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METRIC VERSIONS OF POSNER'S THEOREMS

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Abstract. Let S and T be continuous linear operators on an ultraprime Banach algebra A. We show that if S, T, and ST are close to satisfy the derivation identity on A, then either S or T approaches to zero. If T is close to satisfy the derivation identity and $[T(a), a]$ is near the centre of A for each $a \in A$, then either T approaches to zero or A is nearly commutative. Further, we give quantitative estimates of these phenomena.

1. INTRODUCTION

In [7], E. C. Posner proved two theorems about derivations on prime rings which have turned out to be very influential. A number of authors have refined and extended these theorems in several ways (see [3, Subsection 2.1], where further references can be found). In this paper we follow the pattern of [2]. To this end we restrict our attention to ultraprime Banach algebras. The ultraprimeness is a metric version of the primeness which was introduced by M. Mathieu in [4]. Let A be a Banach algebra. For each $a, b \in A$, we write $M_{a,b}$ for the two-sided multiplication operator on A defined by

$$M_{a,b}(x) = axb \quad (x \in A).$$

Recall that A is prime if $M_{a,b} = 0$ implies $a = 0$ or $b = 0$. We define

$$\kappa(A) = \inf\{\|M_{a,b}\| : \ a, b \in A, \ \|a\| = \|b\| = 1\}.$$

The Banach algebra A is said to be *ultraprime* if $\kappa(A) > 0$. It is clear that each finite-dimensional prime Banach algebra is ultraprime. For a Banach space X we denote by $\mathcal{L}(X)$ the Banach algebra of all continuous linear operators from X into itself. The Banach algebra $\mathcal{L}(X)$ is ultraprime and, more generally, every closed subalgebra of

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$\mathcal{L}(X)$ containing the finite rank operators is ultraprime [4]. Every prime C^*-algebra is ultraprime [5].

In [2], a metric version of the first Posner's theorem is obtained by giving an estimate of the distance from the composition D_1D_2 of two derivations D_1 and D_2 on an ultraprime Banach algebra A to the set of all generalized derivations on A. In this paper we measure the "derivativity" of a given continuous linear operator T on an ultraprime Banach algebra A through the constant $der(T) = \sup\{\|T(ab) - T(a)b - aT(b)\|\colon \|a\| = \|b\| = 1\}$ and we estimate $\|S\|\|T\|$ in terms of $der(S)$, $der(T)$, and $der(ST)$ for arbitrary continuous linear operators S and T on A. Further, we present a metric version of the second Posner's theorem by estimating $\|T\|\sup\{\|ab-ba\|\colon \|a\| = \|b\| = 1\}$ in terms of $der(T)$ and $\sup\big\{dist\big([T(a),a],\mathcal{Z}(A)\big)\colon \|a\| = 1\big\}$.

2. First Posner's Theorem

Let us recall that an additive map D from a ring R into itself is said to be a *derivation* if

$$D(ab) = D(a)b + aD(b) \quad (a, b \in R). \tag{1}$$

The first Posner's theorem states that if R is a prime ring with characteristic different from 2, and D_1, D_2 are derivations on R such that the composition D_1D_2 is also a derivation, then either D_1 or D_2 is zero. The purpose of this section is to give a quantitative estimate of this result. Let A be a Banach algebra and let $T \in \mathcal{L}(A)$. We define a continuous bilinear map $T^\delta\colon A \times A \to A$ by

$$T^\delta(a, b) = T(ab) - T(a)b - aT(b) \quad (a, b \in A).$$

The constant $\|T^\delta\|$ can be thought of as a measure of how much T satisfies the derivation identity (1). From now on, we write $der(T)$ (the *derivativity* of T) for $\|T^\delta\|$, i.e.,

$$der(T) = \sup\big\{\|T(ab) - T(a)b - aT(b)\|\colon\ a, b \in A,\ \|a\| = \|b\| = 1\big\}.$$

The map $T \mapsto der(T)$ gives a seminorm on $\mathcal{L}(A)$ which vanishes precisely on the linear subspace $Der(A)$ of $\mathcal{L}(A)$ consisting of all continuous derivations on A. This seminorm has shown to be extremely useful for analysing the hyperreflexivity of the space $Der(A)$ [1].

Theorem 2.1. *Let A be a Banach algebra and let $S, T \in \mathcal{L}(A)$. then*

$$\kappa(A)^2\|S\|\|T\| \le 3\,der(ST) + \frac{15}{2}\,der(S)\|T\| + \frac{9}{2}\,der(T)\|S\|.$$

Proof. The arguments are similar to those in [2].

For all $a, b, c \in A$ we have

$$\begin{aligned} S(a)bT(c)+T(a)bS(c) &= (ST)^\delta(ab,c) - a(ST)^\delta(b,c) \\ &\quad - T^\delta(a,b)S(c) - S^\delta(T(ab),c) \\ &\quad - S^\delta(a,b)T(c) - S^\delta(ab,T(c)) - S(T^\delta(ab,c)) \\ &\quad + aS^\delta(T(b),c) + aS^\delta(b,T(c)) + aS(T^\delta(b,c)) \end{aligned}$$

and taking norms we arrive at

$$\|S(a)bT(c)+T(a)bS(c)\| \le \big(2\|(ST)^\delta\| + 5\|S^\delta\|\|T\| + 3\|T^\delta\|\|S\|\big)\|a\|\|b\|\|c\|.$$

To shorten notation, we write $\mu = 2\|(ST)^\delta\| + 5\|S^\delta\|\|T\| + 3\|T^\delta\|\|S\|$.

On account of [2, Observation 2], we have

$$\begin{aligned} 2\,S(a)uT(b)vS(c) &= \big(S(a)uT(b)+T(a)uS(b)\big)vS(c) \\ &\quad + S(a)u\big(T(b)vS(c)+S(b)vT(c)\big) \\ &\quad - \big(S(a)(uS(b)v)T(c) + T(a)(uS(b)v)S(c)\big), \end{aligned}$$

and hence $2\|S(a)uT(b)vS(c)\| \le 3\mu\|S\|\|a\|\|b\|\|c\|\|u\|\|v\|$ for all a, b, c, u, $v \in A$. This gives $\left\|M_{S(a),T(b)vS(c)}\right\| \le \frac{3}{2}\mu\|S\|\|a\|\|b\|\|c\|\|v\|$ for all $a,b,c,v \in A$. Since $\kappa(A)\|S(a)\|\|T(b)vS(c)\| \le \left\|M_{S(a),T(b)vS(c)}\right\|$, it follows that

$$\kappa(A)\|S(a)\|\|T(b)vS(c)\| \le \frac{3}{2}\mu\|S\|\|a\|\|b\|\|c\|\|v\|$$

for all $a,b,c,v \in A$ and therefore that

$$\kappa(A)\|S(a)\|\left\|M_{T(b),S(c)}\right\| \le \frac{3}{2}\mu\|S\|\|a\|\|b\|\|c\|$$

for all $a,b,c \in A$. From $\kappa(A)\|T(b)\|\|S(c)\| \le \left\|M_{T(b),S(c)}\right\|$ we now deduce that $\kappa(A)^2\|S(a)\|\|T(b)\|\|S(c)\| \le \frac{3}{2}\mu\|S\|\|a\|\|b\|\|c\|$ for all $a,b,c \in A$ and hence that $\kappa(A)^2\|S\|^2\|T\| \le \frac{3}{2}\mu\|S\|$, which clearly establishes the theorem. ■

Corollary 2.2. *Let A be a Banach algebra and let $S,T \in \mathcal{L}(A)$. Then*

$$\kappa(A)^2\min\{\|S\|,\|T\|\} \le \kappa(A)\sqrt{3der(ST)} + \frac{15}{2}der(S) + \frac{9}{2}der(T).$$

Proof. Of course, we can assume that $\kappa(A), \|S\|, \|T\| \ne 0$.

By applying Theorem 2.1 we arrive at

$$1 \le \frac{\alpha}{\|S\|\|T\|} + \frac{\beta}{\|S\|} + \frac{\gamma}{\|T\|},$$

where $\alpha = 3der(ST)\kappa(A)^{-2}$, $\beta = \frac{15}{2}der(S)\kappa(A)^{-2}$, and $\gamma = \frac{9}{2}der(T)\kappa(A)^{-2}$. We now write $\lambda = \min\{\|S\|,\|T\|\}$. Then $1 \le \frac{\alpha}{\lambda^2} + \frac{\beta}{\lambda} + \frac{\gamma}{\lambda}$ and therefore

$$\lambda^2 - (\beta+\gamma)\lambda - \alpha \leq 0.$$

This implies that

$$\lambda \leq \frac{\beta+\gamma+\sqrt{(\beta+\gamma)^2+4\alpha}}{2} \leq \beta+\gamma+\sqrt{\alpha},$$

which establishes the inequality in the corollary. ■

3. Second Posner's Theorem

Let R be a ring. In the sequel, we write $[a,b] = ab - ba$ for all $a,b \in R$ and we denote by $\mathcal{Z}(R)$ the centre of R. A map $T\colon R \to R$ is said to be *commuting* if

$$[T(a),a] = 0 \quad (a \in R) \tag{2}$$

and, more generally, it is said to be *centralizing* if

$$[T(a),a] \in \mathcal{Z}(R) \quad (a \in R). \tag{3}$$

The second Posner's theorem states that if D is a centralizing derivation on a prime ring R, then either D is zero or R is commutative. Our next concern is to give a quantitative estimate of this result. Our method is motivated by [6]. To this end, we measure how much a linear operator T from a Banach algebra A into itself satisfies conditions (2) and (3) by considering the constants

$$com(T) = \sup\bigl\{\|[T(a),a]\| \colon\ a \in A,\ \|a\| = 1\bigr\}$$

and

$$cen(T) = \sup\bigl\{dist\bigl([T(a),a],\mathcal{Z}(A)\bigr) \colon\ a \in A,\ \|a\| = 1\bigr\},$$

respectively. Note that both com and cen are seminorms on $\mathcal{L}(A)$ vanishing precisely on the commuting maps and the centralizing maps, respectively. Further, we measure the commutativity of A through the constant

$$\chi(A) = \sup\bigl\{\|[a,b]\| \colon\ a,b \in A,\ \|a\| = \|b\| = 1\bigr\}.$$

Let us recall that $\mathcal{Z}(A)$ is closed so that the quotient linear space $A/\mathcal{Z}(A)$ turns into a Banach space with respect to the norm given by $\|a+\mathcal{Z}(A)\| = dist\bigl(a,\mathcal{Z}(A)\bigr)$ $(a \in A)$.

Lemma 3.1. *Let A be a Banach algebra. Then*

$$\|[a,b]\| \leq 2\,\|a+\mathcal{Z}(A)\|\,\|b+\mathcal{Z}(A)\|$$

for all $a,b \in A$.

Proof. Let $a,b \in A$. For all $u,v \in \mathcal{Z}(A)$ we have $[a,b] = [a+u,b+v]$ and so $\|[a,b]\| \leq 2\|a+u\|\|b+v\|$. By taking the infima in u and v we arrive at the claimed inequality. ■

Lemma 3.2. *Let A a Banach algebra and let $T \in \mathcal{L}(A)$. Then*

$$\kappa(A) com(T)^2 \leq \big(8cen(T) + der(T)\big)\|T\|$$

Proof. For all $a, b \in A$, we have

$$[T(a), b] + [T(b), a] = \frac{1}{2}[T(a+b), a+b] - \frac{1}{2}[T(a-b), a-b].$$

We thus get

$$\|[T(a), b] + [T(b), a] + \mathcal{Z}(A)\| \leq 4\, cen(T) \tag{4}$$

for all $a, b \in A$ with $\|a\| = \|b\| = 1$.

Let $a \in A$ with $\|a\| = 1$. Then

$$\begin{aligned} 4[T(a), a]^2 = {} & 2\big[[T(a), a], T(a)\big]a + 2a\big[[T(a), a], T(a)\big] \\ & - \big[[T(a), a^2] + [T(a^2), a], T(a)\big] + \big[[T^\delta(a, a), a], T(a)\big] \end{aligned}$$

and therefore

$$\begin{aligned} 4\,\big\|[T(a), a]^2\big\| \leq {} & 4\,\big\|\big[[T(a), a], T(a)\big]\big\| \\ & + \big\|\big[[T(a), a^2] + [T(a^2), a], T(a)\big]\big\| + \Big\|\big[[T^\delta(a, a), a], T(a)\big]\Big\|. \end{aligned}$$

From Lemma 3.1 and (4) we now deduce that

$$\begin{aligned} \big\|[T(a), a]^2\big\| \leq {} & 2\big\|[T(a), a] + \mathcal{Z}(A)\big\|\|T\| \\ & + \frac{1}{2}\big\|[T(a), a^2] + [T(a^2), a] + \mathcal{Z}(A)\big\|\|T\| + \|T^\delta\|\|T\| \\ \leq {} & \big(4cen(T) + der(T)\big)\|T\|. \end{aligned}$$

For each $x \in A$ with $\|x\| = 1$, we have

$$[T(a), a]x[T(a), a] = [T(a), a]^2 x + [T(a), a]\big[x, [T(a), a]\big]$$

and so

$$\begin{aligned} \|[T(a), a]x[T(a), a]\| \leq {} & \big\|[T(a), a]^2 x\big\| + \big\|[T(a), a]\big[x, [T(a), a]\big]\big\| \\ \leq {} & \big(4cen(T) + der(T)\big)\|T\| + \|[T(a), a]\|\, 2\,\big\|[T(a), a)] + \mathcal{Z}(A)\big\| \\ \leq {} & \big(8cen(T) + der(T)\big)\|T\|. \end{aligned}$$

We thus get $\big\|M_{[T(a),a],[T(a),a]}\big\| \leq \big(8cen(T) + der(T)\big)\|T\|$ and hence

$$\kappa(A)\|[T(a), a]\|^2 \leq \big(8cen(T) + der(T)\big)\|T\|.$$

Taking the supremum in a we finally obtain the inequality in the lemma. ■

Theorem 3.3. *Let A be a Banach algebra and let $T \in \mathcal{L}(A)$. Then*

$$\kappa(A)^2\chi(A)\|T\| \leq 36\,com(T) + \frac{9}{2}\,der(T)\chi(A)$$

and

$$\kappa(A)^{5/2}\chi(A)\|T\| \leq 36\big(8cen(T) + der(T)\big)^{1/2}\|T\|^{1/2} + \frac{9}{2}\kappa(A)^{1/2}der(T)\chi(A).$$

Proof. Let $a, b \in A$ with $\|a\| = \|b\| = 1$. We write $ad(a)$ for the inner derivation on A implemented by a, i.e. $ad(a)(x) = [a, x]$ for each $x \in A$. Since $\big(-ad(a)T + ad(T(a))\big)(b) = \frac{1}{2}[T(a+b), a+b] - \frac{1}{2}[T(a-b), a-b]$, it follows that $\|ad(a)T - ad(Ta))\| \leq 4com(T)$, and consequently $dist\big(ad(a)T, Der(A)\big) \leq 4com(T)$. On account of [1, Proposition 2.2], we have

$$der\big(ad(a)T\big) \leq 3dist\big(ad(a)T, Der(A)\big) \leq 12com(T)$$

and Theorem 2.1 now yields

$$\kappa(A)^2\|ad(a)\|\|T\| \leq 36com(T) + \frac{9}{2}der(T)\|ad(a)\|.$$

Taking the supremum in a we arrive at the first inequality in the theorem. From this inequality together with Lemma 3.2 we get the second inequality in the theorem. ■

Corollary 3.4. *Let A be a Banach algebra and let $T \in \mathcal{L}(A)$. Then*

$$\kappa(A)^2\min\{\chi(A), \|T\|\} \leq \frac{9}{2}der(T) + 6\kappa(A)\sqrt{com(T)}$$

and

$$\kappa(A)^{5/4}\min\{\chi(A), \|T\|^{1/2}\} \leq \sqrt{36\big(8cen(T) + der(T)\big)^{1/2} + \frac{9}{2}\kappa(A)^{1/2}der(T)}.$$

Proof. Of course, we can assume that $\kappa(A), \chi(A), \|T\| \neq 0$.

By applying the first inequality in Theorem 3.3 we arrive at

$$1 \leq \frac{\alpha}{\chi(A)\|T\|} + \frac{\beta}{\|T\|},$$

where $\alpha = 36com(T)\kappa(A)^{-2}$ and $\beta = \frac{9}{2}der(T)\kappa(A)^{-2}$. Write $\lambda = \min\{\chi(A), \|T\|\}$. Then $1 \leq \frac{\alpha}{\lambda^2} + \frac{\beta}{\lambda}$ and therefore $\lambda^2 - \beta\lambda - \alpha \leq 0$, which implies that

$$\lambda \leq \frac{\beta + \sqrt{\beta^2 + 4\alpha}}{2} \leq \beta + \sqrt{\alpha}$$

and this gives the first inequality in the corollary.

We now apply the second inequality in Theorem 3.3 to get

$$1 \leq \frac{\alpha}{\chi(A)\|T\|^{1/2}} + \frac{\beta}{\|T\|}$$

where $\alpha = 36\big(8cen(T) + der(T)\big)^{1/2}\kappa(A)^{-5/2}$ and $\beta = \frac{9}{2}der(T)\kappa(A)^{-2}$. Let $\lambda = \min\{\chi(A), \|T\|^{1/2}\}$. Then $1 \leq \frac{\alpha}{\lambda^2} + \frac{\beta}{\lambda^2}$, which implies $\lambda \leq \sqrt{\alpha + \beta}$ and this proves the second inequality in the corollary. ■

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