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GLOBAL NONEXISTENCE OF ARBITRARY INITIAL ENERGY SOLUTIONS OF VISCOELASTIC EQUATION WITH NONLOCAL BOUNDARY DAMPING

Jie Ma* and Hongrui Geng

Abstract. In this paper, we consider the long time behavior of solutions of the initial value problem for the viscoelastic wave equation under boundary damping

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\mathrm{div}(a(x)\nabla u(\tau))d\tau + u_t = 0 \quad \text{in}\, \Omega \times (0,\infty).$$

For the low initial energy case, which is the non-positive initial energy, based on concavity argument we prove the blow up result. As for the high initial energy case, we give out sufficient conditions of the initial datum such that the solution blows up in finite time.

1. Introduction

In this work, we are concerned with the following problem

$$(1.1)\quad \begin{cases} u_{tt}-\Delta u+\displaystyle\int_0^t g(t-\tau)\mathrm{div}(a(x)\nabla u(\tau))d\tau+u_t = 0, & (x,t)\in\Omega\times(0,\infty),\\ u = 0, & (x,t)\in\Gamma_1\times(0,\infty),\\ \dfrac{\partial u}{\partial\nu}-\displaystyle\int_0^t g(t-\tau)(a(x)\nabla u(\tau))\cdot\nu d\tau = f(u), & (x,t)\in\Gamma_0\times(0,\infty),\\ u\,(x,0) = u_0(x), u_t\,(x,0) = u_1(x), & x\in\Omega, \end{cases}$$

where Ω is a bounded domain of $R^n (n \geq 1)$ with a smooth boundary $\Gamma := \partial\Omega$, such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_0, Γ_1 have positive measures, ν is the unit outward normal on $\partial\Omega$.

This problem has its origin in the mathematical description of viscoelastic materials. It is well known that viscoelastic materials exhibit natural damping, which is due to

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the special property of these materials to retain a memory of their past history. From the mathematical point of view, these damping effects are modeled by intro-differential operators. Therefore, the dynamics of viscoelastic materials are of great importance and interest as they have wide applications in nature sciences. From the physical point of view, the problem (1.1) describes the position $u(x,t)$ of the material particle x at time t, which is claimed in the portion Γ_1 of its boundary with its portion Γ_0 supported by elastic bearings with nonlinear boundary responses, represented by the function $f(u)$. (see [3, 8, 14, 20, 21]).

The wave equation with memory has been considered by many mathematicians. Cavalcanti et al. [6] firstly studied

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^\gamma u = 0, & (x,t) \in \Omega \times (0,\infty), \\ u\,(x,0) = u_0(x), u_t\,(x,0) = u_1(x), & x \in \Omega, \\ u\,(x,t) = 0, & x \in \partial\Omega, \end{cases}$$

and obtained an exponential decay rate of the solution under some assumption on $g(s)$ and $a(x)$. At this point it is important to mention some papers in connection with viscoelastic effects, among them, Alves and Cavalcanti [1], Aassila et al.[2] and references therein. Rammaha [19] deals with wave equations that feature two competing forces and analyzes the influence of these forces on the long-time behavior of solutions. Cavalcanti and Oquendo [7] considered

$$u_{tt} - k_0\Delta u + \int_0^t \text{div}[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0,$$

under the restrictive assumptions on both the damping function h and the kernel g. And then Messaoudi [15] obtained the global existence of solutions for the viscoelastic equation, at same time he also obtained a blow-up result with negative energy. Furthermore, he improved his blow-up result in [16]. Recently, Wang and Wang [23] investigated the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = a_1|u|^{p-1}u, & (x,t) \in \Omega \times (0,\infty), \\ u\,(x,0) = u_0(x), u_t\,(x,0) = u_1(x), & x \in \Omega, \\ u\,(x,t) = 0, & x \in \partial\Omega, \end{cases}$$

and showed that the global existence of the solutions if the initial data are small enough. Moreover, they derived decay estimate for the energy functional. And then, in [24] Wang established the blow-up result for the above problem when the initial energy is high. Also, in [25] Wang studied blow-up of solutions of the Klein-Gordon equation with arbitrary positive initial energy. Zeng, Mu and Zhou [26] studied blow-up of solutions for the Kirchhoff type equation with arbitrary positive initial energy. Ma, Mu and Zeng [17] obtained blow-up of solutions for the viscoelastic equations with arbitrary positive initial energy.

Recently, boundary dispassion problems for wave equation have been considered by many authors. Vitillaro [22] considered the following problem

$$\begin{cases} u_{tt} - \Delta u = 0, & (x,t) \in \Omega \times (0,\infty), \\ u = 0, & (x,t) \in \Gamma_1 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + |u_t|^{m-2}u_t = |u|^{p-2}u, & (x,t) \in \Gamma_0 \times (0,\infty), \\ u\,(x,0) = u_0(x), u_t\,(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

and proved the local existence of the solutions in energy space when $m > \frac{r}{r+1-p}$ or $n = 1, 2$, where $r = \frac{2(n-1)}{n-2}$, and global existence when $p \leq m$ or the initial data was chosen suitably. The authors in [4] considered a semilinear wave equation with a nonlinear boundary dissipation and nonlinear boundary/interior sources and establish a general decay estimate of the energy. Cavalcanti et al. [5] studied a problem of the form

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u = 0, & (x,t) \in \Gamma_1 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^t h(t-\tau)\frac{\partial u}{\partial \nu} d\tau + h(u_t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty), \\ u\,(x,0) = u_0(x), u_t\,(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

for g, h specific functions and established uniform decay rate results under quite restrictive assumptions on both the damping function h and the kernel g. Li ,Zhao and Chen [13] studied a problem of the form

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-\tau)\mathrm{div}(a(x)\nabla u(\tau))d\tau + |u|^\gamma u = 0, & (x,t) \in \Omega \times (0,\infty), \\ u = 0, & (x,t) \in \Gamma_1 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} - \displaystyle\int_0^t g(t-\tau)(a(x)\nabla u(\tau)) \cdot \nu d\tau + g(u_t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty), \\ u\,(x,0) = u_0(x), u_t\,(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

They proved the existence and uniqueness of its global solution by means of the Galerkin method and showed the uniform decay rate of the energy. In [12] Li studied the following system

$$\begin{cases} u_{tt}-k_0\Delta u+\displaystyle\int_0^t g(t-\tau)\mathrm{div}(a(x)\nabla u(\tau))d\tau + b(x)h(u_t)=0, & (x,t)\in\Omega\times(0,\infty),\\ u=0, & (x,t)\in\Gamma_1\times(0,\infty),\\ -\dfrac{\partial u}{\partial\nu}+\displaystyle\int_0^t g(t-\tau)(a(x)\nabla u(\tau))\cdot\nu d\tau=f(u), & (x,t)\in\Gamma_0\times(0,\infty),\\ u\,(x,0)=u_0(x), u_t\,(x,0)=u_1(x), & x\in\Omega, \end{cases}$$

and established the uniform decay estimates of solutions of the above problem. However, they did not give a sufficient condition for the initial data such that the corresponding solution blows up in finite time with arbitrary positive initial energy.

Motivated by the above work, we intend to employing the so called concavity argument which was first introduced by Levine (see [10, 11]), our main purpose is to establish some sufficient conditions for initial data with arbitrary initial energy such that the corresponding solution of (1.1) blows up in finite time.

In the paper, we denote

$$V=\{u\in H^1(\Omega)\,|u=0 \text{ on } \Gamma_1\},$$

and give the assumptions on $g(s)$, $a(x)$ and $f(s)$:

$(A1)$ $g: R^+\to R^+$ is a bounded C^1 function and non-increasing function satisfying $g(0)>0$.

$(A2)$ The function $e^{\frac{t}{2}}g(t)$ is of positive type in the following sense:

$$\int_0^t v(s)\int_0^s e^{\frac{s-\tau}{2}}g(s-\tau)v(\tau)d\tau ds\geq 0$$

for all $v\in C^1([0,\infty))$ and $t>0$.

$(A3)$ $a:\Omega\to R^+$ is a nonnegative bounded function and $a(x)\geq a_0>0$ on Ω with

$$\|a(x)\|_{L^\infty}\int_0^\infty g(s)ds=k<1.$$

$(A4)$ There exists a positive constant $\alpha>0$ such that

$$sf(s)\geq(2+\alpha)F(s), s\in R,$$

where

$$F(s)=\int_0^s f(\tau)d\tau.$$

Remark 1.1. We note that Assumption 1.1 is also used in [12, 13]. For the definition of $b(t)$ of positive type in detail, we refer the readers to [18]. And an example of $b(t)$ of positive type is positive, decreasing, convex $b(t)$ (see [9]). Thus, it is obvious that $g(t)=\epsilon e^{-t}$ $(0<\epsilon<1)$ satisfies the assumptions (A1), (A2) and (A3).

Remark 1.2. It is clear that $f(s) = |s|^p s$, $p > 2$, $\alpha \leq p$ satisfies the assumption (A4).

Our result are based on the following existence and uniqueness theorem of solution to the problem (1.1).

Theorem 1.1. Under the assumptions (A1)-(A4), let the initial data $(u_0, u_1) \in \big(H^2(\Omega) \cap V(\Omega)\big) \times V(\Omega)$, and f satisfying the following conditions: $f(0) = 0$ and

$$|f(u) - f(v)| \leq c(|u|^{p-1} + |v|^{p-1})|u - v|$$

for all $u, v \in R$, some constant $c > 0$ and

$$1 < p \leq \frac{n-1}{n-2} \quad \text{when} \quad n \geq 3$$

then there exists a unique solution $u(t)$ to (1.1) satisfying

$$u \in L^\infty_{\text{loc}}\big(0, \infty; V(\Omega)\big) \cap H^2(\Omega), u_t \in L^\infty_{\text{loc}}\big(0, \infty; V(\Omega)\big), u_{tt} \in L^\infty_{\text{loc}}\big(0, \infty; L^2(\Omega)\big).$$

Moreover, we have

$$u \in C\big([0, \infty); H^1_0(\Omega)\big), u_t \in C\big([0, \infty); L^2(\Omega)\big).$$

Proof. The proof can be obtained by the Faedo-Galerkin method and calculus theorem in an abstract (c.f. [12, 13]).

Our main blow-up result for the problem (1.1) with arbitrarily initial energy is stated as follows.

Theorem 1.2. Under the assumptions (A1)-(A4), if $k < \frac{\alpha}{2+\alpha}$ and either one of the following states is satisfied:

(1) $E(0) < 0$;

(2) $E(0) = 0$ and $\int_\Omega u_0 u_1 dx \geq 0$;

(3) $E(0) > 0, \int_\Omega u_0 u_1 dx \geq 0,\ I(u_0) < 0$ and $\|u_0\|_2^2 > \frac{2(2+\alpha)}{\alpha-(2+\alpha)k} C_p^2 E(0)$,

then the solution of the problem (1.1) blows up at a finite time T, upper bounds for T is estimated by $0 < T \leq \frac{G(0)}{\beta G'(0)}$, here $G(t)$ and β are given in (3.1) and (3.6) respectively. where C_P is the constant of the Poincaré's inequality on Ω, the energy functional $E(t)$ and $I(u, v)$ are defined as

$$I(u, v) := \|\nabla u\|_2^2 - \int_{\Gamma_0} u f(u) d\Gamma, \tag{1.2}$$

$$
\begin{aligned}
E(t) :=& \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{1}{2}\|\nabla u(t,\cdot)\|_2^2 \\
&- \frac{1}{2}\|\sqrt{a(x)}\nabla u(t,\cdot)\|_2^2 \int_0^t g(s)ds + \frac{1}{2}(g \circ \nabla u)(t) \\
&- \int_{\Gamma_0} F(u) d\Gamma,
\end{aligned} \tag{1.3}
$$

and $(g \circ v)(t) = \int_0^t g(t-\tau)\|\sqrt{a(x)}\big(v(t,\cdot) - v(\tau,\cdot)\big)\|_2^2 d\tau$.

The rest of this paper is organized as follows. In section2, we introduce some Lemmas needed for the proof of our main results. The proof of our main results is presented in section 3.

2. Preliminaries

In this section, we introduce some Lemmas which play a crucial role in proof of our main result in next section.

Lemma 2.1. $E(t)$ is a non-increasing function.

Proof. By differentiating (1.3) and using (1.1) and (A1), we get

$$
E'(t) = -\|u_t\|_2^2 - \frac{1}{2}g(t)\|\sqrt{a(x)}\nabla u(t,\cdot)\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) \le 0, \tag{2.1}
$$

thus, Lemma 2.1 follows at once. At the same time, we have the following inequality

$$
E(t) \le E(0) - \int_0^t \|u_\tau\|_2^2 d\tau. \tag{2.2}
$$

Lemma 2.2. Assume that $g(t)$ satisfies assumptions (A1) and (A2), $H(t)$ is a twice continuously differentiable function and satisfies

$$
\begin{cases} H''(t) + H'(t) > 2\int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(\tau,x)\nabla u(t,x)dxd\tau, \\ H(0) > 0, \quad H'(0) > 0, \end{cases} \tag{2.3}
$$

for every $t \in [0, T_0)$, and $u(x,t)$ is the solution of the problem (1.1). Then the function $H(t)$ is strictly increasing on $[0, T_0)$.

Proof. Consider the following auxiliary ODE

$$
\begin{cases} h''(t) + h'(t) = 2\int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(\tau,x)\nabla u(t,x))dxd\tau, \\ h(0) = H(0), \quad h'(0) = 0, \end{cases} \tag{2.4}
$$

for every $t \in [0, T_0)$.

It is easy to see that the solution of (2.4) is written as follows

$$h(t) = h(0) + 2\int_0^t \int_0^\zeta e^{\xi-\zeta} \int_0^\xi g(\xi-\tau) \int_\Omega a(x)\nabla u(\zeta,x)\nabla u(\tau,x)dxd\tau d\xi d\zeta \tag{2.5}$$

for every $t \in [0, T_0)$.

By a direct computation, we obtain

$$\begin{aligned} h'(t) &= 2\int_0^t e^\xi e^{-t} \int_0^\xi g(\xi-\tau) \int_\Omega a(x)\nabla u(\xi,x)\nabla u(\tau,x)dxd\tau d\xi \\ &= 2e^{-t}\int_\Omega a(x)\int_0^t (e^{\frac{\xi}{2}}\nabla u(\xi,x))\int_0^\xi (e^{\frac{\xi-\tau}{2}} g(\xi-\tau))(e^{\frac{\tau}{2}}\nabla u(\tau,x))d\tau d\xi dx \end{aligned}$$

for every $t \in [0, T_0)$.

Because $g(t)$ satisfies (A2) and $a(x)$ satisfies (A3), then $h'(t) \geq 0$, which implies that $h(t) \geq h(0) = H(0)$. Moreover, we see that $H'(0) > h'(0)$.

Next, we show that

$$H'(t) > h'(t) \quad \text{for} \quad t \geq 0. \tag{2.6}$$

Assume that (2.6) is not true, let us take

$$t_0 = \min\{t \geq 0 : H'(t) = h'(t)\}.$$

By the continuity of the solutions for the ODES (2.3) and (2.4), we see that $t_0 > 0$ and $H'(t_0) = h'(t_0)$, and have

$$\begin{cases} H''(t) - h''(t) + H'(t) - h'(t) > 0, \ t \in [0, T_0), \\ H(0) - h(0) = 0, \quad H'(0) - h'(0) \geq 0, \end{cases}$$

which yields

$$H'(t_0) - h'(t_0) > e^{-t_0}(H'(0) - h'(0)) > 0.$$

This contradicts to $H'(t_0) = h'(t_0)$. Thus, we have $H'(t) > h'(t) \geq 0$, which implies our desired result. The proof of Lemma 2.2 is complete.

Lemma 2.3. Suppose that $(u_0, u_1) \in \big(H^2(\Omega) \cap V(\Omega)\big) \times V(\Omega)$ satisfies

$$\int_\Omega u_0 u_1 \mathrm{d}x \geq 0. \tag{2.7}$$

If the local solution $u(t)$ of the problem (1.1) exists on $[0, T)$ and satisfies

$$I\big(u(t)\big) < 0, \tag{2.8}$$

then $H(t) = \|u(t,\cdot)\|_2^2$ is strictly increasing on $[0, T)$.

Proof. Since $I(u) := \|\nabla u\|_2^2 - \int_{\Gamma_0} uf(u)d\Gamma < 0$, and $u(t)$ is the local solution of problem (1.1), by a simple computation, we have

$$\begin{aligned}
\frac{1}{2}\frac{dH}{dt} &= \int_\Omega uu_t dx,\\
\frac{1}{2}\frac{d^2H}{dt^2} &= \int_\Omega |u_t|^2 dx + \int_\Omega uu_{tt} dx\\
&= \|u_t\|^2 - \int_\Omega uu_t dx + \int_{\Gamma_0} uf(u)d\Gamma - \|\nabla u\|^2\\
&\quad + \int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(\tau,x)\nabla u(t,x)dxd\tau\\
&> -\int_\Omega uu_t dx + \int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(\tau,x)\nabla u(t,x)dxd\tau,
\end{aligned}$$

which yields

$$\frac{1}{2}(\frac{d^2H}{dt^2} + \frac{dH}{dt}) > \int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(\tau,x)\nabla u(t,x)dxd\tau.$$

Therefore, by Lemma 2.2, the proof of Lemma 2.3 is complete.

Lemma 2.4. If $(u_0, u_1) \in \big(H^2(\Omega)\cap V(\Omega)\big)\times V(\Omega)$ satisfy the assumptions (3) in Theorem 1.2, then the solution $u(x,t)$ of problem (1.1) satisfies

$$I(u(t,x)) < 0, \tag{2.9}$$

$$\|u(t,x)\|_2^2 > \frac{2(2+\alpha)}{\alpha-(2+\alpha)k}C_p^2E(0). \tag{2.10}$$

for every $t \in [0, T)$.

Proof. We will prove the lemma by a contradiction argument. Firstly we assume that (2.9) is not true over $[0, T)$, it means that there exists a time t_1 such that

$$t_1 = \min\{t \in (0,T) : I\big(u(t,x)\big) = 0\} > 0. \tag{2.11}$$

Since $I\big(u(t,x)\big) < 0$ on $[0, t_1)$, by Lemma 2.3 we see that $H(t) = \|u(t,\cdot)\|_2^2$ is strictly increasing over$[0, t_1)$, which implies

$$H(t) = \|u(t,\cdot)\|_2^2 > \|u_0\|_2^2 > \frac{2(2+\alpha)}{\alpha-(2+\alpha)k}C_p^2E(0).$$

By the continuity of $H(t) = \|u(t,\cdot)\|_2^2$ on t, we have

$$H(t_1) = \|u(t_1,\cdot)\|_2^2 > \frac{2(2+\alpha)}{\alpha-(2+\alpha)k}C_p^2E(0). \tag{2.12}$$

On the other hand, by (2.2) we get

$$\frac{1}{2}\|\nabla u(t_1,\cdot)\|_2^2-\frac{1}{2}\int_0^{t_1} g(s)ds\|\sqrt{a(x)}\nabla u(t_1,\cdot)\|_2^2-\int_{\Gamma_0} F(u(t_1))d\Gamma \le E(0). \tag{2.13}$$

It follows from (A3), (A4) and (2.11) that

$$(\frac{1-k}{2}-\frac{1}{2+\alpha})\|\nabla u(t_1,\cdot)\|_2^2 \le E(0). \tag{2.14}$$

Thus, by the Poincaré's inequality and $k<\frac{\alpha}{2+\alpha}$, we see that

$$H(t_1)=\|u(t_1,\cdot)\|_2^2 \le \frac{2(2+\alpha)}{\alpha-(2+\alpha)k}C_p^2E(0). \tag{2.15}$$

Obviously, (2.15) contradicts to (2.12). Thus, (2.9) holds for every $t\in[0,T)$.

By Lemma 2.3, it follows that $H(t)=\|u(t,\cdot)\|_2^2$ is strictly increasing on $[0,T)$, which implies

$$H(t)=\|u(t,\cdot)\|_2^2>\|u_0\|_2^2>\frac{2(2+\alpha)}{\alpha-(2+\alpha)k}C_p^2E(0)$$

for every $t\in[0,T)$. The proof of Lemma 2.4 is complete.

3. The Proof of Theorem 1.2

To prove our main result, we adopt the concavity method introduced by Levine, and define the following auxiliary function:

$$G(t)=\|u(t,\cdot)\|_2^2+\int_0^t\|u(\tau,\cdot)\|_2^2d\tau+(t_2-t)\|u_0\|_2^2+a(t_3+t)^2, \tag{3.1}$$

where t_2, t_3 and a are certain positive constants determined later.

Proof of Theorem 1.2. By direct computation, we obtain

$$G'(t)=2\int_\Omega uu_tdx+2\int_0^t(u,u_\tau)d\tau+2a(t_3+t), \tag{3.2}$$

and

$$\begin{aligned}\frac{1}{2}G'' =& \|u_t\|_2^2-\|\nabla u\|^2+\int_{\Gamma_0}uf(u)d\Gamma\\ &+\int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(\tau,x)\nabla u(t,x)dxd\tau+a\\ =& \|u_t\|_2^2-\|\nabla u\|^2+\int_{\Gamma_0}uf(u)d\Gamma\\ &+\int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(t,x)(u(\tau,x)-\nabla u(t,x))dxd\tau+a\\ &+\int_0^t g(t-\tau)d\tau\|\sqrt{a(x)}\nabla u(t,x)\|_2^2.\end{aligned} \tag{3.3}$$

By the Young's inequality, for any $\epsilon > 0$, we have

$$\begin{aligned}&\int_0^t g(t-\tau)\int_\Omega a(x)\nabla u(t,x)|\nabla u(\tau,x)-\nabla u(t,x)|dxd\tau\\ &\le \frac{1}{2\epsilon}\int_0^t g(\tau)d\tau\|\sqrt{a(x)}\nabla u(t,\cdot)\|_2^2+\frac{\epsilon}{2}(g\circ\nabla u)(t).\end{aligned}$$

Taking $\epsilon = \frac{1}{2}$ into the above inequality, by (A4), (1.3), (2.2), (3.3), Lemma 2.3 and the Poincaré's inequality, we obtain

$$\begin{aligned}G'' &\ge (4+\alpha)\|u_t\|_2^2+\alpha\|\nabla u\|_2^2-(\frac{1}{\epsilon}+\alpha)\int_0^t g(\tau)d\tau\|\sqrt{a(x)}\nabla u\|_2^2\\ &\quad +(2+\alpha-\epsilon)(g\circ\nabla u)(t)-2(2+\alpha)E(t)+2a\\ &\ge (4+\alpha)\|u_t\|_2^2+\big(\alpha-(\alpha+\frac{1}{\epsilon})k\big)\nabla u\|_2^2+(2+\alpha-\epsilon)(g\circ\nabla u)(t)\\ &\quad +2(2+\alpha)\int_0^t\|u_\tau\|_2^2d\tau-2(2+\alpha)E(0)+2a\\ &\ge (4+\alpha)\|u_t\|_2^2+2(2+\alpha)\int_0^t\|u_\tau\|_2^2d\tau+\frac{\alpha-(\alpha+\frac{1}{\epsilon})k}{C_p^2}\|u\|_2^2\\ &\quad +(2+\alpha-\epsilon)(g\circ\nabla u)(t)-2(2+\alpha)E(0)+2a\\ &\ge (4+\alpha)\|u_t\|_2^2+2(2+\alpha)\int_0^t\|u_\tau\|_2^2d\tau\\ &\quad +\frac{\alpha-(\alpha+2)k}{C_p^2}\|u\|_2^2-2(2+\alpha)E(0)+2a\end{aligned} \tag{3.4}$$

Case(I): $E(0) < 0$. From (3.4) it follows that

$$G'' \ge (4+\alpha)\|u_t\|_2^2+2(2+\alpha)\int_0^t\|u_\tau\|_2^2d\tau-2(2+\alpha)E(0)+2a.$$

which means that $G''(t) > 0$ for every $t \in (0,T)$. Since $G'(0) \ge 0$ and $G(0) \ge 0$, thus we obtain that $G'(t)$ and $G(t)$ are strictly increasing on $[0,T)$.

We now let the constant a satisfy

$$0 < a \le -2E(0).$$

And set

$$\begin{aligned}A &:= \|u(t,\cdot)\|_2^2+\int_0^t\|u(\tau,\cdot)\|_2^2d\tau+a(t_3+t)^2,\\ B &:= \frac{1}{2}G'(t),\\ C &:= \|u_t(t,\cdot)\|_2^2+\int_0^t\|u_\tau(\tau,\cdot)\|_2^2d\tau+a.\end{aligned}$$

By (3.2) and a simple computation, for all $s \in R$, we have

$$As^2 - 2Bs + C = \int_\Omega \big(su(t,x) - u_t(t,x)\big)^2 dx + \int_0^t \|su(\tau,\cdot) - u_\tau(\tau,\cdot)\|_2^2 d\tau + a\big(s(t_3+t) - 1\big)^2 \geq 0,$$

which implies that $B^2 - AC \leq 0$.

Since we assume that the solution $u(t,x)$ to the problem (1.1) exists for every $t \in [0,T)$, then for $t \in [0,T)$, one has

$$G(t) \geq A, \ G''(t) \geq (4+\alpha)C$$

and

$$G''(t)G(t) - \frac{4+\alpha}{4}\big(G'(t)\big)^2 \geq (4+\alpha)(AC - B^2),$$

which yields

$$G''(t)G(t) - \frac{4+\alpha}{4}\big(G'(t)\big)^2 \geq 0.$$

Let $\beta = \frac{\alpha}{4} > 0$. As $\frac{4+\alpha}{4} > 1$, we see that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(G^{-\beta}(t)\Big) &= -\beta G^{-\beta-1}G' < 0, \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(G^{-\beta}(t)\Big) &= -\beta(-\beta-1)G^{-\beta-2}G'^2 - \beta G^{-\beta-1}G'' \\ &= -\beta G^{-\beta-2}\left[G''G - (1+\beta)G'^2\right] \\ &\leq 0 \end{aligned} \tag{3.5}$$

for every $t \in [0,T)$, which means that the function $G^{-\beta}$ is concave.

Let t_2 and t_3 satisfy

$$\begin{aligned} t_3 &\geq \max\{\frac{4}{a\alpha}\|u_0\|_2^2 - \frac{1}{a}\int_\Omega u_0u_1dx, 0\}, \\ t_2 &\geq 1 + \frac{a}{\|u_0\|_2^2}t_3^2, \end{aligned}$$

from which, we deduce that

$$t_2 \geq \frac{G(0)}{\beta G'(0)}.$$

Since $G^{-\beta}$ is a concave function and $G(0) > 0$, we obtain that

$$G^{-\beta} \leq \frac{G(0) - \beta G'(0)t}{G^{1+\beta}(0)}, \tag{3.6}$$

thus

$$G \geq \left[\frac{G^{1+\beta}(0)}{G(0)-\beta G'(0)t}\right]^{1/\beta}. \tag{3.7}$$

Therefore, there exists a finite time $T \leq \frac{G(0)}{\beta G'(0)} \leq t_2$, such that

$$\lim_{t\to T^-} \|u\|_2^2 + \int_0^t \|u_\tau(\tau,x)\|_2^2 \mathrm{d}\tau = \infty,$$

$$i.e.\ \lim_{t\to T^-} \|u\|_2^2 = \infty.$$

Case(II): $E(0) = 0$. By (A3), (1.3) and (2.2) we have

$$(1-k)\|\nabla u\|_2^2 - 2\int_{\Gamma_0} F(u)d\Gamma < 0, \tag{3.8}$$

for every $t \in [0,T)$.

By (A4) and $k < \frac{\alpha}{2+\alpha}$, we obtain

$$I(u(t,x)) < 0,$$

for every $t \in [0,T)$.

Thus, by $\int_\Omega u_0 u_1 dx \geq 0$ and Lemma 2.3, we see that $H(t) = \|u(t,\cdot)\|_2^2$ is strictly increasing on $[0,T)$.

As (3.4) we also have

$$\begin{aligned} G'' &\geq (4+\alpha)\|u_t\|_2^2 + 2(2+\alpha)\int_0^t \|u_\tau\|_2^2 d\tau + \frac{\alpha-(\alpha+2)k}{C_p^2}\|u\|_2^2 + 2a \\ &\geq (4+\alpha)\|u_t\|_2^2 + 2(2+\alpha)\int_0^t \|u_\tau\|_2^2 d\tau + \frac{\alpha-(\alpha+2)k}{C_p^2}\|u_0\|_2^2 + 2a \end{aligned} \tag{3.9}$$

which means that $G''(t) > 0$ for every $t \in (0,T)$. Since $G'(0) \geq 0$ and $G(0) \geq 0$, thus we obtain that $G'(t)$ and $G(t)$ are strictly increasing on $[0,T)$.

We now let the constant a, t_2, t_3 satisfy

$$\begin{aligned} &(2+\alpha)a \leq \frac{\alpha-(2+\alpha)k}{C_p^2}\|u_0\|_2^2, \\ &t_3 \geq \max\{\frac{4}{a\alpha}\|u_0\|_2^2 - \frac{1}{a}\int_\Omega u_0u_1dx, 0\}, \\ &t_2 \geq 1 + \frac{2}{\alpha}t_3, \end{aligned}$$

Then by the same argument as Case I, we can claim that the corresponding local solution of the equation (1.1) blows up in finite time.

Case (III): $E(0) > 0$. By (3.4), Lemma 2.3 and Lemma 2.4, we have

$$
\begin{aligned}
G'' &\geq (4+\alpha)\|u_t\|_2^2 + 2(2+\alpha)\int_0^t \|u_\tau\|_2^2 d\tau \\
&\quad + \frac{\alpha-(\alpha+2)k}{C_p^2}\|u\|_2^2 - 2(2+\alpha)E(0) + 2a \\
&\geq (4+\alpha)\|u_t\|_2^2 + 2(2+\alpha)\int_0^t \|u_\tau\|_2^2 d\tau \\
&\quad + \frac{\alpha-(\alpha+2)k}{C_p^2}\|u_0\|_2^2 - 2(2+\alpha)E(0) + 2a
\end{aligned} \tag{3.10}
$$

which means that $G''(t) > 0$ for every $t \in (0, T)$. Since $G'(0) \geq 0$ and $G(0) \geq 0$, thus we obtain that $G'(t)$ and $G(t)$ are strictly increasing on $[0, T)$.

It follows from the assumptions (3) in Theorem 1.2 and $k < \frac{\alpha}{2+\alpha}$ that, we can choose a, t_2, t_3 to satisfy

$$
\begin{aligned}
&(2+\alpha)a \leq \frac{\alpha-(\alpha+2)k}{C_p^2}\|u_0\|_2^2 - 2(2+\alpha)E(0).,\\
&t_3 \geq \max\{\frac{4}{a\alpha}\|u_0\|_2^2 - \frac{1}{a}\int_\Omega u_0u_1 dx, 0\},\\
&t_2 \geq 1 + \frac{2}{\alpha}t_3,
\end{aligned}
$$

As the proof of Case I, by a concavity argument we can obtain that, there exists a finite time $T < \infty$, such that

$$
\lim_{t\to T^-}\|u\|_2^2 = \infty.
$$

The proof of Theorem 1.2 is complete.

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