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OBSERVABILITY ESTIMATE AND NULL CONTROLLABILITY FOR ONE-DIMENSIONAL FOURTH ORDER PARABOLIC EQUATION

Zhongcheng Zhou

Abstract. This paper studies the observability and null controllability for a class of one-dimensional fourth order parabolic equation. By means of establishing the global Carleman estimates, we derive the observability inequalities for one-dimensional linear fourth order parabolic equation with potentials. The null controllability results for one-dimensional fourth order semilinear equation are also presented.

1. INTRODUCTION

This study concerns the observability and null controllability of one-dimensional fourth order parabolic system. Such fourth order parabolic equation, sometimes known as Cahn-Hilliard type equation, appear in the study of phase separation in cooling binary solutions and in other contexts generating spatial pattern formation (see [2]).

Let $T > 0$, $\Omega = (0, 1)$ and ω be a nonempty open subset of Ω. Let ω_0 be another open and nonempty subset of Ω such that $\overline{\omega}_0 \subset \omega$. Throughout this study, notations Q, Q^ω and Q^{ω_0} stand for $\Omega \times (0, T)$, $\omega \times (0, T)$ and $\omega_0 \times (0, T)$, respectively.

Consider the following one-dimensional linear fourth order parabolic system

$$(1.1) \qquad \begin{cases} u_t + u_{xxxx} = \xi, & (x, t) \in Q, \\ u(0, t) = u(1, t) = 0, & t \in (0, T), \\ u_x(0, t) = u_x(1, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

Corresponding to each $u_0 \in L^2(\Omega)$ and each $\xi \in L^2(Q)$, system (1.1) admits a unique function $u \in C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^2(\Omega))$. Moreover, $u \in L^2((\delta, T); H^4(\Omega))$ and $u_t \in L^2((\delta, T) \times \Omega)$ for all $\delta \in (0, T)$ (see, for instance, [11]).

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We begin with the Carleman inequality for the solution of system (1.1). It is well known that in general, Carleman inequalities imply the observability of certain systems, and observability estimates play important roles for solving control problems. For instance, we can obtain the controllability for linear and nonlinear partial differential equations of both parabolic type and hyperbolic type (see [7, 9, 12, 14, 15]). This class of estimates are also useful for solving a variety of inverse problems (see [13]).

Let $\psi \in C^\infty(\overline{\Omega})$ satisfy that $\psi > 0$ in Ω, $\psi(0) = \psi(1) = 0$, $\|\psi\|_{C(\overline{\Omega})} = 1$, $|\psi_x| > 0$ in $\overline{\Omega}\backslash\omega_0$, $\psi_x(0) > 0$ and $\psi_x(1) < 0$. For any given positive constants λ and μ, we set $a(x,t) = \dfrac{e^{\mu(\psi(x)+3)} - e^{5\mu}}{t(T-t)}$, $\theta(x,t) = e^{\lambda a(x,t)}$ and $\varphi(x,t) = \dfrac{e^{\mu(\psi(x)+3)}}{t(T-t)}$, $\forall (x,t) \in Q$.

The first main result is on the following Carleman estimate for system (1.1).

Theorem 1.1. *There exist three constants $\mu_0 > 1$, $C_0 > 0$ and $C > 0$ such that for $\mu = \mu_0$ and for all numbers $\lambda \geq C_0(T + T^2)$, it holds that*

$$
\begin{aligned}
&\lambda^7 \int_Q \theta^2\varphi^7 u^2 dxdt + \lambda^5 \int_Q \theta^2\varphi^5 u_x^2 dxdt + \lambda^3 \int_Q \theta^2\varphi^3 u_{xx}^2 dxdt \\
\leq\ & C\Big(\lambda^7 \int_{Q^\omega} \theta^2\varphi^7 u^2 dxdt + \int_Q \theta^2\xi^2 dxdt\Big),
\end{aligned}
\tag{1.2}
$$

where the constants μ_0, C_0 and C depend only on ω, u is the solution of system (1.1) *corresponding to $u_0 \in L^2(\Omega)$ and $\xi \in L^2(Q)$.*

In Section 3, we shall see that Theorem 1.1 implies the observability of the one-dimensional linear fourth order parabolic system as follows

$$
\begin{cases}
p_t - p_{xxxx} - gp = 0, & (x,t) \in Q, \\
p(0,t) = p(1,t) = 0, & t \in (0,T), \\
p_x(0,t) = p_x(1,t) = 0, & t \in (0,T), \\
p(x,T) = p_0(x). & x \in \Omega.
\end{cases}
\tag{1.3}
$$

Corresponding to each $p_0 \in L^2(\Omega)$ and each $g \in L^\infty(Q)$, system (1.3) admits a unique function $p \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^2(\Omega))$ (see, for instance, [11]). We have the following theorem.

Theorem 1.2. *There exists a positive constant C, depending on ω, such that*

$$
\int_\Omega p^2(x,0)dx \leq \exp\Big\{C\Big(1 + \frac{1}{T} + \|g\|_{L^\infty(Q)}^{\frac{2}{7}} + \|g\|_{L^\infty(Q)}T\Big)\Big\} \int_{Q^\omega} p^2 dxdt,
\tag{1.4}
$$

where p is the solution of system (1.3) corresponding to $p_0 \in L^2(\Omega)$ and $g \in L^\infty(Q)$.

The (boundary and/or internal) observability estimates for linear heat and wave equations have been studied from many past publications in recent years (see [7, 13, 14, 15]). As far as fourth order PDEs are concerned, the explicit observability estimates for multidimensional plate system $p_{tt} + \Delta^2 p + gp = 0$ with g potential have been well understood under the boundary conditions $p = \Delta p = 0$ and suitable initial data, by means of pointwise weighted estimates for the Schrödinger operator (see [14, 15]). However, similar problems under the boundary conditions $p = \frac{\partial p}{\partial v} = 0$, where ν is the unit outward normal vector, have not been solved on plate equation. The reason is that the techniques to deal with the boundary conditions $p = \Delta p = 0$ do not adapt to the same problems with the boundary conditions $p = \frac{\partial p}{\partial v} = 0$. Moreover, to our best knowledge, explicit observability estimates for higher order parabolic system have not been well understood, we can not adapt to Carleman estimates for the Schrödinger system to obtain the Carleman estimates for higher order parabolic system.

Stimulated by [14], our tool to prove the inequality (1.2) is to present a weighted pointwise estimate for system (1.1). Based on this Carleman estimate and the energy estimates for system (1.3) , the explicit observability inequality (1.4) can be obtained.

It should be pointed out that the technique to prove Theorem 1.1 and Theorem 1.2 can not be applied to multi-dimensional cases. The main difficulties lie in the boundary estimates in the proof of the corresponding Carleman inequality would be much more complex for multi-dimensional cases than that for the one-dimensional case under consideration.

Our next main result concerns the null controllability for fourth order semilinear parabolic equation. To our best knowledge, there has been limited publications on the controllability of higher order parabolic equation. Among them, the approximate controllability and non-approximate controllability of higher order parabolic equation were studied in [6]. Later, Lin Guo [5, 10] considered the null boundary controllability for a one-dimensional fourth order parabolic equation with nonlinear term f belonging to Gevrey class 2 through reducing the control problem into two well-posed PDEs problems. Recently, Cerpa [3, 4] considered the local boundary controllability for an especial one-dimensional fourth order parabolic equation(Kuramoto-Sivashinsky equation). What we should point out is that the nonlinear terms in [4, 10] are smooth enough in some sense.

Next, we consider the following one-dimensional semilinear fourth order parabolic equation

$$(1.5) \qquad \begin{cases} u_t + u_{xxxx} + f(u) = \chi_\omega h, & (x,t) \in Q, \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where h is the control.

We have the null controllability result of system (1.5) as follows.

Theorem 1.3. *Assume that f is a globally Lipschitz continuous function with $f(0) = 0$. Then for each $u_0 \in L^2(\Omega)$, there exists a control $h \in L^2(\omega \times (0,T))$ such that the corresponding solution of* (1.5) *satisfies*

$$u(\cdot, T) = 0, \quad \text{in } \Omega.$$

Remark 1.1. In Theorem 1.3, we can also deal with the null controllability for semilinear fourth order parabolic equation (1.5) with more stronger nonlinear term by means of explicit observability inequality (1.4).

We shall first establish the null controllability of the linearized system of system (1.5) by making use of the observability estimate (1.4), then applying Kakutani's Fixed Point Theorem (see [1]) to prove Theorem 1.3.

The rest of this paper is organized as follows. In Section 2, we shall prove the Carleman inequality (1.2). Section 3 and Section 4 are devoted to proving observability estimate (1.4) and Theorem 1.3, respectively.

2. Proof of Theorem 1.1

We may as well assume that the solution u is sufficiently smooth. Set $v = \theta u$ and $Lu = u_t + u_{xxxx}$. It is obvious that

$$\begin{aligned}
u_t &= \theta^{-1}(v_t - \lambda a_t v), \\
u_x &= \theta^{-1}(v_x - \lambda a_x v), \\
u_{xx} &= \theta^{-1}\big\{v_{xx} - 2\lambda a_x v_x + (-\lambda a_{xx} + \lambda^2 a_x^2)v\big\}, \\
u_{xxx} &= \theta^{-1}\big\{v_{xxx} - 3\lambda a_x v_{xx} + (3\lambda^2 a_x^2 - 3\lambda a_{xx})v_x + (-\lambda^3 a_x^3 + 3\lambda^2 a_x a_{xx} - \lambda a_{xxx})v\big\}, \\
u_{xxxx} &= \theta^{-1}\big\{v_{xxxx} - 4\lambda a_x v_{xxx} + (6\lambda^2 a_x^2 - 6\lambda a_{xx})v_{xx} + (-4\lambda^3 a_x^3 + 12\lambda^2 a_x a_{xx} \\
&\quad -4\lambda a_{xxx})v_x + (\lambda^4 a_x^4 - 6\lambda^3 a_x^2 a_{xx} + 3\lambda^2 a_{xx}^2 + 4\lambda^2 a_x a_{xxx} - \lambda a_{xxxx})v\big\}.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\theta^2 |Lu|^2 &= \theta^2 (u_t + u_{xxxx})^2 \\
&= \{v_t + v_{xxxx} - 4\lambda a_x v_{xxx} + (6\lambda^2 a_x^2 - 6\lambda a_{xx})v_{xx} \\
&\quad +(-4\lambda^3 a_x^3 + 12\lambda^2 a_x a_{xx} - 4\lambda a_{xxx})v_x \\
&\quad +(\lambda^4 a_x^4 - 6\lambda^3 a_x^2 a_{xx} + 3\lambda^2 a_{xx}^2 + 4\lambda^2 a_x a_{xxx} - \lambda a_{xxxx} - \lambda a_t)v\}^2.
\end{aligned}$$

Define

$$\theta Lu = I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 &= v_t + B_1 v_x - 4\lambda a_x v_{xxx},\\
I_2 &= v_{xxxx} + C_1 v_{xx} + A_1 v,\\
I_3 &= -12\lambda^3 a_x^2 a_{xx} v - 6\lambda a_{xx} v_{xx},\\
A_1 &= \lambda^4 a_x^4 + 6\lambda^3 a_x^2 a_{xx} + 3\lambda^2 a_{xx}^2 + 4\lambda^2 a_x a_{xxx} - \lambda a_{xxxx} - \lambda a_t,\\
B_1 &= -4\lambda^3 a_x^3 + 12\lambda^2 a_x a_{xx} - 4\lambda a_{xxx},\\
C_1 &= 6\lambda^2 a_x^2.
\end{aligned}
$$

The proof of Theorem 1.1 shall be completed in the following five steps.

Step 1. We shall prove the following weighted pointwise estimate for the fourth order parabolic operator L, which is similar to those for second order operators in [8, 14]

$$
\theta^2|Lu|^2 \geq \{\cdot\}_t + \{\cdot\}_x + \{\cdot\}v_t v_x + \{\cdot\}v^2 + \{\cdot\}v_x^2 + \{\cdot\}v_{xx}^2 + \{\cdot\}v_{xxx}^2, \tag{2.1}
$$

where

$$
\begin{aligned}
\{\cdot\}_t =& \{v_{xx}^2 - C_1 v_x^2 + A_1 v^2 - 12\lambda^3 a_x^2 a_{xx} v^2 + 6\lambda a_{xx} v_x^2\}_t,\\
\{\cdot\}_x =& \{2v_t v_{xxx} - 2v_{xt} v_{xx} + 2C_1 v_t v_x + 2B_1 v_x v_{xxx} + B_{1xx} v_x^2 - 2B_{1x} v_x v_{xx} - B_1 v_{xx}^2\\
&+ B_1 C_1 v_x^2 + A_1 B_1 v^2 - 4\lambda a_x v_{xxx}^2 - 4\lambda a_x C_1 v_{xx}^2 + (-4\lambda)[-a_x A_1 v_x^2\\
&+ 2a_x A_1 v v_{xx} - 2(a_x A_1)_x v v_x + (a_x A_1)_{xx} v^2] - 12\lambda a_{xx} v_t v_x - 12\lambda^3 B_1 a_x^2 a_{xx} v^2\\
&+ (-6\lambda) a_{xx} B_1 v_x^2 + (48\lambda^4)[-a_x^3 a_{xx} v_x^2 + 2a_x^3 a_{xx} v_{xx} v - 2(a_x^3 a_{xx})_x v_x v\\
&+ (a_x^3 a_{xx})_{xx} v^2] + 24\lambda^2 a_x a_{xx} v_{xx}^2 + (-12\lambda^3)[2a_x^2 a_{xx} v_{xxx} v - 2(a_x^2 a_{xx})_x v_{xx} v\\
&- 2a_x^2 a_{xx} v_{xx} v_x + (a_x^2 a_{xx})_x v_x^2 + 2(a_x^2 a_{xx})_{xx} v v_x + (a_x^2 a_{xx})_x v_x^2 - (a_x^2 a_{xx})_{xxx} v^2]\\
&+ 6\lambda a_{xxx} v_{xx}^2 - 2a_{xx} v_{xx} v_{xxx} + (-12\lambda^3)[2C_1 a_x^2 a_{xx} v v_x - (C_1 a_x^2 a_{xx})_x v^2]\\
&+ (-6\lambda)[2A_1 a_{xx} v v_x - (A_1 a_{xx})_x v^2]\}_x,\\
\{\cdot\}v_t v_x =& \{-2C_{1x} + 12\lambda a_{xxx}\}v_t v_x,\\
\{\cdot\}v^2 =& \{-A_{1t} - (A_1 B_1)_x + 4\lambda(a_x A_1)_{xxx} + 12\lambda^3(a_x^2 a_{xx})_t + 12\lambda^3(a_x^2 a_{xx} B_1)_x\\
&- 48\lambda^4(a_x^3 a_{xx})_{xxx} - 12\lambda^3(a_x^2 a_{xx})_{xxxx} - 12\lambda^3(a_x^2 C_1 a_{xx})_{xx}\\
&- 24\lambda^3 a_x^2 a_{xx} A_1 - 6\lambda(A_1 a_{xx})_{xx}\}v^2,\\
\{\cdot\}v_x^2 =& \{C_{1t} v_x^2 - B_{1xxx} v_x^2 - (B_1 C_1)_x - 12\lambda(a_x A_1)_x - 6\lambda a_{xxt} + 6\lambda(a_{xx} B_1)_x\\
&+ 144\lambda^4(a_x^3 a_{xx})_x + 48\lambda^3(a_x^2 a_{xx})_{xx} + 24\lambda^3 a_x^2 C_1 a_{xx} + 12\lambda A_1 a_{xx}\}v_x^2,\\
\{\cdot\}v_{xx}^2 =& \{3B_{1x} + 4\lambda(a_x C_1)_x - 24\lambda^2(a_x a_{xx})_x - 24\lambda^3 a_x^2 a_{xx} - 6\lambda a_{xxxx} - 12\lambda a_{xx} C_1\}v_{xx}^2,\\
\{\cdot\}v_{xxx}^2 =& \{16\lambda a_{xx}\}v_{xxx}^2.
\end{aligned}
$$

Indeed, according to the definition of I_1 and I_2, it holds that

$$
2I_1 I_2 = 2(v_t + B_1 v_x - 4\lambda a_x v_{xxx})(v_{xxxx} + C_1 v_{xx} + A_1 v). \tag{2.2}
$$

Calculating each term in the righthand of (2.2), we have

$$2v_t v_{xxxx} = (v_{xx}^2)_t + (2v_t v_{xxx})_x - (2v_{xt}v_{xx})_x, \tag{2.3}$$

$$2v_t(C_1 v_{xx}) = -2C_{1x}v_t v_x + C_{1t}v_x^2 - (C_1 v_x^2)_t + (2C_1 v_t v_x)_x, \tag{2.4}$$

$$2v_t(A_1 v) = -A_{1t}v^2 + (A_1 v^2)_t, \tag{2.5}$$

$$\begin{aligned} 2B_1 v_x v_{xxxx} = {} & (2B_1 v_x v_{xxx})_x + 3B_{1x}v_{xx}^2 + (B_{1xx}v_x^2)_x - B_{1xxx}v_x^2 \\ & - (2B_{1x}v_x v_{xx} + B_1 v_{xx}^2)_x, \end{aligned} \tag{2.6}$$

$$2B_1 v_x(C_1 v_{xx}) = (B_1 C_1 v_x^2)_x - (B_1 C_1)_x v_x^2, \tag{2.7}$$

$$2B_1 v_x(A_1 v) = -(A_1 B_1)_x v^2 + (A_1 B_1 v^2)_x, \tag{2.8}$$

$$2(-4\lambda a_x v_{xxx})v_{xxxx} = (-4\lambda)\{(a_x v_{xxx}^2)_x - a_{xx}v_{xxx}^2\}, \tag{2.9}$$

$$2(-4\lambda a_x v_{xxx})(C_1 v_{xx}) = 4\lambda(a_x C_1)_x v_{xx}^2 - (4\lambda a_x C_1 v_{xx}^2)_x, \tag{2.10}$$

$$\begin{aligned} 2(-4\lambda a_x v_{xxx})(A_1 v) = {} & (-4\lambda)\big\{3(a_x A_1)_x v_x^2 - (a_x A_1)_{xxx}v^2 - (a_x A_1 v_x^2)_x \\ & + (2a_x A_1 v v_{xx})_x - (2(a_x A_1)_x v v_x)_x + ((a_x A_1)_{xx}v^2)_x\big\}. \end{aligned} \tag{2.11}$$

According to the definition of I_1 and I_3, it holds that

$$2I_1 I_3 = 2(v_t + B_1 v_x - 4\lambda a_x v_{xxx})(-12\lambda^3 a_x^2 a_{xx} v - 6\lambda a_{xx} v_{xx}). \tag{2.12}$$

Calculating each term in the righthand of (2.12), we have

$$2v_t(-12\lambda^3 a_x^2 a_{xx} v) = 12\lambda^3(a_x^2 a_{xx})_t v^2 - 12\lambda^3(a_x^2 a_{xx} v^2)_t, \tag{2.13}$$

$$2v_t(-6\lambda a_{xx}v_{xx}) = (-6\lambda)\{-2a_{xxx}v_t v_x + a_{xxt}v_x^2 + (2a_{xx}v_t v_x)_x - (a_{xx}v_x^2)_t\}, \tag{2.14}$$

$$2B_1 v_x(-12\lambda^3 a_x^2 a_{xx} v) = 12\lambda^3(a_x^2 a_{xx} B_1)_x v^2 - 12\lambda^3(B_1 a_x^2 a_{xx} v^2)_x, \tag{2.15}$$

$$2B_1 v_x(-6\lambda a_{xx}v_{xx}) = (-6\lambda)\{(a_{xx}B_1 v_x^2)_x - (a_{xx}B_1)_x v_x^2\}, \tag{2.16}$$

$$\begin{aligned}&2(-4\lambda a_x v_{xxx})(-12\lambda^3 a_x^2 a_{xx} v)\\ &= 48\lambda^4\big\{-(a_x^3 a_{xx})_{xxx}v^2 + 3(a_x^3 a_{xx})_x v_x^2 - (a_x^3 a_{xx} v_x^2)_x + (2a_x^3 a_{xx} v_{xx} v)_x\\ &\quad - (2(a_x^3 a_{xx})_x v_x v)_x + ((a_x^3 a_{xx})_{xx} v^2)_x\big\},\end{aligned} \tag{2.17}$$

$$2(-4\lambda a_x v_{xxx})(-6\lambda a_{xx} v_{xx}) = -24\lambda^2 (a_x a_{xx})_x v_{xx}^2 + 24\lambda^2 (a_x a_{xx} v_{xx}^2)_x. \tag{2.18}$$

According to the definition of I_2 and I_3, it holds that

$$2I_2 I_3 = 2(v_{xxxx} + C_1 v_{xx} + A_1 v)(-12\lambda^3 a_x^2 a_{xx} v - 6\lambda a_{xx} v_{xx}). \tag{2.19}$$

Calculating each term in the righthand of (2.19), we have

$$\begin{aligned}&2v_{xxxx}(-12\lambda^3 a_x^2 a_{xx} v)\\ &= (-12\lambda^3)\big\{(2a_x^2 a_{xx} v_{xxx} v)_x - (2(a_x^2 a_{xx})_x v_{xx} v)_x - (2a_x^2 a_{xx} v_{xx} v_x)_x\\ &\quad + 2a_x^2 a_{xx} v_{xx}^2 + ((a_x^2 a_{xx})_x v_x^2)_x - 4(a_x^2 a_{xx})_{xx} v_x^2 + (2(a_x^2 a_{xx})_{xx} v v_x)_x\\ &\quad + ((a_x^2 a_{xx})_x v_x^2)_x - ((a_x^2 a_{xx})_{xxx} v^2)_x + (a_x^2 a_{xx})_{xxxx} v^2\big\},\end{aligned} \tag{2.20}$$

$$\begin{aligned}&2v_{xxxx}(-6\lambda a_{xx} v_{xx})\\ &= (-6\lambda)(-2a_{xx} v_{xxx}^2 + a_{xxxx} v_{xx}^2) + 6\lambda(a_{xxx} v_{xx}^2 - 2a_{xx} v_{xx} v_{xxx})_x,\end{aligned} \tag{2.21}$$

$$\begin{aligned}&2C_1 v_{xx}(-12\lambda^3 a_x^2 a_{xx} v)\\ &= (-12\lambda^3)\big\{(a_x^2 C_1 a_{xx})_{xx} v^2 - 2a_x^2 C_1 a_{xx} v_x^2 + (2C_1 a_x^2 a_{xx} v v_x)_x\\ &\quad - ((C_1 a_x^2 a_{xx})_x v^2)_x\big\},\end{aligned} \tag{2.22}$$

$$\begin{aligned}&2C_1 v_{xx}(-6\lambda a_{xx} v_{xx})\\ &= (-12\lambda) a_{xx} C_1 v_{xx}^2,\ \ 2A_1 v(-12\lambda^3 a_x^2 a_{xx} v) = (-24\lambda^3) a_x^2 a_{xx} A_1 v^2,\end{aligned} \tag{2.23}$$

$$\begin{aligned}2A_1 v(-6\lambda a_{xx} v_{xx}) &= (-6\lambda)\big\{(2A_1 a_{xx} v v_x)_x - ((A_1 a_{xx})_x v^2)_x\\ &\quad - 2A_1 a_{xx} v_x^2 + (A_1 a_{xx})_{xx} v^2\big\}.\end{aligned} \tag{2.24}$$

Observe that $|\theta Lu|^2 \geq 2I_1 I_2 + 2I_1 I_3 + 2I_2 I_3$. Combining (2.2) to (2.24), we can obtain (2.1).

Step 2. We shall prove the following inequality

$$
\begin{aligned}
&\int_Q \theta^2|Lu|^2dxdt\\
\geq &\int_Q \Big\{\Big[-A_{1t}-(A_1B_1)_x+4\lambda(a_xA_1)_{xxx}+12\lambda^3(a_x^2a_{xx})_t+12\lambda^3(a_x^2a_{xx}B_1)_x\\
&-48\lambda^4(a_x^3a_{xx})_{xxx}-12\lambda^3(a_x^2a_{xx})_{xxxx}-12\lambda^3(a_x^2C_1a_{xx})_{xx}-24\lambda^3a_x^2a_{xx}A_1\\
&-6\lambda(A_1a_{xx})_{xx}-(A_1(C_1-6\lambda a_{xx})_x)_x+12\lambda^3(a_x^2a_{xx}(C_1-6\lambda a_{xx})_x)_x\Big]v^2\\
&+\Big[C_{1t}-B_{1xxx}-(B_1C_1)_x-12\lambda(a_xA_1)_x-6\lambda a_{xxt}+6\lambda(a_{xx}B_1)_x\\
&+144\lambda^4(a_x^3a_{xx})_x+48\lambda^3(a_x^2a_{xx})_{xx}+24\lambda^3a_x^2C_1a_{xx}+12\lambda A_1a_{xx}\\
&+2B_1(C_1-6\lambda a_{xx})_x-((C_1-6\lambda a_{xx})_xC_1)_x+6\lambda(a_{xx}(C_1-6\lambda a_{xx})_x)_x\\
&-4\lambda(a_x(C_1-6\lambda a_{xx})_x)_{xx}-(C_1-6\lambda a_{xx})_{xxxx}\Big]v_x^2\\
&-2(C_1-6\lambda a_{xx})_xv_x\theta Lu\\
&+\Big[3B_{1x}+4\lambda(a_xC_1)_x-24\lambda^2(a_xa_{xx})_x-24\lambda^3a_x^2a_{xx}-6\lambda a_{xxxx}-12\lambda a_{xx}C_1\\
&+8\lambda a_x(C_1-6\lambda a_{xx})_x+3(C_1-6\lambda a_{xx})_{xx}\Big]v_{xx}^2+16\lambda a_{xx}v_{xxx}^2\Big\}dxdt\\
&+\int_0^T\Big\{-4\lambda a_xv_{xxx}^2+\Big[-B_1-4\lambda a_xC_1+24\lambda^2a_xa_{xx}+6\lambda a_{xxx}\\
&-(C_1-6\lambda a_{xx})_x\Big]v_{xx}^2-12\lambda a_{xx}v_{xx}v_{xxx}\Big\}\Big|_0^1dt.
\end{aligned}
\tag{2.25}
$$

In order to prove (2.25), we shall first compute the term $\{\cdots\}v_tv_x$ in the righthand of (2.1).

Since $\theta Lu = I_1 + I_2 + I_3$, it holds that

$$
\begin{aligned}
-2(C_1-6\lambda a_{xx})_xv_tv_x &= 2(C_1-6\lambda a_{xx})_xv_x(A_1v-12\lambda^3a_x^2a_{xx}v+B_1v_x\\
&\quad +C_1v_{xx}-6\lambda a_{xx}v_{xx}-4\lambda a_xv_{xxx}+v_{xxxx}-\theta Lu).
\end{aligned}
\tag{2.26}
$$

Calculating each term in the righthand of (2.26), we obtain

$$
2(C_1-6\lambda a_{xx})_xv_xA_1v=\{(C_1-6\lambda a_{xx})_xA_1v^2\}_x-\big\{A_1(C_1-6\lambda a_{xx})_x\big\}_xv^2,
\tag{2.27}
$$

$$
\begin{aligned}
2(C_1-6\lambda a_{xx})_xv_x(-12\lambda^3a_x^2a_{xx}v) &= \{-12\lambda^3a_x^2a_{xx}(C_1-6\lambda a_{xx})_xv^2\}_x\\
&\quad +12\lambda^3\{a_x^2a_{xx}(C_1-6\lambda a_{xx})_x\}_xv^2,
\end{aligned}
\tag{2.28}
$$

$$
2(C_1-6\lambda a_{xx})_xv_xB_1v_x = 2B_1(C_1-6\lambda a_{xx})_xv_x^2,
\tag{2.29}
$$

$$
\begin{aligned}
2(C_1-6\lambda a_{xx})_xv_xC_1v_{xx} &= \big\{(C_1-6\lambda a_{xx})_xC_1v_x^2\}_x\\
&\quad -\{(C_1-6\lambda a_{xx})_xC_1\big\}_xv_x^2,
\end{aligned}
\tag{2.30}
$$

$$\begin{aligned}&2(C_1-6\lambda a_{xx})_x v_x(-6\lambda a_{xx}v_{xx})\\&=-6\lambda\big\{a_{xx}(C_1-6\lambda a_{xx})_x v_x^2\big\}_x+6\lambda\big\{a_{xx}(C_1-6\lambda a_{xx})_x\big\}_x v_x^2,\end{aligned}\tag{2.31}$$

$$\begin{aligned}&2(C_1-6\lambda a_{xx})_x v_x(-4\lambda a_x v_{xxx})\\&=8\lambda a_x(C_1-6\lambda a_{xx})_x v_{xx}^2+4\lambda\big\{(a_x(C_1-6\lambda a_{xx})_x)_x v_x^2\big\}_x\\&\quad-4\lambda\{a_x(C_1-6\lambda a_{xx})_x\}_{xx}v_x^2-8\lambda\big\{(C_1-6\lambda a_{xx})_x a_x v_x v_{xx}\big\}_x,\end{aligned}\tag{2.32}$$

$$\begin{aligned}&2(C_1-6\lambda a_{xx})_x v_x v_{xxxx}\\&=2\{(C_1-6\lambda a_{xx})_x v_x v_{xxx}\}_x+3(C_1-6\lambda a_{xx})_{xx}v_{xx}^2\\&\quad+\{(C_1-6\lambda a_{xx})_{xxx}v_x^2\}_x-(C_1-6\lambda a_{xx})_{xxxx}v_x^2\\&\quad-\{2(C_1-6\lambda a_{xx})_{xx}v_x v_{xx}+(C_1-6\lambda a_{xx})_x v_{xx}^2\}_x.\end{aligned}\tag{2.33}$$

On the other hand, by the definition of $\psi,\ \varphi,\ a,\ v$ and noticing that $v(0,t)=v(1,t)=0$, $v_t(0,t)=v_t(1,t)=0$, $v_x(0,t)=v_x(1,t)=0$ and $v_{xt}(0,t)=v_{xt}(1,t)=0$, it is easy to check that

$$\int_Q\big\{\ \cdot\ \big\}_t\,dxdt=0,\tag{2.34}$$

where $\big\{\ \cdot\ \big\}_t$ is defined in (2.1).

Then, integrating both sides of (2.1) over Q and combining (2.1), (2.26)–(2.34), we can obtain the estimate (2.25).

Step 3. We claim that there exist positive constants $\mu_0>1$, C_* and C^*, depending only on ω, such that for $\mu=\mu_0$ and for all numbers $\lambda\geq C_*(T+T^2)$,

$$\begin{aligned}&C^*\bigg(\int_Q\theta^2|Lu|^2dxdt-\int_Q V_x dxdt+\lambda^7\int_{Q^{\omega_0}}\theta^2\varphi^7u^2dxdt\\&\quad+\lambda^5\int_{Q^{\omega_0}}\theta^2\varphi^5u_x^2dxdt+\lambda^3\int_{Q^{\omega_0}}\theta^2\varphi^3u_{xx}^2dxdt+\lambda\int_{Q^{\omega_0}}\theta^2\varphi u_{xxx}^2dxdt\bigg)\\&\geq\lambda^7\int_Q\theta^2\varphi^7u^2dxdt+\lambda^5\int_Q\theta^2\varphi^5u_x^2dxdt\\&\quad+\lambda^3\int_Q\theta^2\varphi^3u_{xx}^2dxdt+\lambda\int_Q\theta^2\varphi u_{xxx}^2dxdt.\end{aligned}\tag{2.35}$$

Here $V_x(x,t)$ is the gradient term in (2.1), $\displaystyle\int_0^1 V_x(x,t)dx=V(1,t)-V(0,t)$,

$$V(1,t)\geq J_1\lambda^3\mu^3\varphi^3(1,t)v_{xx}^2(1,t)+K_1\lambda\mu\varphi(1,t)v_{xxx}^2(1,t),$$

$$V(0,t)\leq -J_2\lambda^3\mu^3\varphi^3(0,t)v_{xx}^2(0,t)-K_2\lambda\mu\varphi(0,t)v_{xxx}^2(0,t),$$

where J_1, J_2, K_1 and K_2 are positive numbers.

We shall prove (2.35) by further estimates for each term in (2.25).

For the term $\{\cdots\}v^2$ in (2.25), we take $\mu = 2(C_1(\psi)+1)$, where $C_1(\psi) > 0$ will be fixed later. By the definition of a, φ, ψ and μ, it is obvious that

$$
\begin{array}{lll}
|a_x| \le C(\psi)\mu\varphi, & |a_{xt}| \le C(\psi)\mu T\varphi^2, & |a_{xx}| \le C(\psi)\mu^2\varphi, \\
|a_{xxt}| \le C(\psi)T\mu^2\varphi^2, & |a_{xxx}| \le C(\psi)\mu^3\varphi, & |a_{xxxt}| \le C(\psi)\mu^3 T\varphi^2, \\
|a_{xxxx}| \le C(\psi)\mu^4\varphi, & |a_{xxxxt}| \le C(\psi)\mu^4 T\varphi^2, & |a_{xxxxx}| \le C(\psi)\mu^5\varphi, \\
|a_{xxxxxx}| \le C(\psi)\mu^6\varphi, & |a_{xxxxxxx}| \le C(\psi)\mu^7\varphi, & \\
|a_t| \le CT\varphi^2, & |a_{tt}| \le CT^4\varphi^4. &
\end{array}
$$

Observe that $\varphi \le \frac{T^2}{4}\varphi^2 \le \frac{T^4}{16}\varphi^3 \le \frac{T^6}{64}\varphi^4 \le \frac{T^8}{256}\varphi^5 \le \frac{T^{10}}{1024}\varphi^6$. If we choose $\lambda \ge \mu C(\psi)(T+T^2)$ with $C(\psi)$ is large enough, then it holds that

$$
\begin{aligned}
&\Big| -A_{1t} + 4\lambda(a_x A_1)_{xxx} + 12\lambda^3(a_x^2 a_{xx})_t + 12\lambda^3(a_x^2 a_{xx} B_1)_x - 48\lambda^4(a_x^3 a_{xx})_{xxx} \\
&- 12\lambda^3(a_x^2 a_{xx})_{xxxx} - 12\lambda^3(a_x^2 C_1 a_{xx})_{xx} - 6\lambda(A_1 a_{xx})_{xx} - (A_1(C_1 - 6\lambda a_{xx})_x)_x \\
&+ 12\lambda^3(a_x^2 a_{xx}(C_1 - 6\lambda a_{xx})_x)_x\Big| \le C(\psi)\lambda^7\mu^7\varphi^7, \\
&-(A_1 B_1)_x = 28\lambda^7 a_x^6 a_{xx} + D_1, \\
&-24\lambda^3 a_x^2 a_{xx} A_1 = -24\lambda^7 a_x^6 a_{xx} + D_2, \\
&|D_1| + |D_2| \le C(\psi)\lambda^7\mu^7\varphi^7.
\end{aligned}
$$

Hence,

$$\{\cdots\}v^2 = 4\lambda^7\mu^8|\psi_x|^8\varphi^7 v^2 + D_3 v^2, \tag{2.36}$$

where

$$|D_3| \le C(\psi)\lambda^7\mu^7\varphi^7. \tag{2.37}$$

Similarly, if we choose $\lambda \ge \mu C(\psi)(T+T^2)$, where $C(\psi)$ is large enough, we have

$$\{\cdots\}v_x^2 = 120\lambda^5\mu^6\varphi^5|\psi_x|^6 v_x^2 + E_2 v_x^2, \tag{2.38}$$

where

$$|E_2| \le C(\psi)\lambda^5\mu^5\varphi^5. \tag{2.39}$$

$$\{\cdots\}v_{xx}^2 = 36\lambda^3\mu^4|\psi_x|^4\varphi^3 v_{xx}^2 + F_1 v_{xx}^2, \tag{2.40}$$

where

$$|F_1| \le C(\psi)\lambda^3\mu^3\varphi^3. \tag{2.41}$$

$$\{\cdots\}v_{xxx}^2 = 16\lambda\mu^2|\psi_x|^2\varphi v_{xxx}^2 + 16\lambda\mu\psi_{xx}\varphi v_{xxx}^2, \tag{2.42}$$

where

$$|16\lambda\mu\psi_{xx}\varphi| \le C(\psi)\lambda\mu\varphi. \tag{2.43}$$

Moreover,

$$|-2(C_1 - 6\lambda a_{xx})_x v_x \theta Lu| \le C(C_1 - 6\lambda a_{xx})_x^2 v_x^2 + \frac{1}{2}|\theta Lu|^2.$$

Thus,

$$-2(C_1 - 6\lambda a_{xx})_x v_x \theta Lu \ge -C(C_1 - 6\lambda a_{xx})_x^2 v_x^2 - \frac{1}{2}|\theta Lu|^2, \tag{2.44}$$

where

$$|-C(C_1 - 6\lambda a_{xx})_x^2 v_x^2| \le C(\psi)\lambda^5\mu^5\varphi^5 v_x^2. \tag{2.45}$$

Now, we estimate the term $[\cdots]\big|_0^1 = [\cdots](1,t) - [\cdots](0,t)$ in (2.25).
By the definition of B_1 and C_1, it holds that

$$\begin{aligned} &-4\lambda a_x v_{xxx}^2(1,t) + [-B_1 - 4\lambda a_x C_1 + 24\lambda^2 a_x a_{xx} + 6\lambda a_{xxx} \\ &-(C_1 - 6\lambda a_{xx})_x] v_{xx}^2(1,t) - 12\lambda a_{xx} v_{xx} v_{xxx}(1,t) \\ =& -4\lambda a_x v_{xxx}^2(1,t) + [4\lambda^3 a_x^3 - 12\lambda^2 a_x a_{xx} \\ &+4\lambda a_{xxx} - 4\lambda a_x(6\lambda^2 a_x^2) + 24\lambda^2 a_x a_{xx} \\ &+6\lambda a_{xxx} - (6\lambda^2 a_x^2 - 6\lambda a_{xx})_x] v_{xx}^2(1,t) - 12\lambda a_{xx} v_{xx} v_{xxx}(1,t). \end{aligned} \tag{2.46}$$

Using the similar argument of proving (2.36), it holds that for any $\varepsilon_1 > 0$, if we choose $\lambda \ge \mu C(\varepsilon_1, \psi)(T + T^2)$, where $C(\varepsilon_1, \psi)$ is large enough, then

$$\begin{aligned} &\big|\{-12\lambda^2 a_x a_{xx} + 4\lambda a_{xxx} + 24\lambda^2 a_x a_{xx} \\ &\quad + 6\lambda a_{xxx} - (6\lambda^2 a_x^2 - 6\lambda a_{xx})_x\} v_{xx}^2(1,t)\big| \\ &\le \varepsilon_1 \lambda^3\mu^3\varphi^3 v_{xx}^2(1,t), \end{aligned} \tag{2.47}$$

$$4\lambda^3 a_x^3 v_{xx}^2(1,t) = 4\lambda^3\mu^3\psi_x^3\varphi^3 v_{xx}^2(1,t), \tag{2.48}$$

$$-4\lambda a_x(6\lambda^2 a_x^2)v_{xx}^2(1,t) = -24\lambda^3\mu^3\psi_x^3\varphi^3 v_{xx}^2(1,t), \tag{2.49}$$

$$-4\lambda a_x v_{xxx}^2(1,t) = -4\lambda\mu\psi_x\varphi v_{xxx}^2(1,t), \tag{2.50}$$

$$\begin{aligned}
&|-12\lambda a_{xx}v_{xx}v_{xxx}(1,t)| \\
&\le C\lambda\mu^2\varphi|v_{xx}(1,t)||v_{xxx}(1,t)| \\
&\le C\lambda\mu^2 T^2\varphi^2|v_{xx}(1,t)||v_{xxx}(1,t)| \\
&\le \varepsilon_1\lambda^3\mu^3\varphi^3 v_{xx}^2(1,t) + \varepsilon_1\lambda\mu\varphi v_{xxx}^2(1,t).
\end{aligned} \tag{2.51}$$

From (2.46)-(2.51), we get $[\cdots](1,t) = V(1,t) = V_1(1,t) + G_1(1,t)$, where

$$\begin{aligned}
V_1(1,t) &= -20\lambda^3\mu^3\psi_x^3(1)\varphi^3(1,t)v_{xx}^2(1,t) - 4\lambda\mu\psi_x(1)\varphi(1,t)v_{xxx}^2(1,t), \\
|G_1(1,t)| &\le \varepsilon_1\lambda^3\mu^3\varphi^3(1,t)v_{xx}^2(1,t) + \varepsilon_1\lambda\mu\varphi(1,t)v_{xxx}^2(1,t).
\end{aligned} \tag{2.52}$$

Note that $\psi_x(1) < 0$, if we choose ε_1 small sufficiently and $\lambda \ge \mu C(\varepsilon_1, \psi)(T + T^2)$, then there exist $J_1 > 0$ and $K_1 > 0$ such that

$$\begin{aligned}
&\int_0^T [\cdots](1,t)dt \\
&= \int_0^T V(1,t)dt = \int_0^T \Big(V_1(1,t) + G_1(1,t)\Big)dt \\
&\ge \int_0^T \Big(J_1\lambda^3\mu^3\varphi^3(1,t)v_{xx}^2(1,t) + K_1\lambda\mu\varphi(1,t)v_{xxx}^2(1,t)\Big)dt \ge 0.
\end{aligned} \tag{2.53}$$

In view of $\psi_x(0) > 0$, we have the similar estimate for $\int_0^T -[\cdots](0,t)dt$. Hence, there exist two positive constants J_2 and K_2 such that

$$\begin{aligned}
&\int_0^T [\cdots](0,t)dt \\
&= \int_0^T V(0,t)dt \\
&\le \int_0^T \Big(-J_2\lambda^3\mu^3\varphi^3(0,t)v_{xx}^2(0,t) - K_2\lambda\mu\varphi(0,t)v_{xxx}^2(0,t)\Big)dt \le 0.
\end{aligned} \tag{2.54}$$

From (2.36)–(2.45), (2.53) and (2.54), if we choose $\lambda \ge \mu C(\psi)(T + T^2)$ with $C(\psi)$ large sufficiently, it holds that

$$
\begin{aligned}
&C(\psi)\Big(\int_Q \theta^2|Lu|^2dxdt-\int_Q V_xdxdt+\lambda^7\mu^7\int_Q\varphi^7v^2dxdt\\
&\quad+\lambda^5\mu^5\int_Q\varphi^5v_x^2dxdt+\lambda^3\mu^3\int_Q\varphi^3v_{xx}^2dxdt+\lambda\mu\int_Q\varphi v_{xxx}^2dxdt\Big)\\
&\geq\lambda^7\mu^8\int_Q\varphi^7\psi_x^8v^2dxdt+\lambda^5\mu^6\int_Q\varphi^5\psi_x^6v_x^2dxdt\\
&\quad+\lambda^3\mu^4\int_Q\varphi^3\psi_x^4v_{xx}^2dxdt+\lambda\mu^2\int_Q\varphi\psi_x^2v_{xxx}^2dxdt.
\end{aligned}\tag{2.55}
$$

Recall that $|\psi_x|>0$ in $\bar{\Omega}\backslash\omega_0$. Then, from (2.55), if we choose $\lambda\geq\mu C(\psi)(T+T^2)$ with $C(\psi)$ large sufficiently, it follows that

$$
\begin{aligned}
&C_1(\psi)\Big(\int_Q \theta^2|Lu|^2dxdt-\int_Q V_xdxdt+\lambda^7\mu^7\int_Q\varphi^7v^2dxdt\\
&+\lambda^5\mu^5\int_Q\varphi^5v_x^2dxdt+\lambda^3\mu^3\int_Q\varphi^3v_{xx}^2dxdt+\lambda\mu\int_Q\varphi v_{xxx}^2dxdt\Big)\\
&\geq\lambda^7\mu^8\int_Q\varphi^7v^2dxdt+\lambda^5\mu^6\int_Q\varphi^5v_x^2dxdt+\lambda^3\mu^4\int_Q\varphi^3v_{xx}^2dxdt+\lambda\mu^2\int_Q\varphi v_{xxx}^2dxdt,
\end{aligned}
$$

from which if we choose $\mu=\mu_0=C_1(\psi)+1$ and $\lambda\geq C(\psi)(T+T^2)$ with $C(\psi)$ large sufficiently, it holds that

$$
\begin{aligned}
&C(\psi)\Big(\int_Q \theta^2|Lu|^2dxdt-\int_Q V_xdxdt+\lambda^7\mu^7\int_{Q^{\omega_0}}\varphi^7v^2dxdt\\
&\quad+\lambda^5\mu^5\int_{Q^{\omega_0}}\varphi^5v_x^2dxdt\\
&\quad+\lambda^3\mu^3\int_{Q^{\omega_0}}\varphi^3v_{xx}^2dxdt+\lambda\mu\int_{Q^{\omega_0}}\varphi v_{xxx}^2dxdt\Big)\\
&\geq\lambda^7\mu^8\int_{Q\backslash Q^{\omega_0}}\varphi^7v^2dxdt+\lambda^7\mu^7\int_{Q^{\omega_0}}\varphi^7v^2dxdt\\
&\quad+\lambda^5\mu^6\int_{Q\backslash Q^{\omega_0}}\varphi^5v_x^2dxdt+\lambda^5\mu^5\int_{Q^{\omega_0}}\varphi^5v_x^2dxdt\\
&\quad+\lambda^3\mu^4\int_{Q\backslash Q^{\omega_0}}\varphi^3v_{xx}^2dxdt+\lambda^3\mu^3\int_{Q^{\omega_0}}\varphi^3v_{xx}^2dxdt\\
&\quad+\lambda\mu^2\int_{Q\backslash Q^{\omega_0}}\varphi v_{xxx}^2dxdt+\lambda\mu\int_{Q^{\omega_0}}\varphi v_{xxx}^2dxdt.
\end{aligned}
$$

This implies

$$C\left(\int_Q \theta^2|Lu|^2dxdt - \int_Q V_xdxdt + \lambda^7\int_{Q^{\omega_0}}\varphi^7v^2dxdt\right.$$
$$\left. + \lambda^5\int_{Q^{\omega_0}}\varphi^5v_x^2dxdt + \lambda^3\int_{Q^{\omega_0}}\varphi^3v_{xx}^2dxdt + \lambda\int_{Q^{\omega_0}}\varphi v_{xxx}^2dxdt\right)$$
$$\geq \lambda^7\int_Q\varphi^7v^2dxdt + \lambda^5\int_Q\varphi^5v_x^2dxdt + \lambda^3\int_Q\varphi^3v_{xx}^2dxdt + \lambda\int_Q\varphi v_{xxx}^2dxdt.$$

Returning v to $e^{\lambda a}u$, we can obtain (2.35).

Step 4. We shall eliminate the terms $\lambda^3\int_{Q^{\omega_0}}\theta^2\varphi^3u_{xx}^2dxdt$ and $\lambda\int_{Q^{\omega_0}}\theta^2\varphi u_{xxx}^2dxdt$ in the left side of (2.35).

Let ω_1 be a nonempty open subset of Ω such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega$. Take $\chi \in C_0^\infty(\omega_1)$, $\chi = 1$ in ω_0. Multiplying system (1.1) by $\chi\theta^2\varphi u_{xx}$ and integrating it over Q, we get

$$\int_Q u_t(\chi\theta^2\varphi u_{xx})dxdt = \int_Q -(\chi\theta^2\varphi)_xu_xu_tdxdt + \int_Q\frac{1}{2}(\chi\theta^2\varphi)_tu_x^2dxdt,$$
$$\int_Q u_{xxxx}(\chi\theta^2\varphi u_{xx})dxdt = \int_Q\frac{1}{2}(\chi\theta^2\varphi)_{xx}u_{xx}^2dxdt - \int_Q(\chi\theta^2\varphi)u_{xxx}^2dxdt.$$

Since

$$\begin{aligned}&\int_Q -(\chi\theta^2\varphi)_xu_xu_tdxdt\\ =&\int_Q -(\chi\theta^2\varphi)_xu_x(\xi - u_{xxxx})dxdt\\ =&\int_Q\{-(\chi\theta^2\varphi)_xu_x\xi + (\chi\theta^2\varphi)_xu_xu_{xxxx}\}dxdt\\ =&\int_Q\Big\{-(\chi\theta^2\varphi)_xu_x\xi + \frac{3}{2}(\chi\theta^2\varphi)_{xx}u_{xx}^2 - \frac{1}{2}(\chi\theta^2\varphi)_{xxxx}u_x^2\Big\}dxdt,\end{aligned}$$

we get

$$\begin{aligned}&\int_Q\Big\{-(\chi\theta^2\varphi)_xu_x\xi + \frac{3}{2}(\chi\theta^2\varphi)_{xx}u_{xx}^2 - \frac{1}{2}(\chi\theta^2\varphi)_{xxxx}u_x^2 + \frac{1}{2}(\chi\theta^2\varphi)_tu_x^2\\ &+\frac{1}{2}(\chi\theta^2\varphi)_{xx}u_{xx}^2 - (\chi\theta^2\varphi)u_{xxx}^2\Big\}dxdt\\ =&\int_Q\xi(\chi\theta^2\varphi)u_{xx}dxdt\\ =&\int_Q Lu(\chi\theta^2\varphi)u_{xx}dxdt.\end{aligned}$$

Then using similar arguments in Step 1, if we choose $\lambda \geq C(\psi)(T+T^2)$ with $C(\psi)$ sufficiently large, it holds that

$$\begin{aligned}&\int_Q \lambda\chi\theta^2\varphi u_{xxx}^2 dxdt\\ &\leq C\left(\int_{Q^{\omega_1}} \lambda^3\theta^2\varphi^3 u_{xx}^2 dxdt + \int_{Q^{\omega_1}} \lambda^5\theta^2\varphi^5 u_x^2 dxdt + \int_Q \theta^2|Lu|^2 dxdt\right),\end{aligned}$$

where $Q^{\omega_1} = \omega_1 \times (0,T)$.
Further, we have

$$\begin{aligned}\int_{Q^{\omega_0}} \lambda\theta^2\varphi u_{xxx}^2 dxdt \leq C\Big(\int_{Q^{\omega_1}} \lambda^3\theta^2\varphi^3 u_{xx}^2 dxdt + \int_{Q^{\omega_1}} \lambda^5\theta^2\varphi^5 u_x^2 dxdt\\ + \int_Q \theta^2|Lu|^2 dxdt\Big).\end{aligned} \tag{2.56}$$

On the other hand, multiplying system (1.1) by $\chi_1\theta^2\varphi^3 u$, where $\chi_1 \in C_0^\infty(\omega),\ \chi_1 = 1$ in ω_0 and integrating it over Q, we get

$$\begin{aligned}&\int_Q (u_t + u_{xxxx})(\chi_1\theta^2\varphi^3 u)dxdt\\ &= \int_Q \xi(\chi_1\theta^2\varphi^3 u)dxdt\\ &= \int_Q\Big\{-\frac{1}{2}(\chi_1\theta^2\varphi^3)_t u^2 - 2(\chi_1\theta^2\varphi^3)_{xx}u_x^2 + \frac{1}{2}(\chi_1\theta^2\varphi^3)_{xxxx}u^2 + (\chi_1\theta^2\varphi^3)u_{xx}^2\Big\}dxdt.\end{aligned}$$

Hence, we conclude

$$\begin{aligned}&\int_Q (\chi_1\theta^2\varphi^3)u_{xx}^2 dxdt\\ &= \int_Q \Big\{\xi(\chi_1\theta^2\varphi^3 u) + \frac{1}{2}\Big(\chi_1\theta^2\varphi^3\Big)_t u^2 + 2\Big(\chi_1\theta^2\varphi^3\Big)_{xx} u_x^2 - \frac{1}{2}\Big(\chi_1\theta^2\varphi^3\Big)_{xxxx} u^2\Big\}dxdt,\end{aligned}$$

from which, if we choose $\lambda \geq C(\psi)(T+T^2)$ with $C(\psi)$ sufficiently large, it follows that

$$\begin{aligned}\int_Q \chi_1\lambda^3\theta^2\varphi^3 u_{xx}^2 dxdt \leq C\Big(\int_{Q^\omega} \lambda^7\theta^2\varphi^7 u^2 dxdt\\ + \int_{Q^\omega} \lambda^5\theta^2\varphi^5 u_x^2 dxdt + \int_Q \theta^2|Lu|^2 dxdt\Big).\end{aligned}$$

Further, it holds that

$$\begin{aligned}\int_{Q^{\omega_0}} \lambda^3\theta^2\varphi^3 u_{xx}^2 dxdt \leq C\Big(\int_{Q^\omega} \lambda^7\theta^2\varphi^7 u^2 dxdt + \int_{Q^\omega} \lambda^5\theta^2\varphi^5 u_x^2 dxdt\\ + \int_Q \theta^2|Lu|^2 dxdt\Big).\end{aligned} \tag{2.57}$$

Similarly, we have

$$\int_{Q^{\omega_1}} \lambda^3\theta^2\varphi^3 u_{xx}^2 dxdt \le C\Big(\int_{Q^\omega} \lambda^7\theta^2\varphi^7 u^2 dxdt + \int_{Q^\omega} \lambda^5\theta^2\varphi^5 u_x^2 dxdt + \int_Q \theta^2|Lu|^2 dxdt\Big). \tag{2.58}$$

From (2.56), (2.57) and (2.58) and in view of the definition of V, if we choose $\lambda \ge C(\psi)(T+T^2)$ with $C(\psi)$ sufficiently large, then

$$C\left(\int_Q \theta^2|Lu|^2 dxdt - \int_Q V_x dxdt + \lambda^7\int_{Q^\omega}\theta^2\varphi^7u^2dxdt + \lambda^5\int_{Q^\omega}\theta^2\varphi^5u_x^2dxdt\right) \ge \lambda^7\int_Q\theta^2\varphi^7u^2dxdt + \lambda^5\int_Q\theta^2\varphi^5u_x^2dxdt + \lambda^3\int_Q\theta^2\varphi^3u_{xx}^2dxdt,$$

from which and in view of (2.53) and (2.54), namely, $-\displaystyle\int_Q V_x dxdt \le 0$, we have

$$\begin{aligned}&\lambda^7\int_Q\theta^2\varphi^7u^2dxdt + \lambda^5\int_Q\theta^2\varphi^5u_x^2dxdt + \lambda^3\int_Q\theta^2\varphi^3u_{xx}^2dxdt\\ &\le C\left(\lambda^7\int_{Q^\omega}\theta^2\varphi^7u^2dxdt + \lambda^5\int_{Q^\omega}\theta^2\varphi^5u_x^2dxdt + \int_Q\theta^2\xi^2dxdt\right).\end{aligned} \tag{2.59}$$

Step 5. We shall eliminate the terms $\lambda^5\displaystyle\int_{Q^\omega}\theta^2\varphi^5u_x^2dxdt$ in (2.59).

By interpolation inequality, we obtain that for any $\varepsilon > 0$,

$$\int_\omega (\theta u)_x^2 dx \le \varepsilon\int_\omega(\theta u)_{xx}^2dx + \frac{C}{\varepsilon}\int_\omega(\theta u)^2dx, \tag{2.60}$$

where C is depending only on ω.

Take ε as $\varepsilon_2\left(\dfrac{\lambda}{t(T-t)}\right)^{-2}$ in (2.60), where $\varepsilon_2 > 0$ will be fixed later. It holds that

$$\int_\omega\theta^2u_x^2dx \le \varepsilon_2\left(\frac{\lambda}{t(T-t)}\right)^{-2}\int_\omega(\theta u)_{xx}^2dx + \frac{C}{\varepsilon_2\left(\frac{\lambda}{t(T-t)}\right)^{-2}}\int_\omega(\theta u)^2dx - \int_\omega\theta_x^2u^2dx - 2\int_\omega\theta\theta_xuu_xdx,$$

from which it follows that

$$\frac12\int_\omega\theta^2u_x^2dx \le \varepsilon_2\left(\frac{\lambda}{t(T-t)}\right)^{-2}\int_\omega(\theta u)_{xx}^2dx + \frac{C}{\varepsilon_2\left(\frac{\lambda}{t(T-t)}\right)^{-2}}\int_\omega(\theta u)^2dx + C\int_\omega\theta_x^2u^2dx.$$

In view of the definition of ψ, θ and φ and using Hölder inequality and Young inequality, we get

$$
\begin{aligned}
\int_\omega \theta^2 u_x^2 dx \le\ & \varepsilon_2 C(\psi)\left(\frac{\lambda}{t(T-t)}\right)^{-2}\left(\int_\omega \lambda^2\theta^2\varphi^2u^2dx+\int_\omega \lambda^4\theta^2\varphi^4u^2dx\right.\\
&\left.+\int_\omega \lambda^3\theta^2\varphi^3u^2dx+\int_\omega \lambda^2\theta^2\varphi^2u_x^2dx+\int_\omega \theta^2u_{xx}^2dx\right)\\
&+\frac{C}{\varepsilon_2\left(\frac{\lambda}{t(T-t)}\right)^{-2}}\int_\omega \theta^2u^2dx+C(\psi)\int_\omega \lambda^2\theta^2\varphi^2u^2dx.
\end{aligned}\tag{2.61}
$$

Multiplying the both sides of (2.61) by $\dfrac{\lambda^5}{(t(T-t))^5}$ and integrating it from 0 to T, we can deduce that if we choose ε_2 sufficiently small and $\lambda \ge C(\psi)(T+T^2)$ with $C(\psi)$ sufficiently large, then (2.59) and (2.61) implies (1.2).

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Set $m(x) = e^{\mu(\psi(x)+3)}$, $\rho(x) = e^{\mu(\psi(x)+3)} - e^{5\mu}$, we have the following two lemmas.

Lemma 3.1. *If we choose $\lambda \ge T^2$, then*

$$
\|\theta^2\varphi^7\|_{L^\infty(Q)} \le 2^{14}T^{-14}\exp\{-\widetilde{C}_1\lambda T^{-2}\}e^{28\mu}, \tag{3.1}
$$

where $\widetilde{C}_1 = 8\min\limits_{\overline{\Omega}}\{-\rho(x)\}$.

Proof. Observe that

$$
\theta^2\varphi^7 = e^{2\lambda a}t^{-7}(T-t)^{-7}m^7(x) = 1/(k_1(x,t)),\ (x,t)\in Q,
$$

where

$$
\begin{aligned}
k_1(x,t) &= t^7(T-t)^7m^{-7}(x)\exp\Big\{\frac{-2\lambda\rho(x)}{t(T-t)}\Big\}\\
&= \tau^7m^{-7}(x)\exp\Big\{\frac{-2\lambda\rho(x)}{\tau}\Big\} = k_2(x,\tau)
\end{aligned}
$$

and $\tau = t(T-t)\in(0,T^2/4]$.

Let x be fixed, it is obvious that $\widehat{\tau} = -\dfrac{2}{7}\lambda\rho(x)$ is the minimum point of $k_2(x,\tau)$ and $k_2(x,\widehat{\tau}) = -\left(\dfrac{2}{7}\lambda\rho(x)\right)^7 m^{-7}(x)e^7$. On the other hand, $k_2(x,0)=\infty$, $k_2(x,\cdot)$ is decreasing for $\tau\in(0,\widehat{\tau})$ and increasing for $\tau>\widehat{\tau}$. Hence,

$$\min_{0\le t\le T} k_1(x,t) = \min_{0\le\tau\le T^2/4} k_2(x,\tau) = \begin{cases} k_2(x,\widehat{\tau}), & \text{if } T^2/4 \ge -\frac{2}{7}\lambda\rho(x), \\ k_2(x,T^2/4), & \text{if } T^2/4 < -\frac{2}{7}\lambda\rho(x). \end{cases}$$

Notice that $\|\psi\|_{C(\overline{\Omega})} = 1$ and $\mu > 1$, it holds that $-\rho(x) = e^{5\mu} - e^{\mu(\psi(x)+3)} = e^{3\mu}(e^{2\mu} - e^{\mu\psi(x)}) \ge e^{4\mu}(e^{\mu}-1) > 7/8$. Thus, if we choose $\lambda \ge T^2$, then

$$\min_{0\le t\le T} k_1(x,t) \ge k_2(x,T^2/4) = 2^{-14}T^{14}\exp\Big\{-8\lambda\rho(x)T^{-2}\Big\}m^{-7}(x).$$

Hence, (3.1) follows.

Lemma 3.2. *If we set $\lambda \ge T^2$, then*

$$\theta^2\varphi^7 \ge 2^{28}T^{-14}\exp\{-\widetilde{C}_2\lambda T^{-2}\}e^{21\mu},\ x\in(0,1),\ t\in[T/4,3T/4]. \tag{3.2}$$

Here $\widetilde{C}_2 = 32\max_{\overline{\Omega}}\{-\rho(x)\}$.

Proof. Notice that $\tau = t(T-t) \in (0,T^2/4]$. Then $t\in[T/4,3T/4]$ implies $\tau \in [T^2/16, T^2/4]$. Using the similar argument of proving Lemma 3.1, if we choose $\lambda \ge T^2$, then it holds that

$$\max_{T/4\le t\le 3T/4} k_1(x,t) \le 2^{-28}T^{14}\exp\big\{\widetilde{C}_2\lambda T^{-2}\big\}m^{-7}(x),\quad x\in(0,1),\ t\in[T/4,3T/4].$$

Therefore, (3.2) follows.

Proof of Theorem 1.2 It is known that there exists a unique solution $p \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^2(\Omega))$ to equation (1.3) (see [11]). A simple change of variable in time shows that the Carleman inequality (1.2) is also valid for (1.3). By estimate (1.2), if we choose $\lambda \ge C_0(T+T^2)$, it holds that

$$\begin{aligned} &\lambda^7\int_Q \theta^2\varphi^7p^2dxdt + \lambda^5\int_Q\theta^2\varphi^5p_x^2dxdt + \lambda^3\int_Q\theta^2\varphi^3p_{xx}^2dxdt \\ &\le C\left(\lambda^7\int_{Q^\omega}\theta^2\varphi^7p^2dxdt + \int_Q\theta^2|gp|^2dxdt\right), \end{aligned} \tag{3.3}$$

where C_0 and C depend only on ω.

Since

$$\begin{aligned} C\int_Q\theta^2|gp|^2dxdt &\le C2^{-14}T^{14}\|g\|^2_{L^\infty(Q)}\int_Q\theta^2\frac{e^{7\mu(\psi+3)}}{t^7(T-t)^7}p^2dxdt \\ &= C2^{-14}T^{14}\|g\|^2_{L^\infty(Q)}\int_Q\theta^2\varphi^7p^2dxdt, \end{aligned}$$

if we choose $\lambda \geq C(\psi)\|g\|^{\frac{2}{7}}_{L^\infty(Q)}T^2$ with $C(\psi)$ sufficiently large, it holds that

$$C\int_Q \theta^2|gp|^2dxdt \leq \frac{1}{2}\lambda^7\int_Q \theta^2\varphi^7p^2dxdt. \tag{3.4}$$

By (3.3) and (3.4), if $\lambda \geq C(\psi)\Big(T+T^2+\|g\|^{\frac{2}{7}}_{L^\infty(Q)}T^2\Big)$, where $C(\psi)$ sufficiently large, we have

$$\int_Q \theta^2\varphi^7p^2dxdt \leq \widetilde{C}\int_{Q^\omega}\theta^2\varphi^7p^2dxdt,$$

from which and by Lemma 3.1 and Lemma 3.2, if $\lambda=\lambda_1=\widetilde{C}(\psi)\Big(T+T^2+\|g\|^{\frac{2}{7}}_{L^\infty(Q)}T^2\Big)$, then

$$\begin{aligned}&2^{28}T^{-14}\text{exp}\Big\{-\widetilde{C}_2\widetilde{C}(\psi)\Big(1+\frac{1}{T}+\|g\|^{\frac{2}{7}}_{L^\infty(Q)}\Big)\Big\}e^{21\mu}\int_{\Omega\times(T/4,3T/4)}p^2dxdt\\ &\leq C2^{14}T^{-14}\text{exp}\Big\{-\widetilde{C}_1\widetilde{C}(\psi)\Big(1+\frac{1}{T}+\|g\|^{\frac{2}{7}}_{L^\infty(Q)}\Big)\Big\}e^{28\mu}\int_{Q^\omega}p^2dxdt.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\Omega\times(T/4,3T/4)}p^2dxdt\\ &\leq C\text{exp}\Big\{(\widetilde{C}_2-\widetilde{C}_1)\widetilde{C}(\psi)\Big(1+\frac{1}{T}+\|g\|^{\frac{2}{7}}_{L^\infty(Q)}\Big)\Big\}\int_{Q^\omega}p^2dxdt,\end{aligned}$$

from which it follows that

$$\begin{aligned}&\int_{\Omega\times(T/4,3T/4)}p^2dxdt\\ &\leq \text{exp}\Big\{C\Big(1+\frac{1}{T}+\|g\|^{\frac{2}{7}}_{L^\infty(Q)}\Big)\Big\}\int_{Q^\omega}p^2dxdt.\end{aligned} \tag{3.5}$$

Multiplying (1.3) by p and integrating over Ω, we have

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\int_\Omega p^2dx+\int_\Omega p_{xx}^2dx &= -\int_\Omega gp^2dx\\ &\leq \|g\|_{L^\infty(Q)}\int_\Omega p^2dx,\end{aligned}$$

this implies

$$\frac{d}{dt}\int_\Omega p^2dx+2\|g\|_{L^\infty(Q)}\int_\Omega p^2dx \geq 0,$$

that is,

$$\frac{d}{dt}\Big(\exp\{2\|g\|_{L^\infty(Q)}t\}\int_\Omega p^2dx\Big)\geq 0. \tag{3.6}$$

For each $t\in[T/4,3T/4]$, integrating (3.6) over $[T/4,\ t]$, we obtain

$$\begin{aligned}\int_\Omega |p(x,t)|^2dx &\geq \exp\Big\{2\|g\|_{L^\infty(Q)}\big(\frac{T}{4}-t\big)\Big\}\int_\Omega \big|p(x,\frac{T}{4})\big|^2dx\\ &\geq \exp\Big\{-\|g\|_{L^\infty(Q)}T\Big\}\int_\Omega \big|p(x,\frac{T}{4})\big|^2dx.\end{aligned} \tag{3.7}$$

Integrating (3.7) over $[T/4,3T/4]$ and (3.6) over $[0,T/4]$, respectively, it holds that

$$\frac{T}{2}\int_\Omega \big|p(x,\frac{T}{4})\big|^2dx\leq \exp\big\{\|g\|_{L^\infty(Q)}T\big\}\int_{\Omega\times(T/4,3T/4)}|p(x,t)|^2dxdt, \tag{3.8}$$

and

$$\int_\Omega |p(x,0)|^2dx\leq \exp\big\{C\|g\|_{L^\infty(Q)}T\big\}\int_\Omega \big|p(x,\frac{T}{4})\big|^2dx. \tag{3.9}$$

Combining (3.8), (3.9) and (3.5), we have

$$\begin{aligned}&\int_\Omega |p(x,0)|^2dx\\ &\leq \exp\big\{C\|g\|_{L^\infty(Q)}T\big\}\frac{2}{T}\int_{\Omega\times(T/4,3T/4)}|p(x,t)|^2dxdt\\ &\leq \exp\Big\{C\Big(\frac{1}{T}+\|g\|_{L^\infty(Q)}T\Big)\Big\}\int_{\Omega\times(T/4,3T/4)}|p(x,t)|^2dxdt\\ &\leq \exp\Big\{C\Big(1+\frac{1}{T}+\|g\|^{\frac{2}{7}}_{L^\infty(Q)}+\|g\|_{L^\infty(Q)}T\Big)\Big\}\int_{Q^\omega}p^2dxdt.\end{aligned}$$

This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

In this section, we shall first study the null controllability of linear system, then a fixed point argument applying Kakutani's Theorem ([see [1], pp. 126]) will be used to prove Theorem 1.3.

Consider the following linear system:

$$\begin{cases} u_t+u_{xxxx}+gu=\chi_\omega h, & (x,t)\in Q,\\ u(0,t)=u(1,t)=0, & t\in(0,T),\\ u_x(0,t)=u_x(1,t)=0, & t\in(0,T),\\ u(x,0)=u_0(x), & x\in\Omega.\end{cases} \tag{4.1}$$

We have the following proposition.

Proposition 4.1. *Suppose that* $u_0 \in L^2(\Omega)$ *and* $g \in L^\infty(Q)$. *Then there exists a control* $h \in L^2(\omega \times (0,T))$ *such that the corresponding solution* u *of (4.1) satisfies*

$$u(\cdot, T) = 0 \quad \textit{in } \Omega.$$

Moreover,

$$\|h\|_{L^2(\omega\times(0,T))} \le \exp\Big\{C(1 + \frac{1}{T} + \|g\|^{\frac{2}{7}}_{L^\infty(Q)} + \|g\|_{L^\infty(Q)} T)\Big\}\|u_0\|_{L^2(\Omega)}. \tag{4.2}$$

Proof. It is well known (see [11]) that under the assumption of Proposition 4.1, system (4.1) allows a unique solution $u \in L^2(0,T; H_0^2(\Omega)) \cap C([0,T]; L^2(\Omega))$ with the property that $u_t \in L^2(0,T; H^{-2}(\Omega))$ and

$$\|u\|_{L^2(0,T;H_0^2(\Omega))} + \|u_t\|_{L^2(0,T;H^{-2}(\Omega))} \le C(\|h\|_{L^2(\omega\times(0,T))} + \|u_0\|_{L^2(\Omega)}),$$

where C depends on $\|g\|_{L^\infty(Q)}$.

For each $\varepsilon > 0$, we introduce the functional $J_\varepsilon : L^2(\Omega) \to \mathbb{R}$,

$$J_\varepsilon(p_0) = \frac{1}{2}\int_0^T \int_\omega p^2 dxdt + \varepsilon\|p_0\|_{L^2(\Omega)} + \int_\Omega u_0 p(x,0)dx,$$

where p is the solution to (1.3) with data p_0.

Since $p \in L^2(0,T; H_0^2(\Omega)) \cap C([0,T]; L^2(\Omega))$ (see [11]), it is easy to check that $J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ is continuous and strictly convex. Moreover, we claim that it is also coercive. More precisely,

$$\liminf_{\|p_0\|_{L^2(\Omega)}\to\infty} \frac{J_\varepsilon(p_0)}{\|p_0\|_{L^2(\Omega)}} \ge \varepsilon. \tag{4.3}$$

Indeed, given a sequence $\{p_{0j}\} \in L^2(\Omega)$ with $\|p_{0j}\|_{L^2(\Omega)} \to \infty$, we normalize it

$$\widetilde{p}_{0j} = \frac{p_{0j}}{\|p_{0j}\|_{L^2(\Omega)}}.$$

Thus, we have

$$\frac{J_\varepsilon(p_{0j})}{\|p_{0j}\|_{L^2(\Omega)}} = \frac{\|p_{0j}\|_{L^2(\Omega)}}{2}\int_0^T \int_\omega |\widetilde{p}_j|^2 dxdt + \varepsilon + \int_\Omega u_0 \widetilde{p}_j(x,0)dx,$$

where $\widetilde{p}_j$ is the solution to (1.3) with data $\widetilde{p}_{0j}$.

We distinguish the following two cases.

Case 1.

$$\liminf_{j\to\infty} \int_0^T \int_\omega |\widetilde{p}_j|^2 dxdt > 0,$$

when this holds it is obvious that

$$\liminf_{j\to\infty} \frac{J_\varepsilon(p_{0j})}{\|p_{0j}\|_{L^2(\Omega)}} = \infty.$$

Case 2.

$$\liminf_{j\to\infty} \int_0^T \int_\omega |\widetilde{p}_j|^2 dxdt = 0.$$

In this case, by extracting subsequences, still denoted in the same way, it holds that as $j \to \infty$,

$$\int_0^T \int_\omega |\widetilde{p}_j|^2 dxdt \to 0, \tag{4.4}$$

and

$$\widetilde{p}_{0j} \rightharpoonup p_0 \text{ weakly in } L^2(\Omega). \tag{4.5}$$

In view of (4.4) and (4.5), the solution to (1.3) with data p_0 satisfies

$$p = 0 \quad \text{in } \omega \times (0, T).$$

By the unique continuation property for the solution of (1.3) (see [6]), we have $p \equiv 0$ in Q and

$$\widetilde{p}_j(0) \rightharpoonup 0 \text{ weakly in } L^2(\Omega).$$

Hence,

$$\liminf_{j\to\infty} \frac{J_\varepsilon(p_{0j})}{\|p_{0j}\|_{L^2(\Omega)}} \geq \varepsilon.$$

This proves the claim (4.3).

Then J_ε has a unique critical point which is its minimizer:

$$\widetilde{p}_{0\varepsilon} \in L^2(\Omega): \quad J_\varepsilon(\widetilde{p}_{0\varepsilon}) = \min_{p_0 \in L^2(\Omega)} J_\varepsilon(p_0).$$

Given $\psi_0 \in L^2(\Omega)$ and $\rho \in \mathbb{R}$, we have

$$J_\varepsilon(\widetilde{p}_{0\varepsilon}) \leq J_\varepsilon(\widetilde{p}_{0\varepsilon} + \rho\psi_0),$$

that is,

$$\begin{aligned} \varepsilon\|\widetilde{p}_{0\varepsilon}\|_{L^2(\Omega)} \leq\ & \frac{\rho^2}{2} \int_0^T \int_\omega |\psi|^2 dxdt + \int_0^T \int_\omega \rho \widetilde{p}_\varepsilon \psi dxdt \\ & + \varepsilon\|\widetilde{p}_{0\varepsilon} + \rho\psi_0\|_{L^2(\Omega)} + \rho \int_\Omega u_0 \psi(x, 0) dx, \end{aligned}$$

where $\widetilde{p}_\varepsilon$ and ψ are solutions to (1.3) with data $\widetilde{p}_{0\varepsilon}$ and ψ_0, respectively.

Dividing this inequality by $\rho > 0$ and letting $\rho \to 0^+$, we obtain that

$$\begin{aligned}
&-\int_\Omega u_0\psi(x,0)dx \\
&\le \int_0^T\int_\omega \widetilde{p}_\varepsilon\psi dxdt + \varepsilon\liminf_{\rho\to 0^+}\frac{\|\widetilde{p}_{0\varepsilon}+\rho\psi_0\|_{L^2(\Omega)}-\|\widetilde{p}_{0\varepsilon}\|_{L^2(\Omega)}}{\rho} \\
&\le \int_0^T\int_\omega \widetilde{p}_\varepsilon\psi dxdt + \varepsilon\|\psi_0\|_{L^2(\Omega)}.
\end{aligned}$$

Reproducing this argument with $\rho < 0$, we obtain finally that

$$\Big|\int_0^T\int_\omega \widetilde{p}_\varepsilon\psi dxdt + \int_\Omega u_0\psi(x,0)dx\Big| \le \varepsilon\|\psi_0\|_{L^2(\Omega)}. \tag{4.6}$$

On the other hand, multiplying (4.1) (with right hand side $h = \chi_\omega\widetilde{p}_\varepsilon$) by ψ and integrating by parts, we deduce that

$$\int_\Omega u_\varepsilon(x,T)\psi(x,T)dx = \int_0^T\int_\omega \widetilde{p}_\varepsilon\psi dxdt + \int_\Omega u_0\psi(x,0)dx. \tag{4.7}$$

Combining (4.6) and (4.7), we get

$$\Big|\int_\Omega u_\varepsilon(x,T)\psi(x,T)dx\Big| \le \varepsilon\|\psi_0\|_{L^2(\Omega)}.$$

Hence, the solution u_ε to (4.1) with $h = \chi_\omega\widetilde{p}_\varepsilon$ satisfies

$$\|u_\varepsilon(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon. \tag{4.8}$$

Since $J_\varepsilon(\widetilde{p}_{0\varepsilon}) \le J_\varepsilon(0) = 0$, by the definition of J_ε, we have

$$\begin{aligned}
\frac{1}{2}\int_0^T\int_\omega |\widetilde{p}_\varepsilon|^2 dxdt &\le -\int_\Omega u_0\widetilde{p}_\varepsilon(x,0)dx \\
&\le \|\widetilde{p}_\varepsilon(x,0)\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)},
\end{aligned}$$

from which and (1.4), it holds that

$$\begin{aligned}
&\frac{1}{2}\int_0^T\int_\omega |\widetilde{p}_\varepsilon|^2 dxdt \\
&\le \exp\Big\{C\Big(1+\frac{1}{T}+\|g\|_{L^\infty(Q)}^{\frac{2}{7}}+\|g\|_{L^\infty(Q)}T\Big)\Big\}\Big(\int_{Q^\omega}\widetilde{p}_\varepsilon^2 dxdt\Big)^{\frac{1}{2}}\|u_0\|_{L^2(\Omega)}.
\end{aligned}$$

Hence,

$$\int_0^T\int_\omega |\widetilde{p}_\varepsilon|^2 dxdt \le \exp\Big\{C\Big(1+\frac{1}{T}+\|g\|_{L^\infty(Q)}^{\frac{2}{7}}+\|g\|_{L^\infty(Q)}T\Big)\Big\}\|u_0\|_{L^2(\Omega)}^2. \tag{4.9}$$

From (4.9), by extracting subsequences, still denoted in the same way, we have that there exists a function $\widetilde{p} \in L^2(\omega \times (0,T))$ such that

$$\widetilde{p}_\varepsilon \to \widetilde{p} \quad \text{weakly in} \quad L^2(\omega \times (0,T)) \text{ as } \varepsilon \to 0. \tag{4.10}$$

Hence, let $h = \widetilde{p}$. Combining (4.8), (4.9) and (4.10), the solution u to (4.1) with $h = \widetilde{p}$ as the control satisfies

$$u(\cdot, T) = 0 \text{ in } \Omega,$$

and h satisfies (4.2). This completes the proof of proposition 4.1.

Proof of Theorem 1.3

We may as well assume that f is in $C^1(\mathbb{R})$ and we shall use a fixed point argument applying Kakutani's Theorem. The general case of a globally Lipschitz function f can be easily obtained by a density argument.

Let

$$g(s) = \begin{cases} \frac{f(s)}{s}, & s \neq 0, \\ f'(0), & s = 0. \end{cases}$$

then g is continuous in $\mathbb{R}$ and

$$\|g\|_{L^\infty(\mathbb{R})} \leq \|f'\|_{L^\infty(\mathbb{R})} \leq L. \tag{4.11}$$

by our hypotheses f be globally Lipschitz function, where L be globally Lipschitz constant.

Write Z for the space $L^2(Q)$. For each $z \in Z$, we consider the linear system

$$\begin{cases} u_t + u_{xxxx} + g(z)u = \chi_\omega h, & (x,t) \in Q, \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{4.12}$$

Obviously, (4.12) is of the form (4.1), with $g = g(z) \in L^\infty(Q)$. By Proposition 4.1 and (4.11), there exists a control $h_z \in L^2(\omega \times (0,T))$ such that the corresponding solution of (4.12) satisfies

$$u_z^{h_z}(\cdot, T) = 0 \text{ in } \Omega$$

and

$$\begin{aligned} \|h_z\|_{L^2(\omega\times(0,T))} &\leq \exp\Big\{C(1 + \frac{1}{T} + \|g(z)\|_{L^\infty(Q)}^{\frac{2}{7}} + \|g(z)\|_{L^\infty(Q)}T)\Big\}\|u_0\|_{L^2(\Omega)} \\ &\leq \exp\Big\{C(1 + \frac{1}{T} + \|f'\|_{L^\infty(Q)}^{\frac{2}{7}} + \|f'\|_{L^\infty(Q)}T)\Big\}\|u_0\|_{L^2(\Omega)}, \end{aligned} \tag{4.13}$$

where C is independent of z.

On the other hand, for each $0 < t \leq T$, multiplying (4.12) (with right hand side $h = h_z$) by u_z and integrating by parts over $\Omega \times (0, t)$, we deduce that

$$\frac{1}{2}\int_0^t \int_\Omega ((u_z^{h_z})^2)_t dxdt + \int_0^t \int_\Omega (u_z^{h_z})^2_{xx} dxdt + \int_0^t \int_\Omega g(z)(u_z^{h_z})^2 dxdt$$
$$= \int_0^t \int_\omega h_z u_z^{h_z} dxdt.$$

Hence,

$$\frac{1}{2}\int_\Omega (u_z^{h_z})^2(x,t)dx + \int_0^t \int_\Omega (u_z^{h_z})^2_{xx} dxdt$$
$$\leq \frac{1}{2}\int_\Omega u_0^2(x)dx + \Big(\frac{1}{2} + \|g(z))\|_{L^\infty(Q)}\Big)\int_0^t \int_\Omega (u_z^{h_z})^2 dxdt + \frac{1}{2}\int_0^t \int_\omega (h_z)^2 dxdt,$$

from which and by Gronwall inequality, we get

$$\int_Q (u_z^{h_z})^2 dxdt \leq CT \exp\Big\{\Big(\frac{1}{2} + \|g(z))\|_{L^\infty(Q)}\Big)T\Big\}(\|u_0\|^2_{L^2(\Omega)} + \|h_z\|^2_{L^2(\omega\times(0,T))}),$$

where C is independent of z. Therefore, from the above inequality, (4.11) and (4.13), we have that there exists a positive constant R, which is independent of z, such that

$$\int_Q (u_z^{h_z})^2 dxdt \leq R.$$

Now, for each $z \in L^2(Q)$, set

$$\begin{aligned} U(z) &= \Big\{h \in L^2(\omega \times (0,T))\Big|\ u_z^h(\cdot, T) = 0,\ \|h\|_{L^2(\omega\times(0,T))} \\ &\leq \exp\Big(C(1 + \frac{1}{T} + \|f'\|^{\frac{2}{7}}_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})}T)\Big)\|u_0\|_{L^2(\Omega)}\Big\} \end{aligned} \tag{4.14}$$

and

$$\Lambda(z) = \{u_z^h | h \in U(z),\ \|u_z^h\|_{L^2(Q)} \leq R\},$$

where u_z^h is the solution of (4.12) with control h.

In this way, we have been able to introduce a set-valued mapping on $L^2(Q)$:

$$z \to \Lambda(z),$$

we shall prove that this mapping possesses at least one fixed point u.

From the above argument, and using the regularity of the solution of (4.12) and Aubin's Compact Theorem, we have that for any $z \in Z$, $\Lambda(z)$ is a nonempty compact convex set, we also see that there exists a fixed compact subset $K \in L^2(Q)$ such that

$$\Lambda(z) \subset K, \quad \text{for any } z \in L^2(Q).$$

We are sufficient to prove Λ is upper semicontinuous.

Indeed, if $z_k \to z$ in $L^2(Q)$, $u_{z_k}^{h_k} \in \Lambda(z_k)$ and $h_{z_k} \in U(z_k)$, then from the definition of $U(z_k)$, we have that there exists a subsequence of $\{h_{z_k}\}$, still denoted in the same way, such that

$$h_{z_k} \rightharpoonup h \text{ weakly in } L^2(\omega \times (0,T)),$$

and

$$\|h\|_{L^2(\omega\times(0,T))} \leq \exp\Big\{C\Big(1+\frac{1}{T}+\|f'\|_{L^\infty(\mathbb{R})}^{\frac{2}{7}}+\|f'\|_{L^\infty(\mathbb{R})}T\Big)\Big\}\|u_0\|_{L^2(\Omega)}. \tag{4.15}$$

Furthermore, from (4.12) ($h = h_{z_k}$), we have

$$\begin{aligned} &\|u_{z_k}^{h_{z_k}}\|_{L^2(0,T;H_0^2(\Omega))} + \|(u_{z_k}^{h_{z_k}})_t\|_{L^2(0,T;H^{-2}(\Omega))} \\ &\leq C(\|h_{z_k}\|_{L^2(Q)} + \|u_0\|_{L^2(\Omega)}) \leq C, \end{aligned} \tag{4.16}$$

and

$$\|u_{z_k}^{h_{z_k}}\|_{L^2(\delta,T;H^4(\Omega))} + \|(u_{z_k}^{h_{z_k}})_t\|_{L^2((\delta,T)\times\Omega)} \leq C(\delta), \tag{4.17}$$

for any δ with $0<\delta<T$. Here the constants C and $C(\delta)$ are independent of z.

From (4.16), (4.17) and Aubin's Compact Theorem, we have that as $k \to \infty$,

$$\begin{aligned} &u_{z_k} \to u && \text{in } L^2(Q), \\ &u_{z_k} \rightharpoonup u && \text{weakly in } L^2(0,T;H_0^2(\Omega)), \\ &u_{z_k}(\cdot,T) \to u(\cdot,T) && \text{in } L^2(\Omega), \\ &g(z_k)u_{z_k} \rightharpoonup g(z)u && \text{weakly in } L^2(Q). \end{aligned} \tag{4.18}$$

Hence, $u \in \Lambda(z)$.

By Kakutani's Fix Point Theorem, Theorem 1.3 follows.

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Zhongcheng Zhou
School of Mathematics and Statistics
Southwest University
Chongqing 400715
P. R. China
E-mail: zhouzc@amss.ac.cn