\mathcal{L}f, g)_{\mathcal{H}}$ must be equal to zero:

$$i\varphi_-(0)\overline{\psi_-(0)} - i\varphi_+(0)\overline{\psi_+(0)} + [y, z]_b - [y, z]_a = 0 \tag{3.5}$$

for all $f = \langle \varphi_-, y, \varphi_+ \rangle \in D(\mathcal{L}_{h_1h_2})$. Further, solving the boundary conditions (3.2) for $y(a)$ and $(py')(a)$, we find that $y(a) = -\frac{i}{\alpha}(\varphi_+(0) - \varphi_-(0)$, $(py')(a) = \alpha\varphi_-(0) - \frac{ih_1}{\alpha}(\varphi_+(0) - \varphi_-(0))$. Therefore, using (2.7), we find that (3.5) is equivalent to the equality

$$\begin{aligned} &i\varphi_-(0)\overline{\psi_-(0)} - i\varphi_+(0)\overline{\psi_+(0)} = [y, z]_a - [y, z]_b \\ &= -\frac{i}{\alpha}(\varphi_+(0) - \varphi_-(0))(\overline{pz'})(a) - \alpha[\varphi_-(0) - \frac{ih_1}{\alpha^2}(\varphi_+(0) - \varphi_-(0)]\overline{z}(a) \\ &-[y, u]_b[\overline{z}, u]_b + [y, v]_b[\overline{z}, v]_b = -\frac{i}{\alpha}(\varphi_+(0) - \varphi_-(0))(\overline{pz'})(a) \\ &-\alpha[\varphi_-(0) - \frac{ih_1}{\alpha^2}(\varphi_+(0) - \varphi_-(0))]\overline{z}(a) - ([\overline{z}, u]_b - h_2[\overline{z}, v]_b)[y, v]_b. \end{aligned}$$

Since the values $\varphi_\pm(0)$ can be arbitrary complex numbers, a comparison of the coefficient of $\varphi_\pm(0)$ on the left and right of the last equality gives us that the vector $g = \langle \psi_-, z, \psi_+ \rangle$ satisfies the boundary conditions $(pz')(a) - h_1 z(a) = \alpha\psi_-(0)$, $(pz')(a) - \overline{h_1} z(a) = \alpha\psi_+(0)$, $[z, u]_b - h_2[z, v]_b = 0$. Consequently, the inclusion $\mathcal{L}^*_{h_1h_2} \subseteq \mathcal{L}_{h_1h_2}$ is established, and hence $\mathcal{L}_{h_1h_2} = \mathcal{L}^*_{h_1h_2}$.

It is well-known that the selfadjoint operator $\mathcal{L}_{h_1h_2}$ generates in $\mathcal{H}$ a unitary group $U_t = \exp[i\mathcal{L}_{h_1h_2}t]$ $(t \in \mathbf{R}:= (-\infty,\infty))$. Let $Z_t = PU_tP_1$ $(t \geq 0)$. Then the family $\{Z_t\}$ $(t \geq 0)$ of operators becomes a strongly continuous semigroup of nonunitary contraction on H. Denote by $B_{h_1h_2}$ the generator of this semigroup, $B_{h_1h_2}y = \lim_{t\to+0}(it)^{-1}(Z_ty - y)$. The domain of $B_{h_1h_2}$ consists of all the vectors for which the limit exists. The operator $B_{h_1h_2}$ is maximal dissipative. The operator $\mathcal{L}_{h_1h_2}$ is called the *selfadjoint dilation* of $B_{h_1h_2}$. We shall show that $B_{h_1h_2} = L_{h_1h_2}$, and hence, $\mathcal{L}_{h_1h_2}$ is a selfadjoint dilation of $L_{h_1h_2}$. To do this, we first verify the equality [14, 15]

$$P(\mathcal{L}_{h_1h_2} - \lambda I)^{-1}P_1y = (L_{h_1h_2} - \lambda I)^{-1}y, y \in H,\ \mathrm{Im}\lambda < 0. \tag{3.6}$$

With this purpose, let us set $(\mathcal{L}_{h_1h_2} - \lambda I)^{-1}P_1y = g = \langle\psi_-, z, \psi_+\rangle$. Then $(\mathcal{L}_{h_1h_2} - \lambda I)g = P_1y$, and hence, $\ell(z)-\lambda z = y, \psi_-(\xi) = \psi_-(0)e^{-i\lambda\xi}$, and $\psi_+(\varsigma) = \psi_+(0)e^{-i\lambda\varsigma}$. Since $g \in D(\mathcal{L}_{h_1h_2})$, and hence $\psi_- \in L^2(\mathbf{R}_-)$; it follows that $\psi_-(0) = 0$, and consequently, z satisfies the boundary conditions $(py')(a)-h_1y(a) = 0$, $[y, u]_b-[y, v]_a = 0$. Therefore, $z \in D(L_{h_1h_2})$, and since a point λ with $\mathrm{Im}\lambda < 0$ cannot be an eigenvalue of a dissipative operator, it follows that $z = (L_{h_1h_2}-\lambda I)^{-1}y$. We remark that $\psi_+(0)$ is found from the formula $\psi_+(0) = \alpha^{-1}((pz')(a) - \overline{h_1}z(a))$. Thus, $(\mathcal{L}_{h_1h_2} - \lambda I)^{-1}P_1y$ $= \langle 0, (L_{h_1h_2} - \lambda I)^{-1}y, \alpha^{-1}((pz')(a) - \overline{h_1}z_1(a))e^{-i\lambda\varsigma}\rangle$, for $y \in H$ and $\mathrm{Im}\lambda < 0$. By applying the mapping P, one obtains (3.6).

It is now easy to show that $B_{h_1h_2} = L_{h_1h_2}$. Indeed, by (3.6),

$$\begin{aligned}(L_{h_1h_2} - \lambda I)^{-1} &= P(\mathcal{L}_{h_1h_2} - \lambda I)^{-1}P_1 = -iP\int_0^\infty U_te^{-i\lambda t}dtP_1\\ &= -i\int_0^\infty Z_te^{-i\lambda t}dt = (B_{h_1h_2} - \lambda I)^{-1}\ (\mathrm{Im}\lambda < 0),\end{aligned}$$

and therefore $L_{h_1h_2} = B_{h_1h_2}$. Hence Theorem 3.1 is proved.

4. Scattering Theory of the Dilation and Functional Model of the Maximal Dissipative Operator in the Case of 'Dissipative at a'

According to the Lax-Phillips scattering theory one can construct a scattering function acting from the incoming subspace D_- to the outgoing subspace D_+ only when the unitary group U_t has the following properties

(1) $U_tD_- \subset D_-, t \leq 0$, and $U_tD_+ \subset D_+, t \geq 0$;

(2) $\bigcap_{t\leq 0} U_tD_- = \bigcap_{t\geq 0} U_tD_+ = \{0\}$;

(3) $\overline{\bigcup_{t\geq 0} U_tD_+} = \overline{\bigcup_{t\leq 0} U_tD_+} = \mathcal{H}$;

(4) $D_{-}\perp D_{+}$.

Let $D_{-}:=\left\langle L^{2}(\mathbf{R}_{-}),0,0\right\rangle$ and $D_{+}:=\left\langle 0,0,L^{2}(\mathbf{R}_{+})\right\rangle$. Now we shall show that the properties (1)-(4) are satisfied. Property (**4**) is obvious.

Let us set $R_{\lambda}=(\mathcal{L}_{h_1h_2}-\lambda I)^{-1}$, for all λ with $\mathrm{Im}\lambda<0$ to prove property (**1**) for D_{+} (the proof for D_{-} is similar). Then, for any $f=\langle 0,0,\varphi_{+}\rangle\in D_{+}$, we have

$$R_{\lambda}f=\langle 0,0,-ie^{-i\lambda\xi}\int_{0}^{\xi}e^{i\lambda s}\varphi_{+}(s)ds\rangle.$$

So $R_{\lambda}f\in D_{+}$. If $g\perp D_{+}$, then the equality

$$0=(R_{\lambda}f,g)_{\mathcal{H}}=-i\int_{0}^{\infty}e^{-i\lambda t}(U_{t}f,g)_{\mathcal{H}}dt,\mathrm{Im}\lambda<0.$$

holds. From this it follows that $(U_{t}f,g)_{\mathcal{H}}=0$ for all $t\geq 0$. Hence, $U_{t}D_{+}\subset D_{+}$ for $t\geq 0$, and property (**1**) is proved.

To prove property (**2**), we set $U_{t}^{+}=P^{+}U_{t}P_{1}^{+},t\geq 0$. Note that the semigroup of isometries $U_{t}^{+}=P^{+}U_{t}P_{1}^{+},t\geq 0$, is a one-sided shift in $L^{2}\left(\mathbf{R}_{+}\right)$. Indeed, the generator of the semigroup of the one-sided shift V_{t} in $L^{2}\left(\mathbf{R}_{+}\right)$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\varphi\left(0\right)=0$. On the other hand, the generator S of the semigroup of isometries $U_{t}^{+},t\geq 0$, is the operator $S\varphi=P^{+}\mathcal{L}_{h_1h_2}P_{1}^{+}\varphi$ $=P^{+}\mathcal{L}_{h_1h_2}\left\langle 0,0,\varphi\right\rangle=P^{+}\left\langle 0,0,i\frac{d\varphi}{d\xi}\right\rangle=i\frac{d\varphi}{d\xi}$, where $\varphi\in W_{2}^{1}\left(\mathbf{R}_{+}\right)$ and $\varphi\left(0\right)=0$. Since a semigroup is determined by its generator, it follows that $U_{t}^{+}=V_{t}$, and hence, $\bigcap_{t\geq 0}U_{t}D_{+}=\langle 0,0,\bigcap_{t\geq 0}V_{t}L^{2}\left(\mathbf{R}_{+}\right)\rangle=\{0\}$, so proof is completed.

The linear operator A (with domain $D(A)$) acting in the Hilbert space $\mathbf{H}$ is called *completely nonselfadjoint (*or *simple)* if there is no invariant subspace $M\subseteq D(A)$ ($M\neq\{0\}$) of the operator A on which the restriction of A to M is selfadjoint.

In this scheme of the Lax-Phillips scattering theory, the scattering matrix is defined in terms of the theory of spectral representations. We proceed to their construction. Along the way, we also prove property (**3**) of the incoming and outgoing subspaces.

We first prove the following lemma.

Lemma 4.1. *The operator* $L_{h_1h_2}$*is completely nonselfadjoint (simple).*

Proof. Let $H'\subset H$ be a nontrivial subspace in which $L_{h_1h_2}$ induces a selfadjoint operator $L'_{h_1h_2}$ with domain $D(L'_{h_1h_2})=H'\cap D(L_{h_1h_2})$. If $f\in D(L'_{h_1h_2})$, then $f\in D(L'^{*}_{h_1h_2})$, and $(py')\,(a)-h_1y\,(a)=0$, $(py')\,(a)-\bar{h}_1y\,(a)=0$, $[y,u]_b-h_2[y,v]_b=0$. From this discussion, for the eigenfunctions $y(x,\lambda)$ of the operator $L_{h_1h_2}$ that lie in H' and are eigenvectors of $L'_{h_1h_2}$ we have $y(a,\lambda)=0$, $(py')(a,\lambda)=0$, and then by the uniqueness theorem of the Cauchy problem for the equation $\ell(y)=\lambda y$, $x\in\mathbf{I}$, we have $y(x,\lambda) \equiv 0$. Since all solutions of $\ell(y) = \lambda y\, (x \in \mathbf{I})$ belong to $L_w^2(\mathbf{I})$, it can be concluded that the resolvent $R_\lambda(L_{h_1h_2})$ of the operator $L_{h_1h_2}$ is a Hilbert-Schmidt operator, and hence the spectrum of $L_{h_1h_2}$ is purely discrete. Hence by the theorem on expansion in eigenvectors of the selfadjoint operator $L'_{h_1h_2}$, we have $H' = \{0\}$, i.e., the operator $L_{h_1h_2}$ is simple. The lemma is proved.

We set

$$H_- = \overline{\underset{t\geq 0}{\cup} U_t D_-}, \quad H_+ = \overline{\underset{t\leq 0}{\cup} U_t D_+}.$$

Lemma 4.2. $H_- + H_+ = \mathcal{H}$.

Proof. Considering property (**1**) of the subspace D_+, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H + H_+)$ is invariant relative to group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ (and hence, also H') were nontrivial, then the unitary group $\{U'_t\}$, restricted to this subspace, would be a unitary part of the group $\{U_t\}$, and hence the restriction $L'_{h_1h_2}$ to H' would be a selfadjoint operator in H'. From the simplicity of the operator $L_{h_1h_2}$, it follows that $H' = \{0\}$, i.e. $\mathcal{H}' = \{0\}$. The lemma is proved.

Let us denote by $L_{\infty h_2}$ the selfadjoint operator generated by the expression ℓ and the boundary conditions $y(a) = 0$, $[y,v]_b - h_2[y,u]_b = 0\ (y \in D)$.

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solution of the equation $\ell(y) = \lambda y\ (x \in \mathbf{I})$ satisfying the conditions $\varphi(a,\lambda) = 0, (p\varphi')(a,\lambda) = 1, \psi(a,\lambda) = 1, (p\psi')(a,\lambda) = 0$. The *Titchmarch-Weyl function* $m_{\infty h_2}(\lambda)$ of the selfadjoint operator $L_{\infty h_2}$ is determined by the condition $[\psi + m_{\infty h_2}\varphi, u]_b - h_2[\psi + m_{\infty h_2}\varphi, v]_b = 0$. Then it follows that

$$m_{\infty h_2}(\lambda) = -\frac{[\psi,u]_b - h_2[\psi,v]_b}{[\varphi,u]_b - h_2[\varphi,v]_b}. \tag{4.1}$$

From (4.1), it is clear that $m_{\infty h_2}(\lambda)$ is a meromorphic function on the complex plane $\mathbf{C}$ with a countable number of poles on the real axis and these poles coincide with the eigenvalues of the operator $L_{\infty h_2}$. Further, it is possible to show that the function $m_{\infty h_2}$ has the following properties: $\mathrm{Im}\lambda \mathrm{Im} m_{\infty h_2}(\lambda) > 0$ for $\mathrm{Im}\lambda \neq 0$ and $\overline{m_{\infty h_2}(\lambda)} = m_{\infty h_2}(\bar{\lambda})$ for $\lambda \in \mathbf{C}$, except the reel poles of $m_{\infty h_2}(\lambda)$.

Let us adopt the following notations: $\theta(x,\lambda) := \psi(x,\lambda) + m_{\infty h_2}(\lambda)\varphi(x,\lambda)$,

$$S_{h_1h_2}(\lambda) := \frac{m_{\infty h_2}(\lambda) - h_1}{m_{\infty h_2}(\lambda) - \overline{h_1}}. \tag{4.2}$$

Let $\mathcal{U}_\lambda^-(x,\xi,\varsigma) = \langle e^{-i\lambda\xi}, (m_{\infty h_2}(\lambda) - h_1)^{-1}\alpha\theta(x,\lambda), \bar{S}_{h_1h_2}(\lambda)e^{-i\lambda\varsigma}\rangle$. $\mathcal{U}_\lambda^-(x,\xi,\varsigma)$ satisfies the equation $\mathcal{L}\mathcal{U} = \lambda\mathcal{U}$ and the corresponding boundary conditions for the operator $\mathcal{L}_{h_1h_2}$. But the vectors $\mathcal{U}_\lambda^-(x,\xi,\varsigma)$ for real λ do not belong to the space $\mathcal{H}$. Using $\mathcal{U}_\lambda^-(x,\xi,\varsigma)$, we define the transformation $\mathcal{F}_- : f \to \tilde{f}_-(\lambda)$ by $(\mathcal{F}_- f)(\lambda) := \tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \mathcal{U}_\lambda^-)_{\mathcal{H}}$ on the vector $f = \langle \varphi_-, y, \varphi_+\rangle$ in which φ_-, φ_+, and y are smooth, compactly supported functions.

Lemma 4.3. *The transformation $\mathcal{F}_-$ isometrically maps H_- onto $L^2(\mathbf{R})$. For all vectors $f, g \in H_-$, the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathcal{H}} = (\tilde{f}_-, \tilde{g}_-)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}d\lambda, \; f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\mathcal{U}_\lambda^- d\lambda,$$

where $\tilde{f}_-(\lambda) = (\mathcal{F}_- f)(\lambda)$ and $\tilde{g}_-(\lambda) = (\mathcal{F}_- g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle \varphi_-, 0, 0\rangle$, $g = \langle \psi_-, 0, 0\rangle$, we have that

$$\tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \mathcal{U}_\lambda^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \varphi_-(\xi)e^{i\lambda\xi}d\xi \in H_-^2,$$

and, in view of the usual Parseval equality for Fourier integrals,

$$(f,g)_{\mathcal{H}} = \int_{-\infty}^{0} \varphi_-(\xi)\overline{\psi_-}(\xi)d\xi = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}d\lambda = (\mathcal{F}_- f, \mathcal{F}_- g)_{L^2}.$$

Here and below, $H_\pm^2$ denote the Hardy classes in $L^2(\mathbf{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively.

We now extend the Parseval equality to the whole of H_-. For this purpose, we consider in H_- the dense set H_-' of vectors obtained from the smooth, compactly supported functions in D_- : $f \in H_-'$ if $f = U_T f_0$, $f_0 = \langle \varphi_-, 0, 0\rangle$, $\varphi_- \in C_0^\infty(\mathbf{R}_-)$, where $T = T_f$ is a nonnegative number (depending on f). In this case, if $f, g \in H_-$, then for $T > T_f$ and $T > T_g$ we have that $U_{-T}f, U_{-T}g \in D_-$ and moreover, the first components of these vectors belong to $C_0^\infty(\mathbf{R}_-)$. Therefore, since the operators U_t $(t \in \mathbf{R})$ are unitary, the equality $\mathcal{F}_- U_{-T} f = (U_{-T}f, U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda T}(f, U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda T}\mathcal{F}_- f$, implies that

$$\begin{aligned}(f,g)_{\mathcal{H}} &= (U_{-T}f, U_{-T}g)_{\mathcal{H}} = (\mathcal{F}_- U_{-T}f, \mathcal{F}_- U_{-T}g)_{L^2} \\ &= (e^{-i\lambda T}\mathcal{F}_- f, e^{-i\lambda T}\mathcal{F}_- g)_{L^2} = (\mathcal{F}_- f, \mathcal{F}_- g)_{L^2}.\end{aligned} \tag{4.3}$$

By taking the closure in (4.3), we obtain the Parseval equality for the whole space H_-. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of integrals over finite intervals. Finally we arrive at

$$\mathcal{F}_- H_- = \overline{\bigcup_{t\geq 0} \mathcal{F}_- U_t D_-} = \overline{\bigcup_{t\geq 0} e^{-i\lambda t} H_-^2} = L^2(\mathbf{R}),$$

i.e. $\mathcal{F}_-$ maps H_- onto the whole of $L^2(\mathbf{R})$. The lemma is proved.

We set $\mathcal{U}_\lambda^+(x, \xi, \varsigma) = \langle S_{h_1h_2}(\lambda)e^{-i\lambda\xi}, (m_{\infty h_2}(\lambda) - \overline{h_1})^{-1}\alpha\theta(x,\lambda), e^{-i\lambda\varsigma}\rangle$. $\mathcal{U}_\lambda^+(x, \xi, \varsigma)$ satisfies the equation $\mathcal{L}\mathcal{U} = \lambda\mathcal{U}$ $(\lambda \in \mathbf{R})$ and the boundary conditions (3.2). But the vectors $\mathcal{U}_\lambda^+(x,\xi,\varsigma)$ for real λ do not belong to the space $\mathcal{H}$. Using $\mathcal{U}_\lambda^+(x,\xi,\varsigma)$, we define the transformation $\mathcal{F}_+ : f \to \tilde{f}_+(\lambda)$ on vectors $f = \langle \varphi_-, y, \varphi_+ \rangle$, in which φ_-, φ_+, and y are smooth, compactly supported functions by setting $(\mathcal{F}_+ f)(\lambda) := \tilde{f}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \mathcal{U}_\lambda^+)_\mathcal{H}$.

The proof of the next result is analogous to that of Lemma 4.3.

Lemma 4.4. *The transformation $\mathcal{F}_+$ isometrically maps H_+ onto $L^2(\mathbf{R})$. For all vectors $f, g \in H_+$, the Parseval equality and the inversion formula hold:*

$$(f,g)_\mathcal{H} = (\tilde{f}_+, \tilde{g}_+)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_+(\lambda)\overline{\tilde{g}_+(\lambda)}d\lambda,\ f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}_+(\lambda)U_\lambda^+ d\lambda,$$

where $\tilde{f}_+(\lambda) = (\mathcal{F}_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (\mathcal{F}_+ g)(\lambda)$.

According to (4.2), the function $S_{h_1h_2}(\lambda)$ satisfies $|S_{h_1h_2}(\lambda)| = 1$ for $\lambda \in \mathbf{R}$. So it follows from the explicit formula for the vectors U_λ^+ and U_λ^- that

$$\mathcal{U}_\lambda^- = \bar{S}_{h_1h_2}(\lambda)\mathcal{U}_\lambda^+ \ (\lambda \in \mathbf{R}). \tag{4.4}$$

Therefore, it follows from Lemmas 4.3 and 4.4 that $H_- = H_+$. Together with Lemma 4.2, the equalities $\mathcal{H} = H_- = H_+$ hold, and property (**3**) above is established for the incoming and outgoing subspaces.

These calculations show that the transformation $\mathcal{F}_-$ isometrically maps onto $L^2(\mathbf{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that $\mathcal{F}_-$ is the incoming spectral representation for the group $\{U_t\}$. Similarly, $\mathcal{F}_+$ is the outgoing spectral representation for $\{U_t\}$. It follows from (4.4) that the passage from the $\mathcal{F}_+$-representation of a vector $f \in \mathcal{H}$ to its $\mathcal{F}_-$-representation is realized by multiplication of the function $S_{h_1h_2}(\lambda) : \tilde{f}_-(\lambda) = S_{h_1h_2}(\lambda)\tilde{f}_+(\lambda)$. According to [11], the scattering function (matrix) of the group $\{U_t\}$ with respect to the subspaces D_- and D_+, is the coefficient by which the $\mathcal{F}_-$-representation of a vector $f \in \mathcal{H}$ must be multiplied in order to get the corresponding $\mathcal{F}_+$-representation: $\tilde{f}_+(\lambda) = \bar{S}_{h_1h_2}(\lambda)\tilde{f}_-(\lambda)$. According to [11], we have now proved the following theorem.

Theorem 4.5. *The function $\bar{S}_{h_1h_2}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the selfadjoint operator $\mathcal{L}_{h_1h_2}$).*

To summarize the equivalence of the characteristic function of Sz.-Nagy-Foiaş and the scattering function of Lax-Phillips, let $S(\lambda)$ be an arbitrary nonconstant inner function [12] on the upper half-plane (we recall that a function $S(\lambda)$ that is analytic in the upper half-plane $\mathbf{C}_+$ is called *inner function* on $\mathbf{C}_+$ if $|\ S(\lambda)\ | \leq 1$ for $\lambda \in \mathbf{C}_+$, and $|\ S(\lambda)\ | = 1$ for almost all $\lambda \in \mathbf{R}$). Let us define $\mathcal{K} = H_+^2 \ominus SH_+^2$. It is known that

$\mathcal{K} \neq \{0\}$ and is a subspace of the Hilbert space H_+^2. Let us consider the semigroup of the operators Z_t $(t \geq 0)$ acting in $\mathcal{K}$ according to the formula $Z_t\varphi = P\left[e^{i\lambda t}\varphi\right]$, $\varphi := \varphi(\lambda) \in \mathcal{K}$, where P is the orthogonal projection from H_+^2 onto $\mathcal{K}$. The generator of the semigroup $\{Z_t\}$ is denoted by $T : T\varphi = \lim_{t\to+0}(it)^{-1}(Z_t\varphi - \varphi)$, which is a maximal dissipative operator acting in $\mathcal{K}$ and with the domain $D(T)$ consisting of all functions $\varphi \in \mathcal{K}$, such that the limit exists. The operator T is called a *model dissipative operator*. This model dissipative operator is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş ([12]). The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $\mathbf{K} = \langle 0, H, 0\rangle$, so that $\mathcal{H} = D_- \oplus \mathbf{K} \oplus D_+$. From the explicit form of the unitary transformation $\mathcal{F}_-$ it follows that

$$
\begin{aligned}
&\mathcal{H} \to L^2(\mathbf{R}),\ f \to \tilde{f}_-(\lambda) = (\mathcal{F}_-f)(\lambda),\ D_- \to H_-^2,\ D_+ \to S_{h_1h_2}H_+^2,\\
&\mathbf{K} \to H_+^2 \ominus S_{h_1h_2}H_+^2,\ U_tf \to (\mathcal{F}_-U_t\mathcal{F}_-^{-1}\tilde{f}_-)(\lambda) = e^{i\lambda t}\tilde{f}_-(\lambda).
\end{aligned} \tag{4.5}
$$

The formulae (4.5) are exactly the model studied by Sz.-Nagy-Foiaş. Hence, our operator $L_{h_1h_2}$ is a unitary equivalent to the model dissipative operator with the characteristic function $S_{h_1h_2}(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide (see [12, 14, 15]), we have proved the following result.

Theorem 4.6. *The characteristic function of the maximal dissipative operator $L_{h_1h_2}$ coincides with the function $S_{h_1h_2}(\lambda)$ defined in (4.2).*

5. Selfadjoint Dilation, Scattering Theory of the Dilation and Functional Model of the Maximal Dissipative Operator in the Case of 'Dissipative at b'

In this section we consider maximal dissipative operators $L_{h_1h_2}$ $(\mathrm{Im}h_1 = 0$ or $h_1 = 0$ and $\mathrm{Im}h_2 > 0)$. Since the proofs in case of 'dissipative at b' are largely analogous to those in the case of 'dissipative at a', we shall not give detailed proofs.

Lemma 5.1. *The operator $L_{h_1h_2}$ is completely nonselfadjoint (simple).*

In $\mathcal{H}$ we shall consider the operator $\mathcal{L}_{h_1h_2}$ generated by the expression (3.1) on the set $D(\mathcal{L}_{h_1h_2})$ of vectors $\langle\varphi_-, y, \varphi_+\rangle$, $\varphi_\mp \in W_2^1(\mathbf{R}_\mp)$, $y \in D$,

$$
\begin{aligned}
&(py')(a) - h_1y(a) = 0,\ [y,u]_b - h_2[y,v]_b = \alpha\varphi_-(0),\\
&[y,u]_b - \bar{h}_2[y,v]_b = \alpha\varphi_+(0)\ (\alpha^2{:} = 2\mathrm{Im}h_2,\ \alpha > 0).
\end{aligned} \tag{5.1}
$$

Theorem 5.2. *The operator $\mathcal{L}_{h_1h_2}$ is selfadjoint in $\mathcal{H}$ and it is a selfadjoint dilation of the maximal dissipative operator $L_{h_1h_2}$.*

The selfadjoint operator $\mathcal{L}_{h_1h_2}$ generates in $\mathcal{H}$ a unitary group $U_t = \exp[i\mathcal{L}_{h_1h_2}t]$ $(t \in \mathbf{R})$.

We set

$$H_- = \overline{\bigcup_{t\geq 0} U_t D_-},\ H_+ = \overline{\bigcup_{t\leq 0} U_t D_+}.$$

Lemma 5.3. $H_- + H_+ = \mathcal{H}$.

Let $m_{h_1\infty}(\lambda)$ be the Titchmarsh-Weyl function of the selfadjoint operator $L_{h_1\infty}$ generated by ℓ and the boundary conditions $(py')(a) - h_1 y(a) = 0, [y,v]_b = 0$. Then, $m_{h_1\infty}(\lambda)$ is expressed in terms of the Wronskian of the solutions:

$$m_{h_1\infty}(\lambda) = -\frac{[\psi, v]_b}{[\varphi, v]_b},$$

where $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of the equation $\ell(y) = \lambda y$ $(x \in \mathbf{I})$ and satisfying the initial conditions

$$\varphi(a,\lambda) = -\frac{1}{\sqrt{1+h_1^2}},\ (p\varphi')(a,\lambda) = \frac{h_1}{\sqrt{1+h_1^2}},$$
$$\psi(a,\lambda) = \frac{h_1}{\sqrt{1+h_1^2}},\ (p\psi')(a,\lambda) = \frac{1}{\sqrt{1+h_1^2}}.$$

Let us adopt the following notations:

$$(5.2)\qquad \begin{aligned} k(\lambda) &:= k_{h_1}(\lambda) := -\frac{[\varphi, u]_b}{[\psi, v]_b},\ m(\lambda) := m_{h_1\infty}(\lambda),\\ S(\lambda) &:= S_{h_1h_2}(\lambda) := \frac{m(\lambda)k(\lambda) - h_2}{m(\lambda)k(\lambda) - \bar{h}_2}. \end{aligned}$$

Let $\mathcal{V}_\lambda^-(x,\xi,\varsigma) = \langle e^{-i\lambda\xi}, \alpha m(\lambda)[(m(\lambda)k(\lambda) - h_2)[\psi, v]_b]^{-1}\varphi(x,\lambda), \bar{S}(\lambda)e^{-i\lambda\varsigma}\rangle$. $\mathcal{V}_\lambda^-(x,\xi,\varsigma)$ satisfies the equation $\mathcal{L}\mathcal{V} = \lambda\mathcal{V}$ $(\lambda \in \mathbb{R})$ and the boundary conditions (5.1). But the vector $\mathcal{V}_\lambda^-(x,\xi,\varsigma)$ does not belong to $\mathcal{H}$ for real λ. Using $\mathcal{V}_\lambda^-$, we define the transformation $F_- : f \to \tilde{f}_-(\lambda)$ by $(F_- f)(\lambda) := \tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \mathcal{V}_\lambda^-)_\mathcal{H}$ on the vector $f = \langle\varphi_-, y, \varphi_+\rangle$ in which φ_-, φ_+, and y are smooth, compactly supported functions.

Lemma 5.4. *The transformation F_- isometrically maps H_- onto $L^2(\mathbf{R})$. For all vectors $f, g \in H_-$, the Parseval equality and the inversion formula hold:*

$$(f,g)_\mathcal{H} = (\tilde{f}_-, \tilde{g}_-)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}d\lambda,\ f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\mathcal{V}_\lambda^- d\lambda,$$

where $\tilde{f}_-(\lambda = (F_- f)(\lambda)$ and $\tilde{g}_-(\lambda) = (F_- g)(\lambda)$.

We set $\mathcal{V}_\lambda^+(x,\xi,\varsigma) = \langle S(\lambda)e^{-i\lambda\xi}, \alpha m(\lambda)[(m(\lambda)k(\lambda) - \bar{h}_2)[\psi, v]_b]^{-1}\varphi(x,\lambda), e^{-i\lambda\varsigma}\rangle$. $\mathcal{V}_\lambda^+(x,\xi,\varsigma)$ $\mathcal{V}_\lambda^+$ satisfies the equation $\mathcal{L}\mathcal{V} = \lambda\mathcal{V}$ ($\lambda \in \mathbb{R}$) and the boundary conditions (5.1). But the vector $\mathcal{V}_\lambda^+(x,\xi,\varsigma)$ does not belong to $\mathcal{H}$ for real λ. Using $\mathcal{V}_\lambda^+(x,\xi,\varsigma)$, we define the transformation $F_+ : f \to \tilde{f}_+(\lambda)$ on vectors $f = \langle \varphi_-, y, \varphi_+ \rangle$, in which φ_-, φ_+ and y are smooth, compactly supported functions, by setting $(F_+ f)(\lambda) := \tilde{f}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \mathcal{V}_\lambda^+)_\mathcal{H}$.

Lemma 5.5. *The transformation F_+ isometrically maps H_+ onto $L^2(\mathbf{R})$. For all vectors $f, g \in H_+$, the Parseval equality and the inversion formula hold:*

$$(f,g)_\mathcal{H} = (\tilde{f}_+, \tilde{g}_+)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_+(\lambda)\overline{\tilde{g}_+(\lambda)}d\lambda, f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_+(\lambda)\mathcal{U}_\lambda^+ d\lambda,$$

where $\tilde{f}_+(\lambda) = (F_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (F_+ g)(\lambda)$.

According to (5.2), the function $S_{h_1h_2}(\lambda)$ satisfies $|S_{h_1h_2}(\lambda)| = 1$ for $\lambda \in \mathbf{R}$. Therefore, it follows from the explicit formula for the vectors $\mathcal{V}_\lambda^-$ and $\mathcal{V}_\lambda^+$ that

$$\mathcal{V}_\lambda^- = \bar{S}_{h_1h_2}(\lambda)\mathcal{V}_\lambda^+ \, (\lambda \in \mathbf{R}). \tag{5.3}$$

It follows from Lemmas 5.4 and 5.5 that $H_- = H_+$. Together with Lemma 5.3, this shows that $\mathcal{H} = H_- = H_+$.

So the transformation F_- isometrically maps $\mathcal{H}$ onto $L^2(\mathbf{R})$; the subspace D_- is mapped onto H_-^2, while the operators U_t go over into operators of multiplication by $e^{i\lambda t}$. According to the theory of Lax-Phillips this means that F_- is an incoming spectral representation of the group $\{U_t\}$. Similarly, F_+ is an outgoing spectral representation of $\{U_t\}$. From the formula (5.3) it follows that passage from the F_--representation of an element $f \in \mathcal{H}$ to its F_+-representation is accomplished as follows: $\tilde{f}_+(\lambda) = \bar{S}_{h_1h_2}(\lambda)\, \tilde{f}_-(\lambda)$. According to [11], we have now proved the following statement.

Theorem 5.6. *The function $\bar{S}_{h_1h_2}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the selfadjoint operator $\mathcal{L}_{h_1h_2}$).*

From the explicit form of the unitary transformation F_- it follows that

$$\mathcal{H} \to L^2(\mathbf{R}),\ f \to \tilde{f}_-(\lambda) = (F_- f)(\lambda),\ D_- \to H_-^2, D_+ \to S_{h_1h_2}H_+^2,$$

$$\mathbf{K} \to H_+^2 \ominus S_{h_1h_2}H_+^2,\ U_t f \to (F_- U_t F_-^{-1}\tilde{f}_-)(\lambda) = e^{i\lambda t}\tilde{f}_-(\lambda).$$

These formulae show that the operator $L_{h_1h_2}$ is a unitary equivalent to the model dissipative operator with characteristic functions $S_{h_1h_2}(\lambda)$. We have thus proved the following theorem.

Theorem 5.7. *The characteristic function of the maximal dissipative operator $L_{h_1h_2}$ coincides with the function $S_{h_1h_2}(\lambda)$ defined by (5.2).*

6. Completeness of the System of Eigenfunctions and Associated Functions of the Maximal Dissipative Operators

Let $\mathbf{A}$ denote the linear operator in the Hilbert space $\mathbf{H}$ with the domain $D(\mathbf{A})$. The complex number λ_0 is called an *eigenvalue* of the operator $\mathbf{A}$ if there exists a nonzero element $y_0 \in D(\mathbf{A})$ such that $\mathbf{A}y_0 = \lambda_0 y_0$. Such element y_0 is called the *eigenvector* of the operator $\mathbf{A}$ corresponding to the eigenvalue λ_0. The elements $y_1, y_2, ..., y_k$ are called the *associated vectors* of the eigenvector y_0 if they belong to $D(\mathbf{A})$ and $\mathbf{A}y_j = \lambda_0 y_j + y_{j-1}, j = 1, 2, ..., k$. The element $y \in D(\mathbf{A}), y \neq 0$ is called a *root vector* of the operator $\mathbf{A}$ corresponding to the eigenvalue λ_0, if all powers of $\mathbf{A}$ are defined on this element and $(\mathbf{A} - \lambda_0 I)^n y = 0$ for some integer n. The set of all root vectors of $\mathbf{A}$ corresponding to the same eigenvalue λ_0 with the vector $y = 0$ forms a linear set N_{λ_0} and is called the root lineal. The dimension of the lineal N_{λ_0} is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal N_{λ_0} coincides with the linear span of all eigenvectors and associated vectors of $\mathbf{A}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathbf{A}$ is equivalent to the completeness of the system of all root vectors of this operator.

The studies about the characteristic function of contractions say that one can get some informations about eigenfunctions and associated functions of the characteristic function only when showing the absence of a singular factor of the characteristic function $S_{h_1h_2}(\lambda)$ in the factorization $S_{h_1h_2}(\lambda) = s(\lambda)\mathcal{B}(\lambda)$, where $\mathcal{B}(\lambda)$ is a Blaschke product ([12,14,15]). So we shall show this absence.

Theorem 6.1. *For all values of h_2 with $\mathrm{Imh}_2 > 0$, except possibly for a single value $h_2 = h_2^0$, and for fixed h_1 ($\mathrm{Imh}_1 = 0$ or $h_1 = \infty$), the characteristic function $S_{h_1h_2}(\lambda)$ of the maximal dissipative operator $L_{h_1h_2}$ is a Blaschke product, and the spectrum of $L_{h_1h_2}$ is purely discrete and belongs to the open upper half-plane. The operator $L_{h_1h_2}(h_1 \neq h_2^0)$ has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions (or all root functions) of this operator is complete in the space $L_w^2(\mathbf{I})$.*

Proof. It is clear from the explicit formula (5.2) that the $S_{h_1h_2}(\lambda)$ is an inner function in the upper half-plane and, moreover, it is meromorphic in the whole λ-plane. Therefore, it can be factored as follows

$$S_{h_1h_2}(\lambda) = e^{i\lambda c}\mathcal{B}_{h_1h_2}(\lambda),\ c = c(h_2) \geq 0, \tag{6.1}$$

where $\mathcal{B}_{h_1h_2}(\lambda)$ is a Blaschke product. It follows from (6.1) that

$$|S_{h_1h_2}(\lambda)| \leq e^{-c(h_2)\mathrm{Im}\lambda},\ \mathrm{Im}\lambda \geq 0. \tag{6.2}$$

Further, expressing $n_{h_1}(\lambda):= m_{h_1,\infty}(\lambda)k_{h_1}(\lambda)$ in terms of $S_{h_1h_2}(\lambda)$, we find from (5.2) that

$$n_{h_1}(\lambda) = \frac{\bar{h}_2 S_{h_1h_2}(\lambda) - h_2}{S_{h_1h_2}(\lambda) - 1}. \tag{6.3}$$

If $c(h_2) > 0$ for a given value h_2 $(\mathrm{Imh}_2 > 0)$, then (6.2) implies that $\lim_{t\to+\infty} S_{h_1h_2}(it) = 0$, and then (6.3) gives us that $\lim_{t\to+\infty} n_{h_1}(it) = h_2^0$. Since $n_{h_1}(\lambda)$ does not depend on h_2 this implies that $c(h_2)$ can be nonzero at not more than a single point $h_2 = h_2^0$ (and, further, $h_2^0 = \lim_{t\to+\infty} n_{h_1}(it)$). The theorem is proved.

The next result can be proved similarly.

Theorem 6.2. *For all values of h_1 with $\mathrm{Imh}_1 > 0$, except possibly for a single value $h_1 = h_1^0$, and for fixed h_2 ($\mathrm{Imh}_2 = 0$ or $h_2 = \infty$), the characteristic function $S_{h_1h_2}(\lambda)$ of the maximal dissipative operator $L_{h_1h_2}$ is a Blaschke product, and the spectrum of $L_{h_1h_2}$ is purely discrete and belongs to the open upper half-plane. The operator $L_{h_1h_2}(h_1 \neq h_1^0)$ has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions of this operator is complete in the space $L_w^2(\mathbf{I})$.*

Let $p(x) > 0$ (almost all $x \in \mathbf{I}$) and for positive $\eta > 0$

$$\int_a^x \left|w(t) - \eta p^{-1}(t)\right| dt = o\left(\int_a^x p^{-1}(t)dt\right),\ x \to a^+. \tag{6.4}$$

Then we have [8]

$$m_{\infty h_2}^{-1}(\lambda) = \frac{i}{\eta^{\frac{1}{2}}\sqrt{\lambda}}\left\{1 + o(1)\right\},\ \text{as } \lambda \to \infty \text{ in } S_\varepsilon, \tag{6.5}$$

where $S_\varepsilon = \{\lambda \in \mathbf{C} : 0 < \varepsilon < \arg\lambda < \pi - \varepsilon\}, \varepsilon \in (0, \frac{\pi}{2})$.

Moreover, we prove that $S_{h_1h_2}(\lambda)$ ($\mathrm{Imh}_1 > 0$, $\mathrm{Imh} = 0$ or $h_2 = \infty$) is the Blaschke product. Suppose the contrary, i.e., let $S_{h_1h_2}(\lambda)$ have a singular factor (for $\mathrm{Im}h_1 > 0$, $\mathrm{Imh}_2 = 0$ or $h_2 = \infty$). Then

$$S_{h_1h_2}(\lambda) = e^{i\lambda b(h_1)}\mathcal{B}(\lambda),\ b(h_1) > 0,$$

where $\mathcal{B}(\lambda)$ is the Blaschke product. Then as $\lambda \to \infty$ in S_ε, we have

$$|S_{h_1h_2}(\lambda)| = \left|e^{i\lambda b(h_1)}\right| |\mathcal{B}(\lambda)| \le e^{-\mathrm{Im}\lambda \mathrm{b}(\mathrm{h}_1)} \to 0.$$

However, according to (4.2) and (6.5), we have

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in S_\varepsilon}} |S_{h_1h_2}(\lambda)| = 1.$$

The obtained contradiction shows that $b(h_1) = 0$, i.e. $S_{h_1h_2}(\lambda)$ is the Blaschke product. Thus we have proved the following theorem.

Theorem 6.3. *For all values of* h_1 $(\mathrm{Imh}_1 > 0)$ *and* h_2 $(\mathrm{Imh}_2 = 0$ *or* $h_2 = \infty)$ *the characteristic function* $S_{h_1h_2}(\lambda)$ *of the maximal dissipative operator* $L_{h_1h_2}$ *is a Blaschke product. The spectrum of* $L_{h_1h_2}$ *is purely discrete and belongs to the open upper half-plane. The operator* $L_{h_1h_2}$ *has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity. Moreover, the whole spectrum, except for, possibly, a finite number of points, belongs to the angles* $0 < \arg\lambda < \varepsilon$ *and* $\pi - \varepsilon < \arg\lambda < \pi$, $\varepsilon \in (0, \frac{\pi}{2})$. *The system of all eigenfunctions and associated functions of the operator* $L_{h_1h_2}$ *is complete in the space* $L^2_w(\mathbf{I})$.

Remark. Since a linear operator $\mathbf{S}$ acting in the Hilbert space $\mathbf{H}$ is maximal accretive if and only if $-\mathbf{S}$ is maximal dissipative, all results concerning maximal dissipative operators can be immediately transferred to maximal accretive operators.

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