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CENTRALIZING GENERALIZED DERIVATIONS ON POLYNOMIALS IN PRIME RINGS

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Abstract. Let R be a prime ring, $Z(R)$ its center, U its right Utumi quotient ring, C its extended centroid, G a non-zero generalized derivation of R, $f(x_1, \dots, x_n)$ a non-zero polynomial over C and I a non-zero right ideal of R. If $f(x_1, \dots, x_n)$ is not central valued on R and $[G(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)] \in C$, for all $r_1, \dots, r_n \in I$, then either there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax$ for all $x \in R$, with $(a - \alpha)I = 0$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:

1. $f(x_1, \dots, x_n)$ is central valued in $eRCe$;
2. $char(R) = 2$ and $eRCe$ satisfies the standard identity s_4;
3. $char(R) = 2$ and $f(x_1, \dots, x_n)^2$ is central valued in $eRCe$;
4. $f(x_1, \dots, x_n)^2$ is central valued in $eRCe$ and there exist $a, b \in U$, $\alpha \in C$ such that $G(x) = ax + xb$, for all $x \in R$, with $(a - b + \alpha)I = 0$.

1. INTRODUCTION

Throughout this paper, R always denotes a prime ring with center $Z(R)$ and extended centroid C, U its right Utumi quotient ring. By a generalized derivation on R we mean an additive map $G : R \longrightarrow R$ such that, for any $x, y \in R$, $G(xy) = G(x)y + xd(y)$, for some derivation d in R.

Several authors have studied generalized derivations in the context of prime and semiprime rings (see [6], [10], [14] for references). Here we would like to continue on this line of investigation, by studying some related problems concerning the relationship between the behaviour of generalized derivations in a prime ring and the structure of the ring.

A well known theorem of Posner established that a prime ring R must be commutative if it admits a derivation d such that $[d(x), x] \in Z(R)$, for all $x \in R$ [17]. In [8] T.K. Lee generalized this result and proved that if R is a semiprime ring, I a

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nonzero left ideal, d a nonzero derivation on R and k, n positive integers such that $[d(x^n), x^n]_k = 0$ for all $x \in I$, then $[I, R]d(R) = (0)$. In particular R must be commutative in the case it is prime.

In [9] Lee studied an Engel condition with derivation d for a polynomial $f(x_1, \ldots, x_n)$ which is valued on a non-zero one-sided ideal of R.

He proved that if $[d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]_k = 0$, for all $r_1, \ldots, r_n \in L$, a non-zero left ideal of R, and $k \geq 1$ a fixed integer, then there exists an idempotent element e in the socle of RC, such that $CL = RCe$ and one of the following holds: (i) $f(x_1, \ldots, x_n)$ is central valued in $eRCe$ unless C is finite or $0 < char(R) \leq k + 1$; (ii) in case $char(R) = p > 0$, then $f(x_1, \ldots, x_n)^{p^s}$ is central valued in $eRCe$, for some $s \geq 0$, unless $char(R) = 2$ and $eRCe$ satisfies the identity s_4.

In a recent paper ([4]) we studied the case when the Engel condition is satisfied by a generalized derivation on the evaluations of a multilinear polynomial, more precisely we proved the following:

Theorem. *Let R be a prime ring with extended centroid C, G a non-zero generalized derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C and I a non-zero right ideal of R.*

If $[G(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$, for all $r_1, \ldots, r_n \in I$, then either $G(x) = ax$, with $(a - \gamma)I = 0$ and a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:

1. *$f(x_1, \ldots, x_n)$ is central valued in $eRCe$;*
2. *$G(x) = cx + xb$, where $(c - b + \alpha)e = 0$, for $\alpha \in C$, and $f(x_1, \ldots, x_n)^2$ is central valued in $eRCe$;*
3. *$char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.*

Here we will extend the previous cited result and study what happens in case an Engel-type condition is satisfied by a generalized derivation G which acts on a polynomial, removing the assumption on its multilinearity. More precisely we show the following:

Theorem 1. *Let R be a prime ring, $Z(R)$ its center, U its Utumi quotient ring, C its extended centroid, G a non-zero generalized derivation of R, $f(x_1, \ldots, x_n)$ a non-zero polynomial over C and I a non-zero right ideal of R. If $f(x_1, \ldots, x_n)$ is not central valued on R and $[G(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in C$, for all $r_1, \ldots, r_n \in I$, then either there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax$ for all $x \in R$, with $(a - \alpha)I = 0$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:*

1. *$f(x_1, \ldots, x_n)$ is central valued in $eRCe$;*
2. *$char(R) = 2$ and $eRCe$ satisfies the standard identity s_4;*
3. *$char(R) = 2$ and $f(x_1, \ldots, x_n)^2$ is central valued in $eRCe$;*

4. *$f(x_1,\dots,x_n)^2$ is central valued in $eRCe$ and there exist $a, b \in U$, $\alpha \in C$ such that $G(x) = ax + xb$, for all $x \in R$, with $(a - b + \alpha)I = 0$.*

We also point out that in [10] Lee proves that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U. In particular Lee proves the following result:

Theorem 3. ([10]). *Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.*

Remark 1. In order to investigate on general polynomials $f(x_1,\dots,x_n)$, we need to recall the well known process of linearization (see [9] and also [19], part I, §5): let $m_i(x_1,\dots,x_n) = \sum_i \mu_i(x_1,\dots,x_n)$ be the sum of all monomials of f which involve the indeterminate x_i. The x_i appears in any μ_i with a specific degree h_i. Consider now the following tranformation in any monomial μ_i:

$$\varphi_i : x_i^{h_i} \longmapsto \sum_{n_i+m_i=h_i-1} x_i^{n_i} y_i x_i^{m_i}$$

$$\varphi_i : x_j \longmapsto x_j, \quad \text{for all} \quad j \neq i$$

and $\varphi_i(m_i)$ is a sum of monomials, one for each x_i in m_i replaced with y_i. Thus any polynomial $g_i(y_i, x_1,\dots,x_n) = \varphi_i(m_i)$ is linear with respect to the indeterminate y_i.

We remark that

$$[x, f(x_1,\dots,x_n)] = \sum_{i=1}^{n} g_i([x, x_i], x_1,\dots,x_n).$$

Remark 2. Let d be any derivation of R. We will denote by $f^d(x_1,\dots,x_n)$ the polynomial obtained from $f(x_1,\dots,x_n)$ by replacing each coefficient $\alpha \in C$ of $f(x_1,\dots,x_n)$ with $d(\alpha)$.

Thus $d(f(r_1,\dots,r_n)) = f^d(r_1,\dots,r_n)+\sum_i g(d(r_i), r_1,\dots,r_n)$, for all $r_1, r_2,\dots, r_n$ in R.

2. The Results

We begin with some preliminary results. The first one is contained in [9] (Theorem 11, p.21):

Lemma 1. *Let R be a prime ring, $f(x_1,\dots,x_n)$ a non-zero polynomial over C, d a non-zero derivation of R. If $[d(f(r_1,\dots,r_n)), f(r_1,\dots,r_n)] \in C$, for all $r_1,\dots,r_n \in R$, then one of the following holds:*

1. *$f(x_1,\dots,x_n)$ has values in C;*

2. *char*$(R) = 2$ *and* R *satisfies the standard identity* s_4*;*
3. *char*$(R) = 2$ *and* $f(x_1, \ldots, x_n)^2$ *has values in* C*.*

Now we consider a reduction of main Theorem in [9]:

Lemma 2. *Let* R *be a prime ring,* $f(x_1, \ldots, x_n)$ *a non-zero polynomial over* C *and* I *a non-zero right ideal of* R*. If* $[s_1, f(r_1, \ldots, r_n)]_2 \in C$*, for all* $s_1, r_1, \ldots, r_n \in I$*, then there exists an idempotent e in the socle of* RC *such that* $IC = eRC$ *and one of the following holds:*

1. $f(x_1, \ldots, x_n)$ *is central valued in* $eRCe$*;*
2. *char*$(R) = 2$ *and* $eRCe$ *satisfies the standard identity* s_4*;*
3. *char*$(R) = 2$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued in* $eRCe$*.*

Proof. Since I satisfies the non-trivial polynomial identity $[[x, f(x_1, \ldots, x_n)]_2, y]$, then, by Proposition in [11], there exists an idempotent element $e \in soc(RC)$, such that $IC = eRC$. Therefore we have that $eRCe$ satisfies the polynomial identity $[[x, f(x_1, \ldots, x_n)]_2, y]$. Clearly we suppose that $eRCe$ is not commutative (if not $f(x_1, \ldots, x_n)$ is trivially central valued in $eRCe$) and so there exists an element $s_0 \in eRCe - Z(eRCe)$. Denote by $\delta(x) = [s_0, x]$ the inner derivation of $eRCe$ induced by s_0. Hence by our assumption we have that $eRCe$ satisfies the identity

$$\big[\delta(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)\big] \in Z(eRCe).$$

In this situation, by Lemma 1, we get the required conclusions. ■

Lemma 3. *Let* R *be a prime ring,* $a, b \in U$ *and* $f(x_1, \ldots, x_n)$ *a non-zero polynomial over* C *such that* $[af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b, f(r_1, \ldots, r_n)] \in C$*, for all* $r_1, \ldots, r_n \in R$*. Then either* $a, b \in C$ *or one of the following conclusions holds:*

1. $f(x_1, \ldots, x_n)$ *has values in* C*;*
2. *char*$(R) = 2$ *and* R *satisfies the standard identity* s_4*;*
3. *char*$(R) = 2$ *and* $f(x_1, \ldots, x_n)^2$ *has values in* C*;*
4. $f(x_1, \ldots, x_n)^2$ *has values in* C *and* $a - b \in C$*.*

Proof. It is easy to see that we may rewrite the assumption

$$[af(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)b, f(x_1, \ldots, x_n)] \in C$$

as follows

$$[a, f(x_1, \ldots, x_n)]f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)[b, f(x_1, \ldots, x_n)] \in C.$$

If denote by δ_1 the inner derivation of R induced by the element a and by δ_2 the inner one induced by b, we also have

$$\delta_1(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)\delta_2(f(x_1,\ldots,x_n))\in C.$$

In case $\delta_1=-\delta_2=\Delta$, that is $a+b\in C$, we have that

$$[\Delta(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]\in C,\forall r_1,\ldots,r_n\in R$$

and we are finished by Lemma 1. In the other case, we use the main Theorem in [13]: hence either $char(R)=2$ and R satisfies s_4; or $\delta_1=\delta_2=0$, that is $a,b\in C$; or $f(x_1,\ldots,x_n)^2$ is central valued on R and $\delta_1-\delta_2=0$, that is $a-b\in C$. ∎

An easy application of [13] is also the following:

Corollary 1. *Let R be a prime ring, $b\in U$, $f(x_1,\ldots,x_n)$ a non-zero polynomial over C such that $[f(r_1,\ldots,r_n)b,f(r_1,\ldots,r_n)]\in C$, for all $r_1,\ldots,r_n\in R$. Then either $b\in C$ or one of the following conclusions holds:*

1. *$f(x_1,\ldots,x_n)$ has values in C;*
2. *$char(R)=2$ and R satisfies the standard identity s_4.*

Proof. Here denote by δ the inner derivation of R induced by the element b. Thus $f(r_1,\ldots,r_n)\delta(f(r_1,\ldots,r_n))\in C$, for all $r_1,\ldots,r_n\in R$. Hence by Theorem 2 in [13] we obtain the required conclusions. ∎

Lemma 4. *Let R be a prime ring, G a non-zero generalized derivation of R, I a non-zero right ideal of R and $f(x_1,\ldots,x_n)$ a non-central polynomial over C such that $[G(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]\in C$, for all $r_1,\ldots,r_n\in I$. Then R satisfies a non-trivial generalized polynomial identity, unless $G(x)=ax$, for a suitable $a\in U$ and there exists $\lambda\in C$ such that $(a-\lambda)I=0$.*

Proof. Consider the generalized derivation G assuming the form

$$G(x)=ax+d(x)$$

for a derivation d of R. By our hypothesis, R satisfies the identity

$$\Big[[af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)],x_{n+1}\Big].$$

Let B be a basis of U over C and $U*_C C\{x_1,\ldots,x_n\}$ be the free product of the C-algebra U and the free C-algebra $C\{x_1,\ldots,x_n\}$. Then any element of $T=U*_C C\{x_1,\ldots,x_n\}$ can be written in the form $g=\sum_i\alpha_i m_i$. In this decomposition the coefficients α_i are in C and the elements m_i are B-monomials, that is $m_i=q_0y_1\cdots y_hq_h$, with $q_i\in B$ and $y_i\in\{x_1,\ldots,x_n\}$. In [1] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if any α_i is zero. As a consequence, let $a_1, \ldots, a_k \in U$ be linearly independent over C and $a_1 g_1(x_1, \ldots, x_n) + \ldots + a_k g_k(x_1, \ldots, x_n) = 0 \in T$, for some $g_1, \ldots, g_k \in T$. If, for any i, $g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j h_j(x_1, \ldots, x_n)$ and $h_j(x_1, \ldots, x_n) \in T$, then $g_1(x_1, \ldots, x_n)$,..., $g_k(x_1, \ldots, x_n)$ are the zero element of T. The same conclusion holds if $g_1(x_1, \ldots, x_n)a_1 + \ldots + g_k(x_1, \ldots, x_n)a_k = 0 \in T$, and $g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} h_j(x_1, \ldots, x_n)x_j$ for some $h_j(x_1, \ldots, x_n) \in T$.

We assume that R does not satisfy any non-trivial generalized polynomial identity and obtain a number of contradictions.

Suppose first that $d = 0$. Then I satisfies $[af(x_1, \ldots, x_n), f(x_1, \ldots, x_n)] \in C$. In particular let $x_0 \in I$, then R satisfies $[af(x_0x_1, \ldots, x_0x_n), f(x_0x_1, \ldots, x_0x_n)] \in C$, which is a non-trivial generalized polynomial identity, unless ax_0 and x_0 are linearly C-dependent. Since we assume that R does not satisfy any non-trivial generalized polynomial identity, then for all $x_0 \in I$ there exists $\alpha_0 \in C$ such that $ax_0 = \alpha_0 x_0$. In this case standard arguments show that there exists an unique $\alpha \in C$ such that $ax_0 = \alpha x_0$, for all $x_0 \in I$, that is $(a - \alpha)I = 0$.

Now consider the case $d \neq 0$. Here we divide the proof into two cases:

Case 1. Suppose that the derivation $d \neq 0$ is inner, induced by some element $q \in U - C$, that is $d(x) = [q, x]$. Thus we have, for all $r_1, \ldots, r_n \in I$

$$\begin{aligned}&[af(r_1, \ldots, r_n) + d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]\\ &= [(a + q)f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)q, f(r_1, \ldots, r_n)] \in C\end{aligned}$$

and denote $a + q = c$, so that

$$[cf(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)q, f(r_1, \ldots, r_n)] \in C.$$

Let $u \in I$ such that cu and u are linearly C-independent.

By our assumption R satisfies

$$\begin{aligned}P(x_1, \ldots, x_n) &= \big[[cf(ux_1, \ldots, ux_n) - f(ux_1, \ldots, ux_n)q, f(ux_1, \ldots, ux_n)], uy\big]\\ &= \big[cf(ux_1, \ldots, ux_n)^2 + f(ux_1, \ldots, ux_n)^2 q\\ &\quad -f(ux_1, \ldots, ux_n)(c + q)f(ux_1, \ldots, ux_n), uy\big] = 0 \in T\end{aligned}$$

since R is not a GPI-ring. In this representation consider two kinds of B-monomials: those that have leading coefficient cu, and those that have leading coefficient u. Hence we may write

$$P(x_1, \ldots, x_n) = cuP_1(x_1, \ldots, x_n) + uP_2(x_1, \ldots, x_n) = 0 \in T$$

for $P_1(x_1, \ldots, x_n)$ and $P_2(x_1, \ldots, x_n)$ suitable polynomials. Since cu and u are linearly C-independent, we have that $P_1(x_1, \ldots, x_n) = 0 \in T$, and by calculations it means that R satisfies $cf(ux_1, \dots, ux_n)^2uy$, which is a non trivial generalized polynomial identity for R, a contradiction.

Suppose now that for any $u \in I$ there exists $\alpha \in C$ such that $cu = \alpha u$. Then

$$\begin{aligned} P(x_1, \dots, x_n) &= \big[[cf(ux_1, \dots, ux_n) - f(ux_1, \dots, ux_n)q, f(ux_1, \dots, ux_n)], y\big] \\ &= \big[[\alpha f(ux_1, \dots, ux_n) - f(ux_1, \dots, ux_n)q, f(ux_1, \dots, ux_n)], y\big] \\ &= \big[[-f(ux_1, \dots, ux_n)q, f(ux_1, \dots, ux_n)], y\big] = 0 \in T. \end{aligned}$$

Since $q \notin C$, we consider two kinds of B-monomials: those that have ending coefficient q, and those that have ending coefficient 1. More precisely write

$$P(x_1, \dots, x_n) = M_1(x_1, \dots, x_n)q + M_2(x_1, \dots, x_n) = 0 \in T$$

for $M_1(x_1, \dots, x_n)$ and $M_2(x_1, \dots, x_n)$ suitable polynomials. Since q and 1 are linearly C-independent, we have that $M_1(x_1, \dots, x_n) = 0 \in T$, that is

$$-yf(ux_1, \dots, ux_n)^2q = 0 \in T$$

which is a non trivial generalized polynomial identity for R, a contradiction again.

Case 2. Let now $0 \neq d$ be an outer derivation. Since I satisfies

$$[af(x_1, \dots, x_n) + d(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)] \in C$$

we have that, for $c \in I$, U satisfies the identity

$$\begin{aligned} &\big[[af(cx_1, \dots, cx_n) + f^d(cx_1, \dots, cx_n) \\ &\quad + \sum_i g_i(d(c)x_i + cd(x_i), cx_1, \dots, cx_n), f(cx_1, \dots, cx_n)], y\big]. \end{aligned}$$

Since $d \neq 0$ is an outer derivation, by Kharchenko's theorem (Theorem 2 in [7] and Theorem 1 in [12]), U satisfies the identity

$$\begin{aligned} &\big[[af(cx_1, \dots, cx_n) + f^d(cx_1, \dots, cx_n) \\ &\quad + \sum_i g_i(d(c)x_i + cy_i, cx_1, \dots, cx_n), f(cx_1, \dots, cx_n)], y\big]. \end{aligned}$$

In particular U satisfies

$$\left[\left[\sum_i g_i(cy_i, cx_1, \dots, cx_n), f(cx_1, \dots, cx_n)\right], y\right] \tag{1}$$

Since (1) is a polynomial identity for the right ideal cU, by Proposition in [11], there exists an idempotent element $e \in soc(U)$, such that $cU = eU$. Therefore we have that U satisfies the generalized identity

$$\left[\left[\sum_i g_i(ey_i, ex_1, \ldots, ex_n), f(ex_1, \ldots, ex_n)\right], y\right].$$

For $y_i = [er, ex_i]$, with $r \in U$, we have that U satisfies

$$\begin{aligned}&\left[\left[\sum_i g_i(e[er, ex_i], ex_1, \ldots, ex_n), f(ex_1, \ldots, ex_n)\right], y\right]\\ =&\left[\left[[er, f(ex_1, \ldots, ex_n)], f(ex_1, \ldots, ex_n)\right], y\right]\end{aligned}$$

that is

$$\left[[er, f(ex_1, \ldots, ex_n)]_2, y\right]$$

which is a non-trivial generalized polynomial identity for U as well for R, a contradiction. ∎

Remark 3. In all that follows we will always assume that R satisfies some non-trivial generalized polynomial identity. In fact, in the other case, by Lemma 4, we are done with the conclusion $G(x) = ax$, for some $a \in U$ such that $(a - \alpha)I = 0$, for a suitable $\alpha \in C$.

We would like to point out that the first part of the paper (Lemmas 6 and 7) is dedicated to analyse the case when G is an inner generalized derivation of R: more precisely $G(x) = ax + xb$, for all $x \in R$ and fixed elements $a, b \in U$. In this case the right ideal I satisfies the generalized polynomial identity

$$\big[[af(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)b, f(x_1, \ldots, x_n)], x_{n+1}\big]. \tag{2}$$

Without loss of generality, in Lemmas 5 and 6 we will assume that R is simple and equals to its own socle, $IR = I$. In fact R is GPI and so RC is a primitive ring, having non-zero socle H with non-zero right ideal $J = IH$ (Theorem 3 in [16]). Note that H is simple and $J = JH$ is a completely reducible right H-module since H_H is. It follows from Theorem 2 in [1] that (2) is a generalized polynomial identity for J, more generally J satisfies the same basic conditions as I. Now just replace R by H, I by J and we are done.

Since $R = H$ is a regular ring, then for any $a_1, \ldots, a_n \in I$ there exists $h = h^2 \in R$ such that $\sum_{i=1}^n a_i R = hR$. Then $h \in IR = I$ and $a_i = ha_i$ for each $i = 1, \ldots, n$.

Before proving Lemmas 6 and 7, we premit the following easy result:

Lemma 5. *Let R be a non-commutative prime ring and $a \in R$ such that $a[r_1, r_2]a \in Z(R)$, for any $r_1, r_2 \in R$. Then $a = 0$.*

Proof. Suppose that $0 \neq a \notin Z(R)$. Hence

$$P(x_1, x_2, x_3) = \Big[a[x_1, x_2]a, x_3\Big]$$

is a non-trivial generalized polynomial identity for R.

By Theorem 2 in [1], $P(x_1, x_2, x_3)$ is also a generalized identity for RC. By Martindale's result in [16] RC is a primitive ring with non-zero socle. There exists a vectorial space V over a division ring D such that RC is dense of D-linear transformations over V.

Suppose that $dim_D V \geq 2$. Since a is not central, there exists $v \in V$ such that $\{v, va\}$ are linearly D-independent. By the density of RC, there exist $r_1, r_2, r_3 \in RC$ such that

$$vr_2 = v, \quad vr_3 = 0, \quad (va)r_1 = v, \quad (va)r_2 = 0, \quad (va)r_3 = v.$$

This leads to the contradiction

$$0 = v\Big[a[r_1, r_2]a, r_3\Big] = v \neq 0.$$

Thus we may assume $dim_D V = 1$, that is RC is a division algebra which satisfies a non-trivial generalized polynomial identity. By Theorem 2.3.29 in [18] $RC \subseteq M_t(F)$, for a suitable field F, moreover $M_t(F)$ satisfies the same generalized identity of RC. Hence $a[r_1, r_2]a$ is central in $M_t(F)$, for any $r_1, r_2 \in M_t(F)$. If $t \geq 2$, by the above argument, we get a contradiction. On the other hand, if $t = 1$ then RC is commutative as well as R, and this contradicts the hypothesis.

The previous argument says that a must be central in R. If $a \neq 0$, by the main assumption it follows $[r_1, r_2] \in Z(R)$ for all $r_1, r_2 \in R$, and this means that R is a commutative ring, a contradiction again.

Therefore $a = 0$ and we are done. ■

Lemma 6. *Let $b \in R$, I a non-zero right ideal of R and $f(x_1, \ldots, x_n)$ a non-zero polynomial over C.*

If $[f(r_1, \ldots, r_n)b, f(r_1, \ldots, r_n)] \in C$ for all $r_1, \ldots, r_n \in I$, then either $b \in C$ or there exists an idempotent element $e \in R$ such that $I = eR$ and one of the following holds:

1. *$f(x_1, \ldots, x_n)$ is central valued in eRe;*
2. *$f(x_1, \ldots, x_n)^2$ is an identity for eRe and $(b - \beta)e = 0$, for a suitable $\beta \in C$.*
3. *$char(R) = 2$ and eRe satisfies the standard identity s_4.*

Proof. Suppose by contradiction that there exist $w, v_1, v_2, c_1, .., c_{n+7} \in I$ such that

- bw and w are linearly C-independent;
- $[b, v_1]v_2 \neq 0$;
- $[f(c_1, \dots, c_n), c_{n+1}]c_{n+2} \neq 0$;
- if $char(R) = 2$ then $s_4(c_{n+3}, c_{n+4}, c_{n+5}, c_{n+6})c_{n+7} \neq 0$.

By Remark 3 there exists an idempotent element $h \in I$ such that $hR = bR + wR + v_1R + v_2R + \sum_{i=1}^{n+7} c_iR$ and $b = hb$, $w = hw$, $v_1 = hv_1$, $v_2 = hv_2$, $c_i = hc_i$, for any $i = 1, \dots, n+7$. Since $[[f(hx_1h, \dots, hx_nh)b, f(hx_1h, \dots, hx_nh)], hyh]$ is satisfied by $R = H$, right multiplying by $(1-h)$ we have that

$$hyhf(hx_1h, \dots, hx_nh)^2b(1-h) = 0.$$

By Lemma 3 in [3] we have that either $hb(1-h) = 0$ or $f(hx_1h, \dots, hx_nh)^2$ is an identity for R.

First we prove that in this last case we have a contradiction. In fact, since hRh satisfies the polynomial $f(x_1, \dots, x_n)^2$, then hRh is a finite-dimensional central simple algebra over its center Ch. Moreover we remark that if hRh is a division algebra, then $f(x_1, \dots, x_n)$ is a polynomial identity for hRh, since $f(x_1, \dots, x_n)^2$ is. But this contradicts with $f(c_1, \dots, c_n) \neq 0$.

Therefore hRh is a finite-dimensional central simple algebra containing non-trivial idempotents.

Moreover we also have that $f(x_1, \dots, x_n)(hbh)f(x_1, \dots, x_n) \in Ch$ is satisfied by hRh. Let $B = \{c \in hRh : f(x_1, \dots, x_n)cf(x_1, \dots, x_n) \in Ch\}$. It is easy to see that B is an additive subgroup of hRh which is invariant under the action of all the automorphisms of hRh, and of course $hbh \in B$. Since hRh contains non-trivial idempotent elements, we may apply the main result in [2]. More precisely, since $[f(c_1, \dots, c_n), c_{n+1}]c_{n+2} \neq 0$ and $s_4(c_{n+3}, c_{n+4}, c_{n+5}, c_{n+6})c_{n+7} \neq 0$ when $char(R) = 2$, we have that either $[hRh, hRh] \subseteq B$, that is

$$f(x_1, \dots, x_n)[y_1, y_2]f(x_1, \dots, x_n) \in Ch$$

is satisfied by hRh, or $hbh \in Ch$.

Note that in the first case, by Lemma 5 we get the contradiction that $f(x_1, \dots, x_n)$ is an identity for hRh.

In the other case, since we know that $b = hb$, we have $bh \in Ch$, that is $(b-\beta)h = 0$ for a suitable $\beta \in C$. But this contradicts with $0 \neq (b-\beta)w = (b-\beta)hw$.

Then the conclusion is that $hb = hbh$.

Moreover, by the fact that hRh satisfies $[f(x_1, \dots, x_n)(hbh), f(x_1, \dots, x_n)] \in C$ and by applying Corollary 1, one obtains that either $hbh = hb = b$ is central in hRh or $f(x_1, \dots, x_n)$ is central valued on hRh, unless when $char(R) = 2$ and hR satisfies $s_4(x_1, \dots, x_4)x_5$. Again recall that we assumed $[f(c_1, \dots, c_n), c_{n+1}]c_{n+2} \neq 0$, and, in case $char(R) = 2$, $s_4(c_{n+3}, c_{n+4}, c_{n+5}, c_{n+6})c_{n+7} \neq 0$. Then $b \in hC$, which contradicts with $[b, hv_1]hv_2 \neq 0$.

Thus we get a number of contradictions; hence one of the following conclusions occurs:

- $[b, I]I = 0$;
- $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;
- $char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)x_5$ is an identity for I.

To complete the proof of this Lemma, we have to analyse the case when $[b, I]I = 0$. We know that if $[b, I]I = 0$, then there exists $\beta \in C$ such that $(b - \beta)I = 0$ (for instance see [5]). Denote $b' = b - \beta$, then $b'I = 0$ and I satisfies

$$\big[f(x_1, \ldots, x_n)b', f(x_1, \ldots, x_n)\big] = f(x_1, \ldots, x_n)^2 b'.$$

Again from Lemma 3 in [3], either $b' = 0$, that is $b \in C$, or $f(x_1, \ldots, x_n)^2 x_{n+1} = 0$ in I. In particular in this case, since I satisfies a polynomial identity, there exists an idempotent element $e^2 = e \in R$, such that $I = eR$ and $f(x_1, \ldots, x_n)^2$ is an identity for the finite dimensional simple central algebra eRe. ■

Lemma 7. *Let $a, b \in R$, I a non-zero right ideal of R and $f(x_1, \ldots, x_n)$ a non-zero polynomial over C.*

If $[af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b, f(r_1, \ldots, r_n)] \in C$, for any $r_1, \ldots, r_n \in I$, then either there exists $\gamma \in C$ such that $(a - \gamma)I = 0$ and $b \in C$ or there exists an idempotent element $e \in R$ such that $I = eR$ and one of the following holds:

1. *$f(x_1, \ldots, x_n)$ is central valued in eRe;*
2. *$char(R) = 2$ and eRe satisfies the standard identity s_4;*
3. *$char(R) = 2$ and $f(x_1, \ldots, x_n)^2$ is central valued in eRe;*
4. *$f(x_1, \ldots, x_n)^2$ is central valued in eRe and there exists $\alpha \in C$ such that $(a - b + \alpha)I = 0$.*

Proof. Suppose by contradiction that there exist

$$w, v, c_1, \ldots, c_{n+2}, b_1, \ldots, b_{n+7}, t_1, \ldots, t_{n+2} \in I$$

such that

- v and av are linearly C-independent;
- $[f(c_1, \ldots, c_n), c_{n+1}]c_{n+2} \neq 0$;
- either $[f(b_1, \ldots, b_n)^2, b_{n+1}]b_{n+2} \neq 0$ or $(b-a)w$ and w are linearly C-independent;
- if $char(R) = 2$, then $[f(t_1, \ldots, t_n), t_{n+1}]t_{n+2} \neq 0$;

- if $char(R) = 2$, then $s_4(b_{n+3}, b_{n+4}, b_{n+5}, b_{n+6})b_{n+7} \neq 0$.

There exists an idempotent element $h \in I$ such that

$$hR = aR + bR + wR + vR + \sum_{i=1}^{n+2} c_i R + \sum_{j=1}^{n+7} b_j R + \sum_{i=1}^{n+2} t_i R$$

and $a = ha$, $b = hb$, $w = hw$, $v = hv$, $c_i = hc_i$, $b_j = hb_j$, $t_i = ht_i$ for any $i = 1, \dots, n+2$, $j = 1, \dots, n+7$. Since

(3) $\quad [[af(hx_1h, \dots, hx_nh) + f(hx_1h, \dots, hx_nh)b, f(hx_1h, \dots, hx_nh)], hyh]$

is satisfied by $R = H$, left multiplying by $(1-h)$ we have that

(4) $$(1-h)ahf(x_1h, \dots, x_nh)^2 yh = 0.$$

On the other hand, right multiplying the (2) by $(1-h)$ we also have

(5) $$hyhf(x_1h, \dots, x_nh)^2 b(1-h) = 0.$$

Applying Lemma 3 in [3] to (3) and (4), it follows that either $f(x_1, \dots, x_n)^2$ is an identity for hRh or $(1-h)ah = hb(1-h) = 0$.

Suppose first that $f(x_1, \dots, x_n)^2$ is an identity for hRh. Here we repeat the same argument of Lemma 6, in order to obtain again a contradiction.

Since hRh satisfies the polynomial $f(x_1, \dots, x_n)^2$, then hRh is a finite-dimensional central simple algebra over its center Ch. Moreover hRh is not a division algebra, if not $f(x_1, \dots, x_n)$ is a polynomial identity for hRh, which contradicts with the choices of $c_1, \dots, c_n$.

Therefore hRh is a finite-dimensional central simple algebra containing non-trivial idempotents. Moreover, since $f(x_1, \dots, x_n)^2$ is an identity for hRh, starting from (2) we have that hRh satisfies $f(x_1, \dots, x_n)h(b-a)hf(x_1, \dots, x_n) \in C$.

Let $B = \{c \in hRh : f(x_1, \dots, x_n)cf(x_1, \dots, x_n) \in Ch\}$. It is easy to see that B is an additive subgroup of hRh which is invariant under the action of all the automorphisms of hRh, and of course $h(b-a)h \in B$. In light of [2], and since $[f(c_1, \dots, c_n), c_{n+1}]c_{n+2} \neq 0$ and $s_4(c_{n+3}, c_{n+4}, c_{n+5}, c_{n+6})c_{n+7} \neq 0$ when $char(R) = 2$, we have that either $[hRh, hRh] \subseteq B$, that is

$$f(x_1, \dots, x_n)[y_1, y_2]f(x_1, \dots, x_n) \in Ch$$

is satisfied by hRh, or $h(b-a)h \in Ch$.

In the first case, by Lemma 5, we have the contradiction that $f(x_1, \dots, x_n)$ is an identity for hRh.

In the other case, since we know that $a = ha$ and $b = hb$, we have $bh - ah \in Ch$, that is $(b - a + \alpha)h = 0$ for a suitable $\alpha \in C$. But this contradicts with $0 \neq (b - a + \alpha)w = (b - a + \alpha)hw$.

This means that $ah = hah = ha \in hRh$ and $b = hb = hbh \in hRh$. Therefore hRh satisfies

$$\big[[(hah)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)(hbh), f(x_1,\ldots,x_n)], y\big].$$

By Lemma 3 we have that either $hah = ah$ is central in hRh, or $f(x_1,\ldots,x_n)$ is central in hRh, or $f(x_1,\ldots,x_n)^2$ is central in hRh and $(b-a)h \in Ch$; or $char(R) = 2$ and $f(x_1,\ldots,x_n)^2$ is central in hRh, unless when $char(R) = 2$ and hRh satisfies s_4. Because of our assumptions, the only one conclusion must be $ah = hah \in Z(hRh) = Ch$. Therefore we have $ah = \alpha h$, for some $\alpha \in C$ which contradicts with $ahv = av \neq \alpha v = \alpha hv$.

All these contradictions say that one of the following holds:

1. $(a-\gamma)I = 0$ for a suitable $\gamma \in C$;
2. $[f(x_1,\ldots,x_n), x_{n+1}]x_{n+2}$ is an identity for I;
3. $[f(x_1,\ldots,x_n)^2, x_{n+1}]x_{n+2}$ is an identity for I and $(b-a)I \subseteq CI$;
4. $char(R) = 2$ and $[f(x_1,\ldots,x_n)^2, x_{n+1}]x_{n+2}$ is an identity for I;
5. $char(R) = 2$ and $s_4(x_1,x_2,x_3,x_4)x_5$ is an identity for I.

In case $(a-\gamma)I = 0$, the main hypothesis says that

$$[f(r_1,\ldots,r_n)b, f(r_1,\ldots,r_n)] \in C$$

for all $r_1,\ldots,r_n \in I$, and we end up from Lemma 6.

In all the other cases we remark that, since I satisfies some polynomial identity, there exists an idempotent element $e^2 = e \in R$, such that $I = eR$. ■

Finally we are ready to prove the following:

Theorem 1. *Let G be a non-zero generalized derivation of R, $f(x_1,\ldots,x_n)$ a non-zero polynomial over C and I a non-zero right ideal of R. If $f(x_1,\ldots,x_n)$ is not central valued on R and $[G(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] \in C$, for all $r_1,\ldots,r_n \in I$, then either there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax$ for all $x \in R$, with $(a-\alpha)I = 0$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:*

1. *$f(x_1,\ldots,x_n)$ is central valued in $eRCe$;*
2. *$char(R) = 2$ and $eRCe$ satisfies the standard identity s_4;*
3. *$char(R) = 2$ and $f(x_1,\ldots,x_n)^2$ is central valued in $eRCe$;*
4. *$f(x_1,\ldots,x_n)^2$ is central valued in $eRCe$ and there exist $a, b \in U$, $\alpha \in C$ such that $G(x) = ax + xb$, for all $x \in R$, with $(a-b+\alpha)I = 0$.*

Proof. As we have already remarked, every generalized derivation G on a dense right ideal of R can be uniquely extended to U and assumes the form $G(x) = ax+d(x)$, for some $a \in U$ and a derivation d on U.

If $d = 0$ we conclude by Lemma 7 (in the special case when $b = 0$). Thus we suppose that $d \neq 0$.

For $u \in I$, U satisfies the following

$$[af(ux_1, \dots, ux_n) + d(f(ux_1, \dots, ux_n)), f(ux_1, \dots, ux_n)] \in C.$$

In light of Kharchenko's theory ([7], [12]), we divide the proof into two cases:

Case 1. Let d the inner derivation induced by the element $q \in U$, that is $d(x) = [q, x]$, for all $x \in U$. Thus I satisfies

$$[af(x_1, \dots, x_n) + qf(x_1, \dots, x_n) - f(x_1, \dots, x_n)q, f(x_1, \dots, x_n)] =$$

$$[(a + q)f(x_1, \dots, x_n) + f(x_1, \dots, x_n)(-q), f(x_1, \dots, x_n)] \in C.$$

If denote $-q = b$ and $a + q = c$, the generalized derivation δ is defined as $G(x) = cx + xb$, and we get the conclusion thanks to Lemma 7.

Case 2. Let now d an outer derivation of U. Assume that there exist $c_1, \dots, c_{n+2}$, $b_1, \dots, b_{n+7} \in I$ such that

- $[f(c_1, \dots, c_n), c_{n+1}]c_{n+2} \neq 0$;
- if $char(R) = 2$, $[f(b_1, \dots, b_n)^2, b_{n+1}]b_{n+2} \neq 0$;
- if $char(R) = 2$, $s_4(b_{n+3}, b_{n+4}, b_{n+5}, b_{n+6})b_{n+7} \neq 0$.

We want to show that these assumptions drive us to a contradiction. First we recall that there exists an idempotent element $h \in IH = IR$ such that $hR = \sum_{i=1}^{n+2} c_iR + \sum_{j=1}^{n+7} b_jR$ and $c_i = hc_i$, $b_j = hb_j$, for any $i = 1, .., n + 2$, $j = 1, .., n + 7$.

Since I satisfies

$$\Big[af(x_1, \dots, x_n) + d(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)\Big] \in C$$

then U satisfies

$$\Big[af(hx_1, \dots, hx_n) + d(f(hx_1, \dots, hx_n)), f(hx_1, \dots, hx_n)\Big] \in C.$$

Thus U satisfies the following

$$\Big[af(hx_1, \dots, hx_n) + f^d(hx_1, \dots, hx_n) + \sum_i f(hx_1, \dots, d(h)x_i + hd(x_i), \dots, hx_n), f(hx_1, \dots, hx_n)\Big] \in C.$$

Expansion of this yields that U satisfies

$$\Big[af(hx_1,\dots,hx_n)+f^d(hx_1,\dots,hx_n) + \sum_i g_i(d(h)x_i+hd(x_i),hx_1,\dots,hx_n), f(hx_1,\dots,hx_n)\Big] \in C.$$

Since d is an outer derivation, by Kharchenko's theorem (Theorem 2 in [7] and Theorem 1 in [12]), U satisfies

$$\Big[af(hx_1,\dots,hx_n)+f^d(hx_1,\dots,hx_n) + \sum_i g_i(d(h)x_i+hy_i,hx_1,\dots,hx_n), f(hx_1,\dots,hx_n)\Big] \in C.$$

In particular, by analysing any blended component of the previous condition, U satisfies

$$\left[\sum_i g_i(hy_i,hx_1,\dots,hx_n), f(hx_1,\dots,hx_n)\right] \in C.$$

For $y_i=[hr,hx_i]$, with $r\in U$, we have that U satisfies

$$\left[\sum_i g_i(h[hr,hx_i],hx_1,\dots,hx_n), f(hx_1,\dots,hx_n)\right] = \Big[[hr,f(hx_1,\dots,hx_n)],f(hx_1,\dots,hx_n)\Big] \in C$$

that is

$$\Big[hr,f(hx_1,\dots,hx_n)\Big]_2 \in C.$$

In this situation, the conclusions of Lemma 2 contradict with the choices of elements $c_1,\dots,c_{n+2},b_1,\dots,b_{n+7}\in I$. This contradiction gives us the required conclusion. ■

We would like to conclude the paper by considering the special case when the polynomial $f(x_1,\dots,x_n)$ is multilinear. In fact in this case one of the conclusions in Theorem 1 can be removed. More precisely, when there exists an idempotent element $e\in Soc(RC)$ such that $f(x_1,\dots,x_n)^2$ is central valued in $eRCe$ and $char(R)=2$, we will show that $f(x_1,\dots,x_n)$ must be central valued on $eRCe$ unless $eRCe$ satisfies the standard identity s_4. In light of this we will obtain a complete generalization of the result contained in [4]:

Theorem 2. *Let R be a prime ring, $Z(R)$ its center, U its Utumi quotient ring, C its extended centroid, G a non-zero generalized derivation of R, $f(x_1,\dots,x_n)$ a non-zero multilinear polynomial over C, I a non-zero right ideal of R. If $f(x_1,\dots,x_n)$ is not central valued on R and $[G(f(r_1,\dots,r_n)),f(r_1,\dots,r_n)]\in C$, for all $r_1,\dots,r_n\in I$, then either there exist $a\in U$, $\alpha\in C$ such that $G(x)=ax$ for all $x\in R$, with $(a-\alpha)I=0$ or there exists an idempotent element $e\in soc(RC)$ such that $IC=eRC$ and one of the following holds:*

1. *$f(x_1, \dots, x_n)$ is central valued in $eRCe$;*
2. *$char(R) = 2$ and $eRCe$ satisfies the standard identity s_4;*
3. *$f(x_1, \dots, x_n)^2$ is central valued in $eRCe$ and there exist $a, b \in U$, $\alpha \in C$ such that $G(x) = ax + xb$, for all $x \in R$, with $(a - b + \alpha)I = 0$.*

Proof. By Theorem 1, we are always done, unless in the case there exists an idempotent element $e \in Soc(RC)$ such that $f(x_1, \dots, x_n)^2$ is central valued in $eRCe$ and $char(R) = 2$. Recall that $eRCe$ is a simple finite dimensional algebra over its center. For the sake of clearness we denote $A = eRCe$ and $Ce = Z(eRCe)$ the center of A. A is a PI-ring with $Ce \neq 0$.

Let K be the algebraic closure of C if C is an infinite field and set $K = C$ otherwise. Then $A \otimes_C K \cong M_t(K)$, for some $t \geq 1$. Standard arguments show that $M_t(K)$ and A satisfies the same polynomial identities. In particular $M_t(K)$ satisfies $[f(x_1, \dots, x_n)^2, y]$. If $t = 1$, then A is commutative and we are done. Consider then $t \geq 2$. Let $(r_1, \dots, r_n)$ any even sequence in $M_t(K)$ such that $f(r_1, \dots, r_n) = u = \sum_{i=1}^{t} \lambda_i e_{ii}$, with $\lambda_i \in K$, for all i (see [15] for more details). Denote I_t the identity matrix in $M_t(K)$. Since $u^2 \in K \cdot I_t$, then $\lambda_i^2 = \lambda_j^2$, for all $i \neq j$, which implies $\lambda_i = \lambda_j$, because $char(R) = 2$. Thus, for any even sequence $(r_1, \dots, r_n)$ in $M_t(K)$, we have $f(r_1, \dots, r_n) = \lambda I_t$ and, by Lemma 9 in [15], this means that $f(x_1, \dots, x_n)$ is central valued in $M_t(K)$, as well as in $A = eRCe$, unless when $t = 2$. In this last case $eRCe$ satisfies s_4 and we are done again. ■

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