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# EXISTENCE OF SOLUTIONS FOR NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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**Abstract.** This paper is concerned with the existence of mild solutions, strong solutions and strict solutions for a class of neutral integrodifferential equations with nonlocal conditions in Banach space. Since the nonlinear terms of the systems involve spacial derivatives, the theory of fractional power and $\alpha$-norm is used to discuss the problem. In the end an example is provided to illustrate the applications of the obtained results.

## 1. Introduction

In this paper, we study the existence of solutions for semilinear neutral integrodifferential equations with nonlocal conditions of the following form:

$$(1)\quad \begin{cases} \frac{d}{dt}\Big[x(t)+F(t,x(h_1(t)))\Big]+Ax(t) \\ =\int_0^t B(t-s)x(s)ds+G(t,x(h_2(t))),\ t\in[0,T], x(0)+g(x)=x_0, \end{cases}$$

where $-A$ is the infinitesimal generater of an analytic semigroup on a Banach space $X$, $B(t)$ is a closed linear operator from $X_\alpha$ ( it will be defined later ) into itself, $F$, $G$, $g$, $h_1$ and $h_2$ are given functions to be specified later.

Integro-differential equations can be used to describe a lot of natural phenomena arising from many fields such as electronics, fluid dynamics, biological models, and chemical kinetics. Most of these phenomena cannot be described through classical differential equations. That is why in recent years they have attracted more and more attention of several mathematicians, physicists, and engineers. Some topics for this

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kind of equations, such as existence and regularity , stability, (almost) periodicity of solutions and control problems, have been investigated by many mathematicians, see [1]-[25], for example.

In [18, 19, 20], Grimmer et al. proved the existence of solutions of the following integrodifferential evolution equation:

$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds + g(t), & \text{for } t \geq 0, \\ v(0) = v_0 \in X, \end{cases} \tag{2}$$

where $g : \mathbb{R}^+ \to X$ is a continuous function. The author(s) showed the existence, uniqueness, representation of solutions via resolvent operators associated to the following linear homogeneous equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds, & \text{for } t \geq 0, \\ v(0) = v_0 \in X. \end{cases}$$

The resolvent operator, replacing role of $C_0$-semigroup for evolution equations, plays an important role in solving Eq. (2) in weak and strict senses. In recent years much work on existence problems for nonlinear integrodifferential evolution equations has been done by many authors through applying the theory of resolvent operator. In papers [1, 6, 7, 8, 21], the authors have discussed the (local) existence and regularity of solutions for some partial functional differential equations with finite orinfinite delay in Banach space. And papers [2, 12, 22] have studied the existence problems for semlinear impulsive integrodifferential equations. Meanwhile, as the nonlocal Cauchy problem for evolution equations initiated by Byszewski[26] have offered better effects in discussing practical models than classical Cauchy problems, there are lots of works on various issues of different evolution equations with nonlocal conditions, see [27, 28, 30, 31] for differential evolution systems and [3, 13, 16, 17, 22, 23, 24] for integrodifferential evolution equations.

Particularly, Neutral (integro)differential equations arise in many areas of applied mathematics. For instance, the system of rigid heat conduction with finite wave speeds, studied in [9], can be modeled in the form of integrodifferential equations of neutral type with delay, and for this reason these equations (with initial condition or nonlocal condition) have received much attention in the last few decades. In Paper [6], by using Banach fixed point theorem the authors have studied the existence and regularity of solutions for the following neutral integrodifferential equations with finite delay

$$\begin{cases} \frac{d}{dt}D(t, x_t) = AD(t, x_t) + \int_0^t B(t-s)D(t, x_t)ds + f(t, x_t), \ t \geq 0, \\ x_0 = \phi \in C([-r, 0], X), \end{cases}$$

where $A$ is the generator of a $C_0$-semigroup. See [27, 3, 7, 25] for more relative works.

The purpose of this work is to study existence of mild solutions, strong solutions and strict solutions for nonlocal system (1) by using the theory of resolvent operators and fixed point theorems. As a motivation example for this class of equations we consider the following boundary value problem with nonlocal condition

$$(3)\qquad \begin{cases} \dfrac{\partial}{\partial t}\left[z(t,x)+f\left(t,z(t,x),\dfrac{\partial}{\partial x}z(t,x)\right)\right] \\ =\dfrac{\partial^2}{\partial x^2}z(t,x)+\displaystyle\int_0^t b(t-s)\dfrac{\partial^2}{\partial x^2}z(s,x)ds \\ \quad +g\left(t,z(t,x),\dfrac{\partial}{\partial x}z(t,x)\right), \quad 0\le x\le\pi,\ 0\le t\le T, \\ z(t,0)=z(t,\pi)=0,\ t\in[0,T], \\ z(0,x)+\displaystyle\sum_{i=1}^{p}k_i(x)z(t_i,x)=z_0(x),\ 0\le x\le\pi, \end{cases}$$

This system can also be written into an abstract neutral equation as mentioned above. However, the results established in related papers as [3] become invalid for this situation, since the functions $f,\ g$ in (3) involve spatial derivatives. As one will see in Section 6, if take $X=L^2([0,\pi])$, then the third variables of $f$ and $g$ are defined on $X_{\frac{1}{2}}$ and so the solutions can not be discussed on $X$ like in the appeared references. In this paper, inspired by the work of [32],[33] and [30], we shall discuss this problem by using fractional power operators theory and $\alpha-$norm. That is, we are to restrict this equation in a Banach space $X_\alpha(\subset X)$ and investigate the existence and regularity of mild solutions for Eq. (1). It means that the obtained theorems have more general application than the existed results. On the other hand, we don't need the compactness of the function $g$ in the nonlocal condition. we do not require the function $g$ in the nonlocal condition satisfy the compactness condition or Lipschitz condition, instead, it is continuous and is completely determined on $[\tau,T]$ for some small $\tau>0$. The compactness condition or Lipschitz condition for $g$ appear, respectively, in almost all the above-stated papers on the topics of nonlocal problem of integrodifferential equations.

The paper is organized as follows: in Section 2 we recall some concepts, hypotheses and basic results about resolvent operator. Particularly, we verify in this section the uniform continuity of analytic resolvent for our discussion. It is worthy to mention that compactness of resolvent operator does not imply the uniform continuity. In Section 3, we study the existence of mild solutions for Eq.(1) using Sadovskii fixed point principle. it can be seen that our discussion is quite different from the works of [2, 3, 6, 7, 8, 22] and other existed papers. The existence of strong solutions is discussed in Section 4. To obtain the existence of strong solutions we only require that $F$ and $G$ satisfy Lipschitz conditions which are very weak. Then the existence of strict solutions is studied in Section 5 with Gronwall inequality applied. Finally, in Section 6, an example is provided to illustrate the applications of the obtained results.

## 2. Preliminaries

Let $X$ be a Banach space, throughout this paper, we always assume that $-A : D(A) \subseteq X \to X$ is the infinitesimal generater of a compact analytic semigroup $(S(t))_{t\geq 0}$. $Y$ is the Banach space formed from $D(A)$ with the graph norm $\|y\|_Y = \|Ay\| + \|y\|$, for $y \in D(A)$. Let $0 \in \rho(A)$, the resolvent set of operator $A$, then it is possible to define the fractional power $A^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in $X$ and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \qquad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Denote the space $(D(A^\alpha), \|\cdot\|_\alpha)$ by $X_\alpha$, then it is well known that for each $0 < \alpha \leq 1$, $X_\alpha$ is a Banach space, $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \leq 1$ and the imbedding is compact whenever $R(\lambda, A)$, the resolvent operator of $A$, is compact. Let $\|A^{-\beta}\| \leq M_0$, with $M_0$ a positive constant. Hereafter we denote by $C([0,T], X_\alpha)$ the Banach space of continuous functions from $[0,T]$ to $X_\alpha$ with the norm

$$\|x\|_C = \sup_{0\leq t\leq T} \|A^\alpha x(t)\|, \ x \in C([0,T], X_\alpha).$$

For the theory of operator semigroup we refer to [34] and [35].

The theory of resolvent operator plays an essential role in investigating the existence of solutions of Eq.(1). Next we collect the definition and basic results about this theory, see [18, 19, 20] for more details.

**Definition 2.1.** A family of bounded linear operators $R(t) \in \mathcal{L}(X)$ for $t \in [0,T]$ is called resolvent operators for

$$\begin{cases} \dfrac{d}{dt}x(t) = -Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds, \\ x(0) = x_0 \in X, \end{cases} \tag{4}$$

if

(i) $R(0) = I$ and $\|R(t)\| \leq N_1 e^{\omega t}$ for some $N_1 > 0, \omega \in \mathbb{R}$

(ii) for all $x \in X$, $R(t)x$ is continuous for $t \in [0,T]$.

(iii) $R(t) \in \mathcal{L}(Y)$, for $t \in [0,T]$. For $x \in Y$, $R(t)x \in C^1([0,T], X) \cap C([0,T], Y)$ and for $t \geq 0$ such that

(5)
$$\begin{aligned} R'(t)x &= -AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= -R(t)Ax + \int_0^t R(t-s)B(s)xds. \end{aligned}$$

We shall always assume the following hypotheses on the operators $A$ and $B(\cdot)$:

($V_1$) $A$ generates an analytic semigroup on $X$. $B(t)$ is a closed operator on $X$ with domain at least $D(A)$ a.e. $t \geq 0$ with $B(t)x$ strongly measurable for each $x \in D(A)$ and $\|B(t)\|_{1,0} \leq b(t)$, $b \in L^1(0,\infty)$ with $b^*(\lambda)$ absolutely convergent for Re$\lambda > 0$, where $b^*(\lambda)$ denotes the Laplace transform of $b(t)$.

($V_2$) $\rho(\lambda) := (\lambda I - A_0 - B^*(\lambda))^{-1}$ exists as a bounded operator on $X$ which is analytic for $\lambda$ in the region $\Lambda = \{\lambda \in \mathbb{C} : |arg\lambda| < \frac{\pi}{2} + \delta\}$, where $0 < \delta < \frac{\pi}{2}$. In $\Lambda$ if $|\lambda| \geq \varepsilon > 0$ there exists a constant $M = M(\varepsilon) > 0$ so that $\|\rho(\lambda)\| \leq \frac{M}{|\lambda|}$.

($V_3$) $A\rho(\lambda) \in \mathcal{L}(X)$ for $\lambda \in \Lambda$ and are analytic on $\Lambda$ into $\mathcal{L}(X)$. $B^*(\lambda) \in \mathcal{L}(Y,X)$ and $B^*(\lambda)\rho(\lambda) \in \mathcal{L}(Y,X)$ for $\lambda \in \Lambda$. Given $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ so that for $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$, $\|A\rho(\lambda)\|_{1,0} + \|B^*(\lambda)\rho(\lambda)\|_{1,0} \leq \frac{M}{|\lambda|}$, and $\|B^*(\lambda)\|_{1,0} \to 0$ as $|\lambda| \to \infty$ in $\Lambda$. In addition, $\|A\rho(\lambda)\| \leq \frac{M}{|\lambda|^n}$ for some $n > 0$, $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$. Further, there exists $D \subset D(A^2)$ which is dense in $Y$ such that $A_0(D)$ and $B^*(\lambda)(D)$ are contained in $Y$ and $\|B^*(\lambda)x\|_1$ is bounded for each $x \in D$, $\lambda \in \Lambda$, $|\lambda| \geq \varepsilon$.

Then, it follows from [20] that, under these conditions, there is a resolvent operator $R(t)$ for linear system (4) defined by

$$R(0) = I$$

and

$$R(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - A - B^*(\lambda))^{-1}xd\lambda, \; t > 0,$$

or equivalently, using the notation of ($V_2$),

(6)
$$R(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\rho(\lambda)xd\lambda, \; t > 0,$$

where $\Gamma$ is a contour of the type used to obtain an analytic semigroup. We can select contour $\Gamma$, included in the region $\Lambda$, consisting of $\Gamma_1, \Gamma_2$, and $\Gamma_3$, where

$$\begin{aligned} &\Gamma_1 = \{re^{i\phi} : r \geq 1\}, \; \Gamma_3 = \{re^{-i\phi} : r \geq 1\}, \; \frac{\pi}{2} < \phi < \frac{\pi}{2} + \delta, \\ &\Gamma_2 = \{e^{i\theta} : -\phi \leq \theta \leq \phi\}, \end{aligned}$$

oriented so that $\mathrm{Im}(\lambda)$ is increasing on $\Gamma_1$ and $\Gamma_2$. Moreover, $R(t)$ is also analytic and there exist $N, C_\alpha > 0$ such that

$$\|R(t)\| \le N \text{ and } \|A^\alpha R(t)\| \le \frac{C_\alpha}{t^\alpha},\ 0 < t \le T,\ 0 \le \alpha \le 1. \tag{7}$$

For the resolvent operator $R(t)$ we can further prove the following property:

**Lemma 2.2.** *$AR(t)$ is continuous for $t > 0$ in the uniform operator topology of $\mathcal{L}(X)$.*

*Proof.* From (6) one has that

$$AR(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} A\rho(\lambda)d\lambda,\ t > 0.$$

Let $\lambda t = \mu$ and $J = t\Gamma$ to get

$$AR(t) = \frac{1}{2\pi i}\int_J \frac{1}{t} e^{\mu} A\rho(t^{-1}\mu)x d\mu,\ t > 0,$$

and use Cauchy's theorem to obtain

$$AR(t) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{t} e^{\mu} A\rho(t^{-1}\mu)d\mu,$$

It now follows from $(V_3)$ that

$$\|AR(t)\| \le \frac{M}{2\pi}\int_\Gamma t^{n-1}|e^{\mu}|\frac{1}{|\mu|^n}|d\mu|,$$

which converges absolutely and uniformly for $t > 0$. Thus we conclude the assertion. ∎

In this paper, for the sake of simplicity, we always require that $A^\alpha$ be commutative with $R(t)$ for any $0 \le \alpha \le 1$, that is, for any $x \in D(A^\alpha)$,

$$A^\alpha R(t)x = R(t)A^\alpha x. \tag{8}$$

Generally speaking, this commutation is not always valid although some recent references (such as [11, 27]) have used it readily. We point out, however, that this commutation can be reached many cases. For example, let $B(t-s) = b(t-s)A$ with $b(t)$ a scalar function defined on $(0, +\infty)$, then, the linear problem (4) becomes

$$\begin{cases} \frac{d}{dt}x(t) = -Ax(t) + \int_0^t b(t-s)Ax(s)ds, \\ x(0) = x_0 \in X. \end{cases} \tag{9}$$

If we impose the following conditions on system (9),

$(V_1')$ $A$ generates an analytic semigroup on $X$. In particular

$$\Lambda_1 = \{\lambda \in \mathbb{C} : |arg\lambda| < (\pi/2) + \delta_1\}, 0 < \delta_1 < \pi/2$$

is contained in the resolvent set of $A$ and $\|(\lambda I - A)^{-1}\| \leq M/|\lambda|$ on $\Lambda_1$ for some constant $M > 0$. The scalar function $b(\cdot)$ is in $L^1(0, \infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$, where $b^*(\lambda)$ denotes the Laplace transform of $b(t)$.

$(V_2')$ There exists $\Lambda = \{\lambda \in \mathbb{C} : |arg\lambda| < (\pi/2) + \delta_2\}, 0 < \delta_2 < \pi/2$, so that $\lambda \in \Lambda$ implies $g_1(\lambda) = 1 + b^*(\lambda)$ exists and is not zero. Further $\lambda g_1^{-1}(\lambda) \in \Lambda_1$ for $\lambda \in \Lambda$.

$(V_3')$ In $\Lambda$, $b^*(\lambda) \to 0$ as $|\lambda| \to \infty$.

Then, from [20], the conditions $(V_1) - (V_3)$ above are fulfilled and hence the resolvent operator $R(t)$ is analytic. We see that (8) holds in this situation.

Finally, we end this section by state the following fixed point principle which will be used in the sequel.

**Theorem 2.3.** (see [36]). *Assume that $P$ is a condensing operator on a Banach space $X$, i.e., $P$ is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \leq \alpha(B)$ for every bounded set $B$ of $X$ with $\alpha(B) > 0$. If $P(H) \subseteq H$ for a convex, closed, and bounded set $H$ of $X$, then $P$ has a fixed point in $H$ (where $\alpha(\cdot)$) denotes the kuratowski measurable of noncompactness).*

## 3. Existence of Mild Solutions

The mild solution of Eq.(1) expressed by the resolvent operator is defined as follows.

**Definition 3.1.** A function $x(\cdot) \in C([0, T], X_\alpha)$ is said to be a mild solution of Eq. (1), if

$$\begin{aligned} x(t) =& R(t)\left[x_0 + F(0, x(h_1(0))) - g(x)\right] - F(t, x(h_1(t))) \\ &+ \int_0^t R(t-s)\left[AF(s, x(h_1(s)))\right. \\ &\left. - \int_0^s B(s-\tau)F(\tau, x(h_1(\tau)))d\tau + G(s, x(h_2(s)))\right] ds, \end{aligned}$$

for $t \in [0, T]$.

To guarantee the existence of solutions, we impose the following restrictions on Eq.(1). Let $\alpha \in (0, 1)$.

$(H_0)$ $R(t)$ is a compact operator for each $t > 0$.

$(H_1)$ $\{B(t)\}_{t\in[0,T]}$ is a family of operators from $Y$ to $X$ such that $B(t) \in \mathcal{L}(X_{\alpha+\beta},\ X)$ for each $t \in [0,T]$. Then, there exists a positive number $M_1$ such that

$$\|B(t)\|_{\alpha+\beta,\ 0} \leq M_1 \qquad t \in [0,T]. \tag{10}$$

$(H_2)$ There exists a constant $\beta \in (0,1)$ with $\alpha+\beta \leq 1$, such that $F : [0,T] \times X_\alpha \to X_{\alpha+\beta}$ satisfies the Lipschitz condition, i.e., there exists a constant $L_0 > 0$ such that :

$$\Big\|F(t_1,x_1) - F(t_2,x_2)\Big\|_{\alpha+\beta} \leq L_0(|t_1-t_2| + \|x_1-x_2\|_\alpha)$$

for any $0 \leq t_1, t_2 \leq T, x_1, x_2 \in X_\alpha$, and the inequality

$$\Big\|F(t,x)\Big\|_{\alpha+\beta} \leq L_0(\|x\|_\alpha + 1)$$

holds for any $(t,x) \in [0,T] \times X_\alpha$.

$(H_3)$ The function $G : [0,T] \times X_\alpha \to X_\alpha$ satisfies the following conditions:

(i) for each $t \in [0,T]$, the function $G(t,\cdot) : X_\alpha \to X_\alpha$ is continuous and for each $x \in X_\alpha$ the function $G(\cdot,x) : [0,T] \to X_\alpha$ is strongly measurable;

(ii) for each positive number $k \in \mathbb{N}$, there is a positive function $g_k \in L^2([0,T])$ such that

$$\sup_{\|x\|\leq k} \|G(t,x)\|_\alpha \leq g_k(t)$$

and

$$\liminf_{k\to\infty} \frac{1}{k}\int_0^T g_k(x)ds = \gamma < \infty.$$

$(H_4)$ $g : C([0,T], X_\alpha) \to X_\alpha$ is a continuous mapping which maps bounded sets into bounded sets, i.e., there exists a constant $L > 0$ such that for any $x \in C([0,T], X_\alpha)$,

$$\|g(x)\|_\alpha \leq L\|x\|_C.$$

Moreover, there is a $\delta = \delta(k) \in (0,T)$ such that $g(u) = g(v)$ for any $u, v \in B_k$ with $u(s) = v(s), s \in [\delta, T]$, where $B_k = \{x \in C([0,T], X_\alpha), \|x(\cdot)\|_C \leq k\}$.

$(H_5)$ $h_1, h_2 \in C([0,T]; [0,T])$.

$(H_3')$ $G : [0,T] \times X_\alpha \to X$ satisfies the lipschitz condition, that is, there exists $L_1 > 0$ such that

$$\|G(t_1,x_1) - G(t_2,x_2)\| \leq L_1\left[|t_1-t_2| + \|x_1-x_2\|_\alpha\right],$$

for any $0 \leq t_1, t_2 \leq T, x_1, x_2 \in X_\alpha$ and the inequality

$$\|G(t,x)\| \leq L_1(\|x\|_\alpha + 1),$$

holds for any $(t,x) \in [0,T] \times X_\alpha$.

$(H_4')$ $g : C([0,T], X_\alpha) \to X_\alpha$ is a continuous function, and there exists $L_2 > 0$ such that

$$\left\| g(u) - g(v) \right\|_\alpha \le L_2 \|u - v\|_C,$$

for any $u, v \in C([0,T], X_\alpha)$, and the inequality

$$\left\| g(u) \right\|_\alpha \le L_2(\|u\|_C + 1)$$

holds for any $u \in C([0,T], X_\alpha)$.

First we can prove the following existence result by applying Banach fixed pricinple without any compactness condition for $C_0-$semigroup $(S(t))_{t\ge 0}$ or resolvent operator $(R(t))_{t\ge 0}$.

**Theorem 3.2.** *Assume that assumptions* $(H_1)$*,* $(H_2)$*,* $(H_3')$*,* $(H_4')$ *and* $(H_5)$ *hold, then Eq.(1) has a unique mild solution provided that*

$$(11)\quad C_0 := \left( M_0(N+1) + \frac{C_\alpha T^{2-\alpha}}{1-\alpha} M_1 + \frac{T^\beta C_{1-\beta}}{\beta} \right) L_0 + \frac{C_\alpha T^{1-\alpha}}{1-\alpha} L_1 + N L_2 < 1.$$

*Proof.* Define the operator $P$ on $C([0,T], X_\alpha)$ by the formula

$$\begin{aligned}(Px)(t) =& R(t)\left[x_0 + F(0, x(h_1(0))) - g(x)\right] - F(t, x(h_1(t))) \\ &+ \int_0^t R(t-s) \left[ AF(s, x(h_1(s))) \right. \\ &\left. - \int_0^s B(s-\tau) F(\tau, x(h_1(\tau))) d\tau + G(s, x(h_2(s))) \right] ds.\end{aligned}$$

Then, it is easy to see that $P$ maps $C([0,T], X_\alpha)$ into itself. By a direct computation we can show using (11) that $P$ is a strict contraction on $C([0,T], X_\alpha)$. Hence from the Banach fixed point theorem we conclude that $P$ has a unique fixed point in $C([0,T], X_\alpha)$ which is a mild solution of Eq.(1). ∎

Next, we prove the existence of mild solutions when $R(t)$ satisfies the condition $(H_0)$, i.e., it is compact for $t > 0$.

**Theorem 3.3.** *Assume that assumptions* $(H_0) - (H_5)$ *hold, then Eq.(1) has a mild solution provided that*

$$(12)\qquad C_1 := \left( N M_0 + M_0 + \frac{T^\beta C_{1-\beta}}{\beta} + \frac{T^{2-\alpha} C_\alpha}{1-\alpha} M_1 \right) L_0 + N\left(NL + \gamma\right) < 1.$$

To prove the above theorem, we need some lemmas. First, for a fixed $n \in \mathbb{N}$, we consider the following approximate problem:

$$
(13) \qquad \begin{cases} \frac{d}{dt}[x(t) + F(t, x(h_1(t)))] + Ax(t) \\ = \int_0^t B(t-s)x(s)ds + G(t, x(h_2(t))),\ t \in [0, T], \\ x(0) + R(\frac{1}{n})g(x) = x_0. \end{cases}
$$

**Lemma 3.4.** *Assume that all the conditions in Theorem 3.3 are satisfied, then for any $n \in \mathbb{N}$, the nonlocal Cauchy problem (13) has at least one mild solution $x_n \in C([0, T], X_\alpha)$.*

*Proof.* Let $B_k$ be that given in assumption $(H_4)$. It is obvious that $B_k$ is a bounded, closed and convex set in $C([0, T], X_\alpha)$. Define the operator $Q_n$ on $C([0, T], X_\alpha)$ by the formula

$$
\begin{aligned}
(Q_n x)(t) =& R(t)\left[x_0 + F(0, x(h_1(0))) - R(\frac{1}{n})g(x)\right] - F(t, x(h_1(t))) \\
&+ \int_0^t R(t-s)\,[AF(s, x(h_1(s))) \\
&- \int_0^s B(s-\tau)F(\tau, x(h_1(\tau)))d\tau + G(s, x(h_2(s)))\bigg] ds.
\end{aligned}
$$

It is easy to see that the fixed point of $Q_n$ is a mild solution of Eq.(13). Subsequently, we will prove that $Q_n$ has a fixed point on some $B_k$ by using Theorem 2.3.

We claim that there exists a $k \in \mathbb{N}$ such that $Q_n(B_k) \subseteq B_k$, if it is not true, then for each $k \in \mathbb{N}$, there is a function $x_k(\cdot) \in B_k$, but $Q_n x_k \notin B_k$, that is $\|Q_n x_k(t)\|_\alpha > k$ for some $t(k) \in [0, T]$, where $t(k)$ denotes $t$ is dependent on $k$. On the other hand, however, we have

$$
\begin{aligned}
k <& \|(Q_n x_k)(t)\|_\alpha \\
\le& \left\| R(t)\left[x_0 + F(0, x_k(h_1(0))) - R(\frac{1}{n})g(x_k)\right]\right\|_\alpha + \|F(t, x_k(h_1(t)))\|_\alpha \\
&+ \bigg\| \int_0^t R(t-s)\,[AF(s, x_k(h_1(s))) \\
&+ \int_0^s B(s-\tau)F(\tau, x_k(h_1(\tau)))d\tau - G(s, x_k(h_2(s)))\bigg] ds \bigg\|_\alpha \\
\le& N\left[\|x_0\|_\alpha + M_0 L_0(k+1) + N \sup_{x \in B_k} \|g(x)\|_\alpha\right] + M_0 L_0(k+1)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \left\|A^{1-\beta}R(t-s)A^{\beta}F(s,x_k(h_1(s)))\right\|_{\alpha} ds \\
&+\int_0^t \|A^{\alpha}R(t-s)\| \int_0^s \|B(s-\tau)F(\tau,x(h_1(\tau)))\| d\tau ds + N\int_0^t g_k(s)ds \\
\le& N\left[\|x_0\|_{\alpha} + M_0L_0(k+1) + NLk\right] + M_0L_0(k+1) \\
&+\frac{T^{\beta}C_{1-\beta}}{\beta}L_0(k+1) + \frac{T^{2-\alpha}C_{\alpha}}{1-\alpha}M_1L_0(k+1) + N\int_0^t g_k(s)ds.
\end{aligned}
$$

Dividing on both sides by the $k$ and taking the lower limit as $k \to +\infty$, we get

$$
\left(NM_0 + M_0 + \frac{T^{\beta}C_{1-\beta}}{\beta} + \frac{T^{2-\alpha}C_{\alpha}}{1-\alpha}M_1\right)L_0 + N\left(NL+\gamma\right) \ge 1,
$$

this contradicts (12). Hence for some positive integer $k$, $Q_n(B_k) \subseteq B_k$.

Next we will show that $Q_n$ has a fixed point on $B_k$. To this end, we decompose $Q_n = Q_{n1} + Q_{n2}$, where the operators $Q_{n1}, Q_{n2}$ are defined on $B_k$, respectively, by

$$
\begin{aligned}
(Q_{n1}x)(t) =& R(t)F(0,x(h_1(0))) - F(t,x(h_1(t))) + \int_0^t R(t-s)AF(s,x(h_1(s)))ds, \\
(Q_{n2}x)(t) =& R(t)\left[x_0 - R(\frac{1}{n})g(x)\right] - \int_0^t R(t-s)\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau ds \\
&+\int_0^t R(t-s)G(s,x(h_2(s)))ds,
\end{aligned}
$$

for all $0 \le t \le T$, and we verify that $Q_{n1}$ is a contraction while $Q_{n2}$ is a compact operator.

To prove that $Q_{n1}$ is a contraction, we take $x_1, x_2 \in B_k$, then, for each $t \in [0,T]$, we have

$$
\begin{aligned}
&\|(Q_{n1}x_1)(t) - (Q_{n1}x_2)(t)\|_{\alpha} \\
\le& \left\|R(t)A^{-\beta}\left[A^{\beta}F(0,x_1(h_1(0))) - A^{\beta}F(0,x_2(h_1(0)))\right]\right\|_{\alpha} \\
&+\left\|A^{-\beta}\right\| \left\|A^{\beta}F(t,x_1(h_1(t))) - A^{\beta}F(t,x_2(h_1(t)))\right\|_{\alpha} \\
&+\left\|\int_0^t A^{1-\beta}R(t-s)A^{\beta}[F(s,x_1(h_1(s))) - F(s,x_2(h_1(s)))]ds\right\|_{\alpha} \\
\le& NM_0L_0 \sup_{0\le t\le T}\|x_1(t)-x_2(t)\|_{\alpha} + M_0L_0 \sup_{0\le t\le T}\|x_1(t)-x_2(t)\|_{\alpha} \\
&+\int_0^t \frac{C_{1-\beta}L_0}{(t-s)^{1-\beta}}\|x_1(s)-x_2(s)\|_{\alpha}ds \\
\le& \left(NM_0 + M_0 + \frac{C_{1-\beta}T^{\beta}}{\beta}\right)L_0 \sup_{0\le t\le T}\|x_1(t)-x_2(t)\|_{\alpha}.
\end{aligned}
$$

Thus,

$$\|(Q_{n1}x_1) - (Q_{n1}x_2)\|_C \leq \left(NM_0 + M_0 + \frac{C_{1-\beta}T^{\beta}}{\beta}\right) L_0 \ \|x_1 - x_2\|_C.$$

So by assumptions (12), we see that $Q_{n1}$ is a contraction.

To prove that $Q_{n2}$ is compact, firstly we prove that $Q_{n2}$ is continuous on $B_k$, let $\{x_m\} \subseteq B_k$ with $x_m \to x$ in $B_k$, then by $(H_3)$ and $(H_4)$, we have

$$g(x_m) \to g(x), \qquad m \to \infty,$$

and

$$G(s, x_m(h_2(s))) \to G(s, x(h_2(s))), \qquad m \to +\infty.$$

Since

$$\|G(s, x_m(h_2(s))) - G(s, x(h_2(s)))\|_{\alpha} \leq 2g_k(s),$$

by the dominated convergence theorem and the strong continuity of $R(t)$, we have

$$\begin{aligned}
&\|Q_{n2}x_m - Q_{n2}x\|_{\alpha} \\
&= \sup_{0\leq t\leq T} \Bigg\| R(t)R(\frac{1}{n})\Big[g(x_m) - g(x)\Big] \\
&\quad - \int_0^t R(t-s)\int_0^s B(s-\tau)\Big[F(\tau, x_m(h_1(\tau))) - F(\tau, x(h_1(\tau)))\Big]d\tau ds \\
&\quad + \int_0^t R(t-s)\Big[G(s, x_m(h_2(s))) - G(s, x(h_2(s)))\Big]ds \Bigg\|_{\alpha} \\
&\leq \sup_{0\leq t\leq T} \Bigg( \Big\| R(t)R(\frac{1}{n})\Big[g(x_m) - g(x)\Big]\Big\|_{\alpha} \\
&\quad + \Big\|\int_0^t R(t-s)\int_0^s B(s-\tau)\Big[F(\tau, x_m(h_1(\tau))) - F(\tau, x(h_1(\tau)))\Big]d\tau ds\Big\|_{\alpha} \\
&\quad + \Big\|\int_0^t R(t-s)\Big[G(s, x_m(h_2(s))) - G(s, x(h_2(s)))\Big]ds\Big\|_{\alpha} \Bigg) \\
&\to 0 \text{ as } m \to \infty,
\end{aligned}$$

i.e., $Q_{n2}$ is continuous.

Next we prove that $\{(Q_{n2}x)(\cdot), x \in B_k\} \subseteq C([0,T], X_{\alpha})$ is a family of equicontinuous function. To see this we fix $t_1 > 0$, let $t_2 > t_1$ and $\varepsilon > 0$ be small enough, then

$$\begin{aligned}
&\|(Q_{n2}x)(t_2) - (Q_{n2}x)(t_1)\|_{\alpha} \\
&\leq \Big\|(R(t_2) - R(t_1)(x_0 - R(\frac{1}{n})g(x)\Big\|_{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^{t_2} R(t_2-s)\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau ds\right.\\
&\left.-\int_0^{t_1} R(t_1-s)\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau ds\right\|_\alpha\\
&+\left\|\int_0^{t_2} R(t_2-s)G(s,x(h_2(s)))ds-\int_0^{t_1} R(t_1-s)G(s,x(h_2(s)))ds\right\|_\alpha\\
\le\,& \|R(t_2)-R(t_1)\|\,[\|x_0\|_\alpha+N\|g(x)\|_\alpha]\\
&+\int_0^{t_1-\varepsilon}\|A^\alpha R(t_2-s)-A^\alpha R(t_1-s)\|\int_0^s\|B(s-\tau)F(\tau,x(h_1(\tau)))\|\,d\tau ds\\
&+\int_{t_1-\varepsilon}^{t_1}\|A^\alpha R(t_2-s)-A^\alpha R(t_1-s)\|\int_0^s\|B(s-\tau)F(\tau,x(h_1(\tau)))\|\,d\tau ds\\
&+\int_{t_1}^{t_2}\left\|A^\alpha R(t_2-s)\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau\right\|ds\\
&+\int_0^{t_1-\varepsilon}\|R(t_2-s)-R(t_1-s)\|\,\|G(s,x(h_2(s)))\|_\alpha\,ds\\
&+\int_{t_1-\varepsilon}^{t_1}\|R(t_2-s)-R(t_1-s)\|\,\|G(s,x(h_2(s)))\|_\alpha\,ds\\
&+\int_{t_1}^{t_2}\|R(t_2-s)G(s,x(h_2(s)))\|_\alpha\,ds.
\end{aligned}
$$

Noting that $\|G(s,x(h_2(s)))\|_\alpha \le g_k(s)$ and $g_k(s)\in L^2$, we see that $\|(Q_{n2}x)(t_2)-(Q_{n2}x)(t_1)\|_\alpha \to 0$ independently of $x\in B_k$ as $t_2-t_1\to 0$ since, by Lemma 2.2, $(R(t))_{t>0}$ and $(A^\alpha R(t))_{t>0}$ are continuous in $t$ in the uniform operators topology. Similarly, we can prove that the functions $\{Q_{n2}x, x\in B_k\}$ are equicontinuous at $t=0$. Hence $Q_{n2}$ maps $B_k$ into a family of equicontinuous functions.

Now, we verify that for fixed $t\in[0,T]$, the set $\{(Q_{n2}x)(t), x\in B_k\}$ is relatively compact in $X_\alpha$.

If $t=0$, then $(Q_{n2}x)(0)=x_0-R(\frac{1}{n})g(x)$. Clearly $g(x)$ is a bounded set in $B_k$, so it is true for $t=0$.

If $t\in(0,T]$, let $(Q_{n2}x)(t)=R(t)\left[x_0-R(\frac{1}{n})g(x)\right]+(Q'_{n2}x)(t)$, where

$$
\begin{aligned}
&(Q'_{n2}x)(t)\\
=&-\int_0^t R(t-s)\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau ds+\int_0^t R(t-s)G(s,x(h_2(s)))ds.
\end{aligned}
$$

Take $\alpha'\in(0,1)$ such that $0<\alpha'-\alpha<\frac{1}{2}$, then we have

$$
\|A^{\alpha'}(Q'_{n2}x)(t)\| \le \left\|\int_0^t A^{\alpha'}R(t-s)\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau ds\right\|
$$

$$
\begin{aligned}
&+\left\|\int_0^t A^{\alpha'}R(t-s)G(s,x(h_2(s)))ds\right\|\\
\leq&\frac{C_{\alpha'}T^{2-\alpha'}}{1-\alpha'}M_1L_0(\|x\|_\alpha+1)+\frac{T^{\frac{1}{2}-(\alpha'-\alpha)}C_{\alpha'-\alpha}}{\sqrt{1-2(\alpha'-\alpha)}}\|g_k(\cdot)\|_{L^2},
\end{aligned}
$$

which implies $\{A^{\alpha'}(Q'_{n2}x)(t), x\in B_k\}$ is bounded in $X$. Hence we infer that $(Q'_{n2}x)(t)$ is relatively compact in $X_\alpha$ by the compactness of operator $A^{-\alpha'}: X\to X_\alpha$ (noting that the imbedding $X_{\alpha'}\hookrightarrow X_\alpha$ is compact). Thus, $(Q_{n2}x)(t)$ is also relatively compact in $X_\alpha$ because $R(t)\left[x_0-R(\frac{1}{n})g(x)\right]$ does so.

Therefore, from the infinite-dimensional version of the Ascoli-Arzela theorem, $Q_{n2}$ is a completely continuous operator on $C([0,T],X_\alpha)$. Those arguments enable us to conclude that $Q_n=Q_{n1}+Q_{n2}$ is a condensing map on $B_k$, and by the Theorem 2.3 there exists a fixed point $x_n(\cdot)$ for $Q_n$ on $B_k$. Therefore, the nonlocal Cauchy problem (13) has a mild solution $x_n(\cdot)$, and the proof is completed. ■

Now define the solution set $D$ and the sets $D(t)$ by

$$
\begin{aligned}
&D=\{x_n\in C([0,T],X_\alpha): x_n=Q_nx_n, n\geq 1\},\\
&D(t)=\{x_n(t): x_n\in D, n\geq 1\},\ t\in[0,T].
\end{aligned}
$$

**Lemma 3.5.** *Assume that all the conditions of Theorem 3.3 are satisfied. Then for each $t\in(0,T]$, $D(t)$ is relatively compact in $X_\alpha$ and $D$ is equicontinuous on $(0,T]$.*

*Proof.* For $n\geq 1$ and $x_n\in D$, we have, for $t\in(0,T]$,

$$
\begin{aligned}
x_n(t)=&R(t)\left[x_0+F(0,x_n(h_1(0)))-R(\frac{1}{n})g(x_n)\right]-F(t,x_n(h_1(t)))\\
&+\int_0^t R(t-s)\Big[AF(s,x_n(h_1(s)))\\
&-\int_0^s B(s-\tau)F(\tau,x_n(h_1(\tau)))d\tau+G(s,x_n(h_2(s)))\Big]ds\\
=&R(t)\left[x_0+F(0,x_n(h_1(0)))-R(\frac{1}{n})g(x_n)\right]-A^{-\alpha}A^{\alpha}F(t,x_n(h_1(t)))\\
&+\int_0^t R(t-s)\Bigg[AF(s,x_n(h_1(s)))\\
&-\int_0^s B(s-\tau)F(\tau,x_n(h_1(\tau)))d\tau+G(s,x_n(h_2(s)))\Bigg]ds\\
=&\sum_{i=1}^3 I_i.
\end{aligned}
$$

From $(H_2)$ and $(H_4)$ we obtain $F(t, x_n(h_1(t)))$ and $g(x_n(t))$ are bounded in $X_\alpha$. By the compactness of $(R(t))_{t>0}$ and $A^{-\alpha}$, we see $I_1, I_2$ are relatively compact in $X_\alpha$. Now we verify

$$I_3 = \int_0^t R(t-s)\left[AF(s, x_n(h_1(s))) - \int_0^s B(s-\tau)F(\tau, x_n(h_1(\tau)))d\tau\right]ds + \int_0^t R(t-s)G(s, x_n(h_2(s)))ds$$

is relatively compact in $X_\alpha$ too. For this, take $0 < \alpha' < \alpha + \beta$, as above we have

$$\begin{aligned}
\|A^{\alpha'} I_3\| =& \left\| \int_0^t A^{\alpha'} R(t-s)\left[AF(s, x_n(h_1(s))) - \int_0^s B(s-\tau)F(\tau, x_n(h_1(\tau)))d\tau\right]ds \right. \\
&\left. + \int_0^t A^{\alpha'} R(t-s)G(s, x_n(h_2(s)))ds \right\| \\
\leq& \left\| \int_0^t A^{1+\alpha'-\alpha-\beta} R(t-s) A^{\alpha+\beta} F(s, x_n(h_1(s))) \right\| \\
&+ \left\| \int_0^t A^{\alpha'} R(t-s) \int_0^s B(s-\tau)\Big[F(\tau, x_n(h_1(\tau)))\Big] d\tau ds \right\| \\
&+ \left\| \int_0^t A^{\alpha'-\alpha} R(t-s) A^{\alpha} G(s, x_n(h_2(s)))ds \right\| \\
\leq& \frac{T^{\alpha+\beta-\alpha'} C_{1+\alpha'-\alpha-\beta}}{\alpha+\beta-\alpha'} L_0(\|x\|_\alpha + 1) + \frac{T^{1-\alpha'} C_{\alpha'}}{1-\alpha'} M_1 L_0 \left(\|x\|_\alpha + 1\right) \\
&+ \frac{T^{\frac{1}{2}-(\alpha'-\alpha)} C_{\alpha'-\alpha}}{\sqrt{1-2(\alpha'-\alpha)}} \|g_k(\cdot)\|_{L^2},
\end{aligned}$$

which implies $\{A^{\alpha'} I_3, x_n \in D(t)\}$ is bounded in $X$. Hence we infer that $I_3$ is relatively compact in $X_\alpha$. Thus, $D(t)$ is relatively compact in $X_\alpha$.

Due to Lemma 2.2, we can prove the second assertion by using the similar arguments as that in Lemma 3.4. ∎

**Proof of Theorem 3.3.**

*Proof.* We prove that the solution set $D$ is relatively compact in $C([0, T], X_\alpha)$. Due to Lemma 3.5 we only need to verify that $D(0)$ is relatively compact in $X_\alpha$ and $D$ is equicontinuous at $t = 0$.

For $x_n \in D, n \geq 1$, let

$$\overline{x_n}(t) = \begin{cases} x(\delta), & t \in [0, \delta), \\ x_n(t), & t \in [\delta, T], \end{cases}$$

where $\delta$ comes from the condition $(H_4)$. Then, by condition $(H_4)$, $g(x_n) = g(\overline{x_n})$.

Since $D$ is relatively compact in $C((0,T], X_\alpha)$, without loss of generality, we may suppose that there is a subsequence, still denote $\{\overline{x_n}\} \subseteq D$, such that $\overline{x_n} \to x$ in $C((0,T], X_\alpha)$, as $n \to \infty$, for some $x(\cdot)$. Thus, by the continuity of $g$ and the strong continuity of $R(t)$ at $t = 0$, we get

$$\begin{aligned}\|x_n(0) - (x_0 + g(x))\|_\alpha \le& \Big\|R(\frac{1}{n})g(x_n) - R(\frac{1}{n})g(x)\Big\|_\alpha + \Big\|R(\frac{1}{n})g(x) - g(x)\Big\|_\alpha \\ =& \Big\|R(\frac{1}{n})\Big[g(\overline{x_n}) - g(x)\Big]\Big\|_\alpha + \Big\|\Big[R(\frac{1}{n}) - I\Big]g(x)\Big\|_\alpha \\ \to& \ 0 \text{ as } n \to \infty,\end{aligned}$$

i.e. $D(0)$ is relatively compact in $X_\alpha$.

On the other hand, for $t \in (0,T]$

$$\begin{aligned}&\|x_n(t) - x_n(0)\|_\alpha \\ =&\Bigg\|R(t)\Big[x_0 + F(0, x_n(h_1(0))) - R(\frac{1}{n})g(x_n)\Big] - F(t, x_n(h_1(t))) \\ &+ \int_0^t R(t-s)\Bigg[AF(s, x_n(h_1(s))) \\ &- \int_0^s B(s-\tau)F(\tau, x_n(h_1(\tau)))d\tau + G(s, x_n(h_2(s)))\Bigg]ds \\ &- \Big[x_0 + F(0, x_n(h_1(0))) - R(\frac{1}{n})g(x_n) - F(0, x_n(h_1(0)))\Big]\Bigg\|_\alpha \\ \le&\Bigg\|\int_0^t R(t-s)\Bigg[AF(s, x_n(h_1(s))) \\ &- \int_0^s B(s-\tau)F(\tau, x_n(h_1(\tau)))d\tau + G(s, x_n(h_2(s)))\Bigg]ds\Bigg\|_\alpha \\ &+ \Big\|R(t)x_0 - x_0\Big\|_\alpha + \Big\|(R(t) - I)R(\frac{1}{n})g(x_n)\Big\|_\alpha + \Big\|(R(t) - I)F(0, x_n(h_1(0)))\Big\|_\alpha \\ &+ \Big\|F(t, x_n(h_1(t))) - F(0, x_n(h_1(0)))\Big\|_\alpha \\ \to& \ 0,\end{aligned}$$

uniformly in $n$ as $t \to 0$, since $D(0) = \big\{x_0 - R(\frac{1}{n})g(x_n) : x_n \in D\big\}_{n=1}^{\infty}$ is relatively compact and $A^{-\beta}$ is compact. Thus, we obtain that the set $D \subseteq C([0,T], X_\alpha)$ is equicontinuous at $t = 0$, and hence $D$ is relatively compact in $C([0,T], X_\alpha)$. We may suppose that $x_n \to x^* \in C([0,T], X_\alpha)$ as $n \to \infty$.

By the expression of mild solution for the system (13), we have

$$
\begin{aligned}
x_n(t) =& R(t)\left[x_0 + F(0, x_n(h_1(0))) - R(\frac{1}{n})g(x_n)\right] - F(t, x_n(h_1(t))) \\
&+ \int_0^t R(t-s)\left[AF(s, x_n(h_1(s)))\right. \\
&\left. - \int_0^s B(s-\tau)F(\tau, x_n(h_1(\tau))) + G(s, x_n(h_2(s)))\right] ds,
\end{aligned}
$$

for $0 \le t \le T$. Taking limit as $n \to \infty$ on both sides, we obtain that

$$
\begin{aligned}
x^*(t) =& R(t)\left[x_0 + F(0, x^*(h_1(0))) - g(x^*)\right] - F(t, x^*(h_1(t))) \\
&\int_0^t R(t-s)\left[AF(s, x^*(h_1(s)))\right. \\
&\left. - \int_0^s B(s-\tau)F(\tau, x^*(h_1(\tau))) + G(s, x^*(h_2(s)))\right] ds,
\end{aligned}
$$

for $t \in [0, T]$, which implies that Eq.(1) has a mild solution $x^*(\cdot)$. ■

**Remark 3.6.** It is easily seen from the above proof that, if the function $g(\cdot)$ is supposed to be completely continuous, then the compact condition $(H_0)$ of resolvent operator $R(t)$ in Theorem 3.3 can be taken off, that is, it is sufficient to assume that the analytic semigroup $(S(t))_{t\ge 0}$ is compact.

## 4. Existence of Strong Solutions

In this section, we prove the existence of strong solutions defined as follows.

**Definition 4.1.** A function $x(\cdot) \in C([0,T]; X_\alpha)$ is said to be a strong solution of Eq.(1), if

$(i)$ $x$ is differentiable a.e.on $[0,T]$ in $X$ , and $x' \in L^1([0,T], X)$;

$(ii)$ $x$ satisfies

$$
\frac{d}{dt}\left[x(t) + F(t, x(h_1(t)))\right] + Ax(t) = \int_0^t B(t-s)x(s)ds + G(t, x(h_2(t)))
$$

a.e. on $[0.T]$, and

$$
x(0) + g(x) = x_0.
$$

**Theorem 4.2.** *Let $X$ be a reflexive Banach space, suppose the conditions $(H_0)-(H_2)$, $(H_3')$ and $(H_4')$ are satisfied with $F([0,T]\times X_\alpha) \subseteq D(A)$, and the function $AF(0,\cdot): X_\alpha \to X$ maps bounded sets into bounded sets. Additionally, the following conditions hold:*

$(H_6)$ $x_0 \in D(A), g(x) \in D(A)$ *and* $g(\cdot)$ *maps bounded sets into bounded sets;*

$(H_7)$ *There exist constants* $0 < l_1, l_2 \leq 1$, *such that* $\|h_i(t) - h_i(\overline{t})\| \leq l_i|t - \overline{t}|, i = 1, 2$;

$(H_8)$ *There holds*

$$M^* := \left(\|A^{-\beta}\| + \frac{1}{\beta}C_{1-\beta}T^{\beta} + T^2 N M_1\right) L_0 l_1 + \frac{C_\alpha T^{1-\alpha}}{1-\alpha} L_1 l_2 < 1. \tag{14}$$

*Then, the nonlocal Cauchy problem (1) has a strong solution on* $[0, T]$.

*Proof.* Let $P$ be the operator defined in the proof of Theorem 3.2. Consider the set $B'_k = \left\{x \in C([0,T], X_\alpha) : \|x\|_C \leq k,\ \|x(t) - x(s)\|_\alpha \leq L^*|t - s|,\ t, s \in [0,T]\right\}$ for some positive constants $k$ and $L^*$ large enough. It is clear that $B'_k$ is a nonempty, closed and convex set. We shall prove that $P$ has a fixed point on $B'_k$. Obviously, from the proof of Theorem 3.3 it is sufficient to show that for any $x \in B'_k$, one has that

$$\|(Px)(t_2) - (Px)(t_1)\|_\alpha \leq L^*|t_2 - t_1|,\ t_2, t_1 \in [0,T].$$

In fact, by the expression of operator $P$, we get

$$\begin{aligned}
&\|Px(t_2) - Px(t_1)\|_\alpha \\
\leq &\|[R(t_2) - R(t_1)]A^\alpha [x_0 + F(0, x(h_1(0))) - g(x)]\| \\
&+ \|A^{-\beta}\|\|F(t_2, x(h_1(t_2))) - F(t_1, x(h_1(t_1)))\|_{\alpha+\beta} \\
&+ \left\| \int_0^{t_1} A^{1-\beta} R(t_1 - s) A^{\beta+\alpha} [F(s + t_2 - t_1, x(h_1(s + t_2 - t_1))) - F(s, x(h_1(s)))] ds \right. \\
&\left. + \int_0^{t_2 - t_1} A^{1-\beta} R(t_2 - s) A^{\beta+\alpha} F(s, x(h_1(s))) ds \right\| \\
&+ \left\| \int_0^{t_1} R(t_1 - s) \int_0^s B(s - \tau)[F(\tau + t_2 - t_1, x(h_1(\tau + t_2 - t_1))) - F(\tau, x(h_1(\tau)))] d\tau ds \right. \\
&+ \int_0^{t_1} R(t_1 - s) \int_0^{t_2 - t_1} B(s + t_2 - t_1 - \tau) F(\tau, x(h_1(\tau))) d\tau ds \\
&\left. + \int_0^{t_2 - t_1} R(t_2 - s) \int_0^s B(s - \tau) F(\tau, x(h_1(\tau))) d\tau ds \right\|_\alpha \\
&+ \left\| \int_0^{t_1} R(t_1 - s)[G(s + t_2 - t_1, x(h_2(s + t_2 - t_1))) - G(s, x(h_2(s)))] ds \right. \\
&\left. + \int_0^{t_2 - t_1} R(t_2 - s) G(s, x(h_2(s))) ds \right\|_\alpha .
\end{aligned}$$

Since

$$
\begin{aligned}
&\|[R(t_2)-R(t_1)]\,[x_0+F(0,x(h_1(0)))-g(x)]\|_\alpha \\
&= \left\| \int_{t_1}^{t_2} R'(t)\,[x_0+F(0,x(h_1(0)))-g(x)]\,dt \right\|_\alpha \\
&\le \left\| \int_{t_1}^{t_2} -R(t)A\,[x_0+F(0,x(h_1(0)))-g(x)]\,dt \right\|_\alpha \\
&\quad + \left\| \int_{t_1}^{t_2} \int_0^t R(t-s)B(s)\,[x_0+F(0,x(h_1(0)))-g(x)]\,dsdt \right\|_\alpha \\
&\le \frac{C_\alpha}{1-\alpha}\,\|A[x_0+F(0,x(h_1(0)))-g(x)]\|\,\left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \\
&\quad + \frac{C_\alpha M_1}{(1-\alpha)(2-\alpha)}\,\|x_0+F(0,x(h_1(0)))-g(x)\|_{\alpha+\beta}\,\left|t_2^{2-\alpha}-t_1^{2-\alpha}\right|.
\end{aligned}
$$

From conditions $(H_2)$, $(H_3')$ and $(H_7)$, it yields that

$$
\begin{aligned}
&\|Px(t_2)-Px(t_1)\| \\
&\le \frac{C_\alpha}{1-\alpha}\,\|A[x_0+F(0,x(h_1(0)))-g(x)]\|\,\left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \\
&\quad + \frac{C_\alpha M_1}{(1-\alpha)(2-\alpha)}\,\|x_0+F(0,x(h_1(0)))-g(x)\|_{\alpha+\beta}\,\left|t_2^{2-\alpha}-t_1^{2-\alpha}\right|, \\
&\quad + \|A^{-\beta}\|L_0|t_2-t_1| + \|A^{-\beta}\|L_0L^*l_1|t_2-t_1| + \frac{1}{\beta}C_{1-\beta}T^\beta L_0|t_2-t_1| \\
&\quad + \frac{1}{\beta}C_{1-\beta}T^\beta L_0L^*l_1|t_2-t_1| + \frac{1}{\beta}C_{1-\beta}L_1(k+1)|t_2^\beta-t_1^\beta| \\
&\quad + \frac{C_\alpha T^{2-\alpha}}{1-\alpha}M_1\|L_0|t_2-t_1| + \frac{C_\alpha T^{2-\alpha}}{1-\alpha}M_1L_0L^*l_1|t_2-t_1| \\
&\quad + \frac{C_\alpha T^{1-\alpha}}{1-\alpha}M_1L_0(k+1)|t_2-t_1|^{1-\alpha} \\
&\quad + TNL_1|t_2-t_1| + TNL_1L^*l_2|t_2-t_1| + N(k+1)|t_2-t_1| \\
&\le \left(C^* + \left[\left(\|A^{-\beta}\| + \frac{1}{\beta}C_{1-\beta}T^\beta + \frac{C_\alpha T^{2-\alpha}}{1-\alpha}M_1\right)L_0l_1 + NTL_1l_2\right]L^*\right)|t_2-t_1|,
\end{aligned}
$$

where $C^*$ is a constant independent of $L^*$ and $x \in B_k'$. So it follows from (14) that $\|Px(t_2)-Px(t_1)\| \le L^*|t_2-t_1|$ as long as $L^*$ is large enough ($\ge \frac{C^*}{1-M*}$). Thus, $P$ has a fixed point $x(\cdot)$ which is a mild solution of Eq.(1). Moreover, $x(\cdot)$ is Lipschitz continuous in $\alpha-$norm and hence is Lipschitz in $X$.

For this $x(\cdot)$, let

$$
\begin{aligned}
f(t) &= R(t)[x_0+F(0,x(h_1(0)))-g(x)], \\
p(t) &= \int_0^t R(t-s)AF(s,x(h_1(s)))ds,
\end{aligned}
$$

$$q(t) = \int_0^t R(t-s) \int_0^s B(s-\tau)F(\tau, x(h_1(\tau)))d\tau ds,$$
$$r(t) = \int_0^t R(t-s)G(s, x(h_2(s)))ds.$$

Then, from the hyprothes, it is not difficult to verify that they are all Lipschitz continuous in $X$, respectively. Since $x$ is Lipschitz continuous in $X$ on $[0, T]$ and the space $X$ is reflexive, we see that $x(\cdot)$ is $a.e.$ differentiable on $[0, T]$ and $x'(\cdot) \in L^1([0, T], X)$. The same argument shows that $f(t), p(t), q(t), r(t)$ also have this property.

On the other hand, by the standard arguments, we can obtain that $f(t)$, $p(t)$, $q(t)$, $r(t) \in D(A)$, and one has that

$$\begin{aligned}
f'(t) &= R'(t)[x_0 + F(0, x(h_1(0))) - g(x)] \\
&= -R(t)A[x_0 + F(0, x(h_1(0))) - g(x)] \\
&\quad + \int_0^t R(t-s)B(s)[x_0 + F(0, x(h_1(0))) - g(x)]ds, \\
p'(t) &= AF(t, x(h_1(t))) + \int_0^t R'(t-s)AF(s, x(h_1(s)))ds \\
&= AF(t, x(h_1(t))) - A\int_0^t R(t-s)AF(s, x(h_1(s)))ds \\
&\quad + \int_0^t \int_0^{t-s} R(t-s-\tau)B(\tau)AF(s, x(h_1(s)))d\tau ds, \\
q'(t) &= \int_0^t B(t-s)F(s, x(h_1(s)))ds + \int_0^t R'(t-s) \int_0^s B(s-\tau)F(\tau, x(h_1(\tau)))d\tau ds \\
&= \int_0^t B(t-s)F(s, x(h_1(s)))ds - \int_0^t R(t-s)A\int_0^s B(s-\tau)F(\tau, x(h_1(\tau)))d\tau ds \\
&\quad + \int_0^t \int_0^{t-s} R(t-s-u)B(u)\int_0^u B(u-\tau)F(\tau, x(h_1(\tau)))d\tau du ds,
\end{aligned}$$

and

$$\begin{aligned}
r'(t) &= G(t, x(h_2(t))) + \int_0^t R'(t-s)G(s, x(h_2(s)))ds \\
&= G(t, x(h_2(t))) - \int_0^t R(t-s)AG(s, x(h_2(s)))ds \\
&\quad + \int_0^t \int_0^{t-s} R(t-s-\tau)B(\tau)G(\tau, x(h_2(\tau)))d\tau ds.
\end{aligned}$$

Using Definition 2.1, we have $x$ satisfies a.e. that

$$
\begin{aligned}
&\frac{d}{dt}\Big[x(t)+F(t,x(h_1(t)))\Big]\\
=&\frac{d}{dt}\Bigg(R(t)\Big[x_0+F(0,x(h_1(0)))-g(x)\Big]+\int_0^t R(t-s)\Bigg[AF(s,x(h_1(s)))\\
&-\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau+G(s,x(h_2(s)))\Bigg]ds\Bigg)\\
=&-AR(t)\Big[x_0+F(0,x(h_1(0)))-g(x)\Big]+AF(t,x(h_1(t)))\\
&-A\int_0^t R(t-s)AF(s,x(h_1(s)))ds\\
&+\int_0^t R(t-s)A\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau ds-\int_0^t R(t-s)AG(s,x(h_2(s)))ds\\
&+\int_0^t R(t-s)B(s)[x_0+F(0,x(h_1(0)))-g(x)]ds-\int_0^t B(t-s)F(s,x(h_1(s)))ds\\
&+\int_0^t\int_0^{t-s} R(t-s-\tau)B(\tau)AF(s,x(h_1(s)))d\tau ds\\
&-\int_0^t\int_0^{t-s} R(t-s-u)B(u)\int_0^u B(u-\tau)F(\tau,x(h_1(\tau)))d\tau du ds\\
&+\int_0^t\int_0^{t-s} R(t-s-\tau)B(\tau)G(\tau,x(h_2(\tau)))d\tau ds+G(t,x(h_2(t))),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{d}{dt}\Big[x(t)+F(t,x(h_1(t)))\Big]\\
=&-A\Bigg[R(t)[x_0+F(0,x(h_1(0)))-g(x)]-F(t,x(h_1(t)))\\
&+\int_0^t R(t-s)AF(s,x(h_1(s)))ds\\
&-\int_0^t R(t-s)\int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau ds+\int_0^t R(t-s)G(s,x(h_2(s)))\Bigg]ds\\
&+\int_0^t B(t-s)R(s)[x_0+F(0,x(h_1(0)))-g(x)]ds-\int_0^t B(t-s)F(s,x(h_1(s)))ds\\
&+\int_0^t\int_0^s B(t-s)R(s-\tau)AF(\tau,x(h_1(\tau)))d\tau ds\\
&-\int_0^t B(t-s)\int_0^s R(s-\tau)\int_0^\tau B(\tau-u)F(u,x(h_1(u)))du d\tau ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t B(t-s)\int_0^s R(s-\tau)G(\tau,x(h_2(\tau)))d\tau ds+G(t,x(h_2(t)))\\
=&-Ax(t)+\int_0^t B(t-s)x(s)ds+G(t,x(h_2(t))).
\end{aligned}
$$

This shows that $x(\cdot)$ is also a strong solution of Eq.(1). Thus the proof is completed. ∎

## 5. Existence of Strict Solutions

In this section, by using the Gronwall's lemma, we give a existence result of strict solutions for nonlocal problem (1).

**Definition 5.1.** A function $x(\cdot):[0,T]\to D(A^\alpha)$ is said to be a strict solution of Eq. (1), if the following conditions hold:

(i) $x(\cdot)+F(\cdot,x(h_1(\cdot)))\in C^1([0,T],X)\cap C([0,T],X_\alpha)$;

(ii) $x(\cdot)$ satisfies Eq.(1) for $t\in[0,T]$.

We assume that:

$(H_9)$ $F\in C^1([0,T]\times X_\alpha;X)$, and the partial derivatives $D_1F(\cdot,\cdot)$ and $D_2F(\cdot,\cdot)$ are locally Lipschitz (in $X$-norm) with respect to the second argument. Additionally, $F([0,T],X_\alpha)\subseteq D(A)$ is continuously differentiable.

$(H_{10})$ $G\in C^1([0,T]\times X_\alpha;X)$, and the partial derivatives $D_1G(\cdot,\cdot)$ and $D_2G(\cdot,\cdot)$ are locally Lipschitz with respect to the second argument.

$(H_{11})$ $h_1(\cdot)$ and $h_2(\cdot)$ are continuously differentiable on $[0,T]$ with $h_i(t)\le t$, for $t\in[0,T]$. Let $|h_i'|=\sup_{0\in[0,T]}|h_i'(t)|$, for $i=1,2$.

**Theorem 5.2.** *Assume that assumptions $(H_1)$, $(H_3')$, $(H_9)$, $(H_{10})$ and $(H_{11})$ hold with $M_0L_0<1$. In addition, suppose that $g:C([0,T],X_\alpha)\to X$ is continuously differentiable. Let $x(\cdot)$ be a mild solution of Eq.(1). If $x(0)+F(0,x(0))\in D(A)$, then $x(\cdot)$ is a strict solution of Eq.(1).*

*Proof.* Let $x(\cdot)$ be the mild solution of Eq.(1) which is obtained by Theorem 3.2 or Theorem 3.3. Then by using the strict contraction principle, one can show that there exists a unique function $y$ satisfying:

$$
\begin{aligned}
y(t)=&R(t)\Big\{-A\Big[x_0-g(x)+F(0,x(0))\Big]+H(0,x(0))\Big\}\\
&+\int_0^t R(t-s)B(s)\Big[x_0-g(x)+F(0,x(0))\Big]ds\\
&+\int_0^t R(t-s)\left[D_1H(s,x(s))+D_2H(s,x(s))y(s)\right]ds\\
&-\left[D_1F(t,x(h_1(t)))+D_2F(t,x(h_1(t)))y(h_1(t))h_1'(t)\right],
\end{aligned}
\tag{15}
$$

where

$$H(t,x(t)) = AF(t,x(h_1(t))) - \int_0^t B(t-s)F(s,x(h_1(s)))ds + G(s,x(h_2(s))).$$

Now, we introduce the function $z(t)$ defined by

$$z(t) = x_0 - g(x) + \int_0^t y(s)ds, \ t \in [0,T].$$

We will prove that $x(t) = z(t)$ on $[0,T]$, which implies that $x(t)$ is a strict solution of Eq.1. Using the expression of $y(t)$, we obtain

$$\begin{aligned}
\int_0^t y(s)ds &= \int_0^t R(s)\Big\{ - A\Big[x_0 - g(x) + F(0,x(0))\Big] + H(0,x(0))\Big\}ds \\
&\quad + \int_0^t \int_0^s R(s-\tau)B(\tau)\Big[x_0 - g(x) + F(0,x(0))\Big]d\tau ds \\
&\quad + \int_0^t \int_0^s R(s-\tau)\Big[D_1H(\tau,x(\tau)) + D_2H(\tau,x(\tau))y(\tau)\Big]d\tau ds \\
&\quad - \int_0^t \Big[D_1F(s,x(h_1(s))) + D_2F(s,x(h_1(s)))y(h_1(s))h_1'(s)\Big]ds \\
&= \int_0^t \Big\{ - R(s)A\Big[x_0 - g(x) + F(0,x(0))\Big] \\
&\quad + \int_0^s R(s-\tau)B(\tau)\Big[x_0 - g(x) + F(0,x(0))\Big]d\tau\Big\}ds \\
&\quad + \int_0^t R(s)H(0,x(0))ds \\
&\quad + \int_0^t \int_0^s R(s-\tau)\Big[D_1H(\tau,x(\tau)) + D_2H(\tau,x(\tau))y(\tau)\Big]d\tau ds \\
&\quad - \int_0^t \Big[D_1F(s,x(h_1(s))) + D_2F(s,x(h_1(s)))y(h_1(s))h_1'(s)\Big]ds,
\end{aligned}$$

or

$$\int_0^t y(s)ds = \int_0^t R'(s)\Big[x_0 - g(x) + F(0,x(0))\Big]ds + \int_0^t R(s)H(0,x(0))ds$$

$$+\int_0^t\int_0^s R(s-\tau)\Big[D_1H(\tau,x(h\tau))+D_2H(\tau,x(\tau))y(\tau)\Big]d\tau ds$$
$$-\int_0^t\Big[D_1F(s,x(h_1(s)))+D_2F(s,x(h_1(s)))y(h_1(s))h_1'(s)\Big]ds$$
$$=R(t)\Big[x_0-g(x)+F(0,x(0))\Big]-x_0+g(x)-F(0,x(0))+\int_0^t R(s)H(0,x(0))ds$$
$$+\int_0^t\int_0^s R(s-\tau)\Big[D_1H(\tau,x(\tau))+D_2H(\tau,x(\tau))y(\tau)\Big]d\tau ds$$
$$-\int_0^t\Big[D_1F(s,x(h_1(s)))+D_2F(s,x(h_1(s)))y(h_1(s))h_1'(s)\Big]ds,$$

then,

$$\begin{aligned} z(t)=&R(t)\Big[x_0-g(x)+F(0,x(0))\Big]-F(0,x(0))+\int_0^t R(s)H(0,x(0))ds\\ &+\int_0^t\int_0^s R(s-\tau)\Big[D_1H(\tau,x(\tau))+D_2H(\tau,x(\tau))y(\tau)\Big]d\tau ds\\ &-\int_0^t\Big[D_1F(s,x(h_1(s)))+D_2F(s,x(h_1(s)))y(h_1(s))h_1'(s)\Big]ds. \end{aligned} \tag{16}$$

Since

$$\begin{aligned} &F(t,z(h_1(t)))\\ &=\int_0^t \frac{d}{ds}F(s,z(h_1(s)))ds+F(0,z(0))\\ &=\int_0^t\Big[D_1F(s,z(h_1(s)))+D_2F(s,z(h_1(s)))y(h_1(s))h_1'(s)\Big]ds+F(0,z(0)), \end{aligned}$$

or

$$\begin{aligned} F(0,z(0))\ =&\int_0^t\Big[D_1F(s,z(h_1(s)))\\ &+D_2F(s,z(h_1(s)))y(h_1(s))h_1'(s)\Big]ds-F(t,z(h_1(t))), \end{aligned} \tag{17}$$

and $t\to H(t,z(t))$ is continuously differentiable on $[0,T]$, we have

$$\begin{aligned}\frac{d}{dt}\int_0^t R(t-s)H(s,z(s))ds =& H(t,z(t)) + \int_0^t R'(t-s)H(s,z(s))ds \\ =& H(t,z(t)) - \int_0^t H(s,z(s))dR(t-s) \\ =& R(t)H(0,z(0)) + \int_0^t R(t-s)\Big[D_1H(s,z(s)) \\ & + D_2H(s,z(s))y(s)\Big]ds,\end{aligned}$$

hence,

$$\begin{aligned}&\int_0^t R(t-s)H(s,z(s))ds \\ =& \int_0^t R(s)H(0,z(0))ds + \int_0^t\int_0^s R(s-\tau)\Big[D_1H(\tau,z(\tau)) + D_2H(\tau,z(\tau))y(\tau)\Big]d\tau ds,\end{aligned}$$

or

$$\begin{aligned}\int_0^t R(s)H(0,z(0))ds =& \int_0^t R(t-s)H(s,z(s))ds \\ -& \int_0^t\int_0^s R(s-\tau)\Big[D_1H(\tau,z(\tau)) + D_2H(\tau,z(\tau))y(\tau)\Big]d\tau ds.\end{aligned} \tag{18}$$

Observing that $x(0) = z(0)$, we substitute (17) and (18) into (16) and get

$$\begin{aligned}z(t) =& R(t)\left[x_0 - g(x) + F(0,x(0))\right] - F(t,z(h_1(t))) \\ &+ \int_0^t \left[D_1F(s,z(h_1(s))) + D_2F(s,z(h_1(s)))y(h_1(s))h_1'(s)\right]ds \\ &+ \int_0^t R(t-s)H(s,z(s))ds \\ &- \int_0^t\int_0^s R(s-\tau)\left[D_1H(\tau,z(\tau)) + D_2H(\tau,z(\tau))y(\tau)\right]d\tau ds \\ &+ \int_0^t\int_0^s R(s-\tau)\left[D_1H(\tau,x(\tau)) + D_2H(\tau,x(\tau))y(\tau)\right]d\tau ds \\ &- \int_0^t \left[D_1F(s,x(h_1(s))) + D_2F(s,x(h_1(s)))y(h_1(s))h_1'(s)\right]ds.\end{aligned}$$

Consequently,

$$\begin{aligned} z(t)-x(t) =& \Big( F(t,x(h_1(t))) - F(t,z(h_1(t))) \Big) \\ &+ \int_0^t R(t-s)\Big( H(s,z(s)) - H(s,x(s)) \Big) ds \\ &+ \int_0^t \Big[ D_1F(s,z(h_1(s))) - D_1F(s,x(h_1(s))) \Big] ds \\ &+ \int_0^t \Big[ D_2F(s,z(h_1(s))) - D_2F(s,x(h_1(s))) \Big] y(h_1(s))h_1'(s)ds \\ &- \int_0^t \int_0^s R(s-\tau)\Big[ D_1H(\tau,z(\tau)) - D_1H(\tau,x(\tau)) \Big] d\tau ds \\ &- \int_0^t \int_0^s R(s-\tau)\Big[ D_2H(\tau,z(\tau)) - D_2H(\tau,x(\tau)) \Big] y(\tau) d\tau ds. \end{aligned}$$

Then from assumptions $(H_1)$, $(H_3')$, $(H_9)$, $(H_{10})$ and $(H_{11})$ we obtain that

$$\sup_{0\le s\le t} \|z(s)-x(s)\| \le M_0L_0 \sup_{0\le s\le t} \|z(s)-x(s)\| + c\int_0^t \sup_{0\le \tau\le s} \|z(\tau)-x(\tau)\| ds,$$

where $c$ is positive constant. Since $M_0L_0 < 1$, we have

$$\sup_{0\le s\le t} \|z(s)-x(s)\| \le \frac{c}{1-M_0L_0} \int_0^t \sup_{0\le \tau\le s} \|z(\tau)-x(\tau)\| ds.$$

By Gronwall's lemma we infer that $x(\cdot) = z(\cdot)$ on $[0,T]$ and we conclude that $x(\cdot) \in C^1([0,T],X)$ and is a strict solution of Eq.(1). ■

## 6. EXAMPLE

Consider the following partial functional indegrodifferential equation

$$(19) \quad \begin{cases} \dfrac{\partial}{\partial t}\left[ z(t,x) + \displaystyle\int_0^\pi a(y,x)\left( z(t\sin t, y) + \sin\left( \frac{\partial}{\partial y} z(t,y) \right)\right) dy \right] \\ = \dfrac{\partial^2}{\partial x^2} z(t,x) + \displaystyle\int_0^t b(t-s)\frac{\partial^2}{\partial x^2} z(s,x)ds + c(t, z(t\cos t, x, \frac{\partial}{\partial x} z(t,x))), \\ \qquad 0\le x\le \pi,\ 0\le t\le T, \\ z(t,0) = z(t,\pi) = 0, t\in[0,T], \\ z(0,x) + \displaystyle\sum_{i=1}^{p} k_i(x) z(t_i,x) = z_0(x),\ 0\le x\le \pi, \end{cases}$$

where $T \leq \pi$, $p$ is a positive integer, $0 < t_0 < t_1 < \cdots < t_p < T$, $z_0(x) \in X := L^2([0,\pi])$, $b(\cdot)$ is continuous, then there exist a constant $M_1 > 0$ such that $\|b(\cdot)\| \leq M_1$ and $k_i(x)$ is a $C^1$ function.

To write system (19) in the form of Eq.(1), let $A$ be defined by

$$Az = -z''$$

with the domain

$$D(A) = \{z(\cdot) \in X : z', z'' \in X, \text{ and } z(0) = z(\pi) = 0\}.$$

Then $-A$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ which is compact, analytic, and self-adjoint. Furthermore, $A$ has a discrete spectrum, the eigenvalues are $n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $\xi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1, 2, \cdots$. Then the following properties hold:

(i) If $z \in D(A)$, then

$$Az = \sum_{n=1}^{\infty} n^2 \langle z, \xi_n \rangle \xi_n.$$

(ii) For each $z \in X$,

$$A^{-1/2} z = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, \xi_n \rangle \xi_n.$$

In particular, $\|A^{-1/2}\| = 1$.

(iii) The operator $A^{1/2}$ is given by

$$A^{1/2} z = \sum_{n=1}^{\infty} n \langle z, \xi_n \rangle \xi_n$$

on the space

$$D(A^{1/2}) = \left\{ z(\cdot) \in X, \sum_{n=1}^{\infty} n \langle z, \xi_n \rangle \xi_n \in X \right\}.$$

For System (19) we assume that the following conditions hold:

$(A_1)$ The function $a$ and $(\partial/\partial x)a(y,x)$ are measurable, $a(y,0) = a(y,\pi) = 0$, let

$$N_0 = \left[ \int_0^{\pi} \int_0^{\pi} \left( \frac{\partial^2}{\partial x^2} a(y,x) \right)^2 dy dx \right]^{1/2} < \infty.$$

$(A_2)$ The function $c : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $C^1$ function.

We take $\alpha = \beta = \frac{1}{2}$, $B(t) = b(t)A$, and define the functions $F : [0,T] \times X_{\frac{1}{2}} \to D(A)$, $G : [0,T] \times X_{\frac{1}{2}} \to X_{\frac{1}{2}}$, and $g : C([0,T], X_{\frac{1}{2}}) \to X_{\frac{1}{2}}$, respectively, by

$$F(t,z)(x) = Z(z(t,x))(x),$$

$$G(t,z)(x) = c(t, z(t,x), \frac{\partial}{\partial x} z(t,x)),$$

and

$$g(z(t,x)) = \sum_{i=1}^{p} k_i(x) z(t_i, x),$$

where

$$Z(z)(x) = \int_0^\pi a(y,x) \left[ z(y) + \sin(z'(y)) \right] dy.$$

Then, with these notations, System (19) can be rewritten into the form (1).

We know that, for the operator $(-A, D(A))$, there is $\varphi \in (0, \pi/2)$ such that

$$\Lambda := \left\{ \lambda \in \mathbb{C} : |arg\lambda| < \frac{\pi}{2} + \varphi \right\} \subset \rho(-A),$$

where $\rho(-A)$ denotes the resolvent set of $-A$. So we may assume that

$(A_3)$ The scalar function $b(\cdot)$ satisfies that $\lambda \in \Lambda$ implies $g(\lambda) := 1 + b^*(\lambda) \neq 0$, and $\lambda g(\lambda) \in \Lambda$ for $\lambda \in \Lambda$, further, $b^*(\lambda) \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Lambda$.

Then the conditions $(V_1') - (V_3')$ are verified, and hence the linear equation (2) for System (19) has a resolvent operator $(R(t))_{t \geq 0}$, which is given by, for $x \in X$,

$$\begin{aligned} R(t)x &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \rho(\lambda) x d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} g^{-1}(\lambda)(\lambda g^{-1}(\lambda) + A)^{-1} x d\lambda, \end{aligned}$$

where $\Gamma$ is described in Section 2. It is readily seen that $R(t)$ is compact for all $t > 0$, since $R(\lambda, -A)$ is compact for any $\lambda \in \rho(A)$.

Clearly, $G$ satisfies $(H_3)$ while $g$ verifies $(H_4)$. From $(A_1)$ it follows that

$$\begin{aligned} \langle Z(z), \xi_n \rangle &= \int_0^\pi \xi_n(x) \left[ \int_0^\pi a(y,x) \left[ z(y) + \sin(z'(y)) \right] dy \right] dx \\ &= \frac{1}{n} \sqrt{\frac{2}{\pi}} \left\langle \int_0^\pi \frac{\partial a(y,x)}{\partial x} \left[ z(y) + \sin(z'(y)) \right] dy, \ \cos(nx) \right\rangle \\ &= \frac{1}{n^2} \sqrt{\frac{2}{\pi}} \left\langle \int_0^\pi \frac{\partial^2 a(y,x)}{\partial x^2} \left[ z(y) + \sin(z'(y)) \right] dy, \ \xi_n \right\rangle, \end{aligned}$$

which shows that $F(t,z) = Z(z)$ takes values in $D(A)$. Furthermore, it is easy to prove (see [32]) that

$$\|z_2 - z_1\| \leq \|z_2 - z_1\|_{\frac{1}{2}}.$$

Thus, we know from $(A_1)$ that $Z: X_{\frac{1}{2}} \to X_1$ is Lipschitz continuous with $\|Z\| \leq N_0$, which implies the function $F$ verifies assumption $(H_2)$. Therefore, all the conditions of theorem 3.3 are all satisfied. Hence from Theorem 3.3, system (19) admits a mild solution on $[0, T]$ under the above assumptions additionally provided that (8) holds.

Moreover, if:

$(A_4)$ The function $(\partial^2/\partial x^2)a(y,x)$ is measurable, and

$$\int_0^\pi \int_0^\pi \left(\frac{\partial^2}{\partial x^2}a(y,x)\right)^2 dydx < \infty.$$

$(A_5)$ Functions $z_0''$ and $(\partial^2/\partial x^2)k_i(x,y)$ are measurable, and $z_0(0) = z_0(\pi) = 0$, $k_i(0) = k_i(\pi) = 0$.

Then system (19) has a strong solution provided that condition (14) is satisfied.

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