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HALF INVERSE PROBLEMS FOR QUADRATIC PENCILS OF STURM-LIOUVILLE OPERATORS

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Abstract. Generally, the coefficients $p(x)$ and $q(x)$ of quadratic pencils of Sturm-Liouville operators are uniquely determined by two spectra or one spectrum and norming constants. In the present paper we show if $p(x)$ and $q(x)$ are known on half of the domain interval, then one spectrum suffices to determine them uniquely on the other half.

1. INTRODUCTION

One of the earliest results on half inverse problems for Sturm-Liouville operators is due to Hochstadt and Lieberman [7]. They consider the problem

$$-y''(x) + q(x)y(x) = \lambda^2 y(x) \ \text{ on } \ (0,\pi), \ \ q \in L^2(0,\pi), \tag{1.1}$$

$$\begin{aligned} y(0)\cos\alpha - y'(0)\sin\alpha &= 0, \ y(\pi)\cos\beta + y'(\pi)\sin\beta \\ &= 0, \ \alpha \in [0,\pi), \ \beta \in [0,\pi), \end{aligned} \tag{1.2}$$

and prove that, for fixed β, if the complex valued function $q(x)$ is known on $(\pi/2,\pi)$, then a single spectrum suffices to determine $q(x)$ on $(0,\pi/2]$ and α.

In this paper we study quadratic pencils $L(y,\lambda)$ of Sturm-Liouville operators of the form

$$-y''(x) + [q(x) + 2\lambda p(x)]\, y(x) = \lambda^2 y(x) \ \text{ on } \ (0,\pi), \tag{1.3}$$

with boundary conditions (1.2), where the complex valued functions q and p satisfy

$$q \in L^2(0,\pi) \ \text{ and } \ p \in W_2^1(0,\pi).$$

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Definition 1. A complex number $\hat{\lambda}$ is called an eigenvalue of the problem $L(y, \lambda)$ if the equation (1.3) with $\lambda = \hat{\lambda}$ has a nontrivial solution $y_0(x)$ satisfying the boundary condition (1.2); then $y_0(x)$ is called the eigenfunction of the problem $L(y, \lambda)$ corresponding to the eigenvalue $\hat{\lambda}$. The number of linearly independent solutions of the problem $L(y, \lambda)$ for a given eigenvalue $\hat{\lambda}$ is called the multiplicity of $\hat{\lambda}$.

Let

$$\sigma(p, q, \alpha, \beta) = \begin{cases} \{\lambda_n(p, q, \alpha, \beta) : n \in \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}\} & \text{for } \alpha = \beta = 0; \\ \{\lambda_n(p, q, \alpha, \beta) : n \in \mathbb{Z}\} & \text{for } \alpha, \beta \in (0, \pi) \end{cases}$$

denote the set of all eigenvalues of problem (1.3), (1.2).

In [9] Koyunbakan uses the Hochstadt's method and proves that the functions $p(x)$ and $q(x)$ can be determined from two spectra uniquely. Moreover, a similar Hochstadt's theorem which is related to the structure of difference for the potential functions is obtained. In [18] the inverse problem of interior spectral data for a quadratic pencil of Schrödinger operator is considered. In [18] authors give two uniqueness theorems from some eigenvalues and information on eigenfunctions at some an internal point in the interval, where $p(x)$ and $q(x)$ are unknown on the whole domain interval. However, in the present paper we show if $p(x)$ and $q(x)$ are known on half of the domain interval, then one spectrum suffices to determine them uniquely on the other half, and so the results in [18] differ from those in the mentioned article. In [18] authors apply Liouville Theorem to complete the proof of main theorem. In [19] we use the Hochstadt and Lieberman's method and Liouville Theorem to show that (a) if $p(x)$ is prescribed on the interval $[\frac{\pi}{2}, \pi]$ and $q(x)$ is full given on $[0, \pi]$, then a single spectrum suffices to determine $p(x)$ on $[0, \frac{\pi}{2}]$; (b) if $q(x)$ is prescribed on the interval $[\frac{\pi}{2}, \pi]$ and $p(x)$ is full given on $[0, \pi]$, then a single spectrum suffices to determine $q(x)$ on $[0, \frac{\pi}{2}]$.

In this work we prove the following two theorems, the first is on the asymptotic form of the eigenvalues and the second on the half inverse problem. The proof goes by developing an analogue of the theory of transformation operators with Goursat-type problem and a Gelfand-Levitan type integral equation in vectorial form, which differ from those in [19]. In addition, in [3, 9, 18, 19] the boundary conditions considered don't include the case $\alpha = 0$ or $\beta = 0$. However, in this paper we consider four cases: (I) $\alpha = \beta = 0$; (II) $\alpha, \beta > 0$; (III) $\alpha = 0 < \beta$; (IV) $\beta = 0 < \alpha$.

Theorem 1. *Let $\sigma(p, q, \alpha, \beta)$ denote the set of all eigenvalues of problem (1.3), (1.2). Then*

(1) *$\sigma(p, q, \alpha, \beta)$ is a countably infinite set.*

(2) *Each eigenvalue is isolated and geometrically simple.*

(3) *There are at most a finite number of eigenvalues with the same real part.*

(4) *The eigenvalues λ_n can be ordered by the index set as follows: Order the real parts as a nondecreasing sequence*

$$\cdots \leq \mathrm{Re}\,(\lambda_{-2}) \leq \mathrm{Re}\,(\lambda_{-1}) \leq \mathrm{Re}\,(\lambda_1) \leq \mathrm{Re}\,(\lambda_2) \leq \cdots, \ \textit{for } \alpha = \beta = 0$$

or

$$\begin{aligned}\cdots &\leq \mathrm{Re}\,(\lambda_{-2}) \leq \mathrm{Re}\,(\lambda_{-1}) \leq \mathrm{Re}\,(\lambda_0) \leq \mathrm{Re}\,(\lambda_1) \leq \mathrm{Re}\,(\lambda_2) \\ &\leq \cdots, \ \textit{for } \alpha, \beta \in (0, \pi)\end{aligned}$$

counting multiplicity, then for each real part with mulitiplicity greater than one, order the (finite number by (2) and (3)) of eigenvalues with this real part by ordering the imaginary parts in nondecreasing order.

(5) *The eigenvalues, ordered according to (4), have the following asymptotic form:*

$$\lambda_n(p, q, \alpha, \beta)$$

$$= \begin{cases} n + c_0 + \dfrac{c_1}{n} + O\left(\dfrac{1}{n^2}\right), \ n \in \mathbb{Z}_0, & \textit{for } \alpha = 0, \ \beta = 0; \\ n + \dfrac{1}{2} + c_0 + \dfrac{c_1 + \dfrac{\cot\beta}{\pi}}{n} + O\left(\dfrac{1}{n^2}\right), \ n \in \mathbb{Z} & \textit{for } \alpha = 0, \ \beta \neq 0; \\ n + \dfrac{1}{2} + c_0 + \dfrac{c_1 + \dfrac{\cot\alpha}{\pi}}{n} + O\left(\dfrac{1}{n^2}\right), \ n \in \mathbb{Z} & \textit{for } \alpha \neq 0, \ \beta = 0; \\ n + c_0 + \dfrac{c_1 + \dfrac{\cot\alpha + \cot\beta}{\pi}}{n} + O\left(\dfrac{1}{n^2}\right), \ n \in \mathbb{Z} & \textit{for } \alpha \neq 0, \ \beta \neq 0, \end{cases} \tag{1.4}$$

where

$$c_0 = \frac{1}{\pi} \int_0^\pi p(x)dx, \qquad c_1 = \frac{1}{2\pi} \int_0^\pi \left[q(x) + p^2(x)\right] dx.$$

Remark 1. From asymptotic formulae (1.4) we see that the imaginary part of $\lambda_n(p, q, \alpha, \beta)$ is bounded as $|n| \to \infty$. Therefore, for $|n|$ large enough, there exists a $\delta > 0$ such that $\lambda_n(p, q, \alpha, \beta)$ locates outside the sector $\arg\lambda \in [\delta, 2\pi - \delta]$.

Theorem 2. *Fix $\beta \in [0, \pi)$. If*

$$\sigma(p, q, \alpha, \beta) = \sigma(p_1, q_1, \alpha_1, \beta) \tag{1.5}$$

and

$$p(x) = p_1(x) \ \textit{ on } \ [\pi/2, \pi] \ \textit{ and } \ q(x) = q_1(x) \ \textit{ a.e. } \ \textit{ on } \ (\pi/2, \pi), \tag{1.6}$$

then $\alpha = \alpha_1$ and

$$p(x) = p_1(x) \ \textit{ on } \ [0, \pi] \ \textit{ and } \ q(x) = q_1(x) \ \textit{ a.e. } \ \textit{ on } \ [0, \pi). \tag{1.7}$$

Remark 2. Also note that there is an analogous theorem with $(\pi/2, \pi)$ replaced by $(0, \pi/2)$, and with fixed β replaced by fixed α : just interchange α and β in (1.2) and replace $p(x)$ with $\widetilde{p}(x) = p(\pi - x)$ and $q(x)$ with $\widetilde{q}(x) = q(\pi - x)$.

Hald [6] generalized a theorem in [7]. Gesztesy and Simon [4] and Malamud [11] found new uniqueness results with partial information on the spectrum for SL operators with scalar and matrix coefficients, respectively. They showed that more information on the potential can compensate for less information about the spectrum. Martinyuk and Pivovarchik [13] proposed a new method for reconstructing the potential on half the interval. Sakhnovich [15] studied the existence of solutions of half inverse problems. Singular potentials were studied by Hryniv and Mykytyuk [8]. Koyunbakan and Panakhov [10] studied half inverse problems for diffusion operators where $q(x)$ is known on only half the interval but $p(x)$ is known on the whole interval. Trooshin and Yamamoto [16] obtained Hochstadt-Lieberman type theorems for nonsymmetric first order systems. Buterin and Shieh [1] considered inverse nodal problem for differential pencils, where reconstruction formulas were given for p and q using nodal data. These references are certainly not intended to be comprehensive but are given to indicate the wide interest in and variety of half inverse type problems.

The contents of the paper are as follows. In Section 2 we prove several lemmas which may be of independent interest. Section 3 contains the proof of Theorem 2. The proof of Theorem 2 is given only for the case $\alpha = 0 = \beta$ but the same method of proof can be used to establish the other cases. In Appendix proofs of Theorem 1 and Lemma 1 are given.

2. Lemmas

In this section we establish several lemmas. Since the proofs are long and technical and the lemmas may be of independent interest we state them before giving the proofs. The first gives a representation of solutions of quadratic pencils of Sturm-Liouville equations (1.3), whose proof is put in appendix; the others give a result on products of solutions of initial value problems.

Lemma 1. *For each $\lambda \in \mathbb{C}$, $\lambda \neq 0$, the solution of the initial value problem*

$$L(y, \lambda) = \lambda^2 y(\cdot, \lambda), \qquad y(0, \lambda) = 0 = y'(0, \lambda) - 1 \tag{2.1}$$

is given, for $x \in [0, \pi]$, by

$$\lambda\, y(x, \lambda) = \sin[\lambda x - \alpha(x)] + \int_0^x A(x, t)\cos(\lambda t)dt + \int_0^x B(x, t)\sin(\lambda t)dt, \tag{2.2}$$

where the kernels $A(x, t), B(x, t)$ are the solution of the problem

$$
\begin{cases}
\dfrac{\partial^2 A(x,t)}{\partial x^2} - 2p(x)\dfrac{\partial B(x,t)}{\partial t} - q(x)A(x,t) = \dfrac{\partial^2 A(x,t)}{\partial t^2}, \\
\dfrac{\partial^2 B(x,t)}{\partial x^2} + 2p(x)\dfrac{\partial A(x,t)}{\partial t} - q(x)B(x,t) = \dfrac{\partial^2 B(x,t)}{\partial t^2}, \\
A(0,0) = p(0),\ B(x,0) = 0,\ \dfrac{\partial A(x,t)}{\partial t}\,|_{t=0} = 0,
\end{cases} \tag{2.3}
$$

with $\alpha(x) = \int_0^x p(t)dt$. Moreover, there holds

$$
A(x,x)\sin\alpha(x) + B(x,x)\cos\alpha(x) = \frac{1}{2}\int_0^x \left[q(x) + p^2(x)\right] dx. \tag{2.4}
$$

Next, in this section, we give a result for products of solutions of initial value problems which is used in the proof of Theorem 2.

By Lemma 1 the solution $y(x,\lambda)$ of (1.3) determined by the initial condition $y(0,\lambda) = 0 = y'(0,\lambda) - 1$ is given by

$$
\lambda y(x,\lambda) = \sin[\lambda x - \alpha(x)] + \int_0^x A(x,t)\cos(\lambda t)dt + \int_0^x B(x,t)\sin(\lambda t)dt. \tag{2.5}
$$

Similarly, the solution $\widetilde{y}(x,\lambda)$ of equation

$$
-\widetilde{y}''(x) + [q_1(x) + 2\lambda p_1(x)]\widetilde{y}(x) = \lambda^2\widetilde{y}(x) \tag{2.6}
$$

determined by the same initial condition is given by:

$$
\lambda\widetilde{y}(x,\lambda) = \sin[\lambda x - \widetilde{\alpha}(x)] + \int_0^x \widetilde{A}(x,t)\cos(\lambda t)dt + \int_0^x \widetilde{B}(x,t)\sin(\lambda t)dt, \tag{2.7}
$$

where kernels $\widetilde{A}(x,t)$ and $\widetilde{B}(x,t)$ have properties similar to those of $A(x,t)$ and $B(x,t)$.

By (2.5) and (2.7), we obtain that

$$
\begin{aligned}
\lambda^2 y(x,\lambda)\widetilde{y}(x,\lambda) &= \sin[\lambda x - \alpha(x)]\sin[\lambda x - \widetilde{\alpha}(x)] \\
&\quad + \int_0^x A(x,t)\cos(\lambda t)\sin[\lambda x - \widetilde{\alpha}(x)]dt \\
&\quad + \int_0^x \widetilde{A}(x,t)\cos(\lambda t)\sin[\lambda x - \alpha(x)]dt \\
&\quad + \int_0^x B(x,t)\sin(\lambda t)\sin[\lambda x - \widetilde{\alpha}(x)]dt \\
&\quad + \int_0^x \widetilde{B}(x,t)\sin(\lambda t)\sin[\lambda x - \alpha(x)]dt
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
&+\int_0^x A(x,t)\cos(\lambda t)dt\int_0^x \widetilde{A}(x,t)\cos(\lambda t)dt\\
&+\int_0^x B(x,t)\sin(\lambda t)dt\int_0^x \widetilde{B}(x,t)\sin(\lambda t)dt\\
&+\int_0^x A(x,t)\cos(\lambda t)dt\int_0^x \widetilde{B}(x,t)\sin(\lambda t)dt\\
&+\int_0^x \widetilde{A}(x,t)\cos(\lambda t)dt\int_0^x B(x,t)\sin(\lambda t)dt.
\end{aligned}
$$

By extending the range of $A(x,t)$, $\widetilde{A}(x,t)$ evenly with respect to the argument t and $B(x,t)$, $\widetilde{B}(x,t)$ oddly with respect to the argument t and some straightforward calculations, for brevity denoting $\theta(t)=\alpha(t)+\widetilde{\alpha}(t)$, we can rewrite () as

$$
\begin{aligned}
2\lambda^2 y(x,\lambda)\widetilde{y}(x,\lambda) = &\cos[\alpha(x)-\widetilde{\alpha}(x)]-\cos[2\lambda x-\theta(x)]\\
&-\int_0^x H_c(x,t)\cos[2\lambda t-\theta(t)]dt\\
&+\int_0^x H_s(x,t)\sin[2\lambda t-\theta(t)]dt,
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{cases}
H_c(x,t) = -2A(x,x-2t)\sin(\theta(t)-\widetilde{\alpha}(x))-2\widetilde{A}(x,x-2t)\sin(\theta(t)-\alpha(x))\\
\qquad -2B(x,x-2t)\cos(\theta(t)-\widetilde{\alpha}(x))-2\widetilde{B}(x,x-2t)\cos(\theta(t)-\alpha(x))\\
\qquad -A_1(t)\cos\theta(t)-A_2(t)\cos\theta(t)-B_1(t)\sin\theta(t)-B_2(t)\sin\theta(t),\\
H_s(x,t) = 2A(x,x-2t)\cos(\theta(t)-\widetilde{\alpha}(x))+2\widetilde{A}(x,x-2t)\cos(\theta(t)-\alpha(x))\\
\qquad -2B(x,x-2t)\sin(\theta(t)-\widetilde{\alpha}(x))-2\widetilde{B}(x,x-2t)\sin(\theta(t)-\alpha(x))\\
\qquad -A_1(t)\sin\theta(t)-A_2(t)\sin\theta(t)+B_1(t)\cos\theta(t)+B_2(t)\cos\theta(t)
\end{cases}
$$

with

$$
\begin{cases}
A_1(t) = \int_{-x+2t}^x A(x,s)\widetilde{A}(x,s-2t)ds+\int_{-x}^{x-2t} A(x,s)\widetilde{A}(x,s+2t)ds,\\
A_2(t) = -\int_{-x+2t}^x B(x,s)\widetilde{B}(x,s-2t)ds-\int_{-x}^{x-2t} B(x,s)\widetilde{B}(x,s+2t)ds,\\
B_1(t) = \int_{-x+2t}^x A(x,s)\widetilde{B}(x,s-2t)ds+\int_{-x}^{x-2t} A(x,s)\widetilde{B}(x,s+2t)ds,\\
B_2(t) = \int_{-x+2t}^x B(x,s)\widetilde{A}(x,s-2t)ds+\int_{-x}^{x-2t} B(x,s)\widetilde{A}(x,s+2t)ds.
\end{cases}
$$

3. Proof of Theorem 2

Proof of Theorem 2. Here we only give the proof for Dirichlet condition (i.e., $\alpha = \beta = 0$). The other case is treated similarly.

If we multiply (2.5) by $\widetilde{y}(x)$ and (2.6) by $y(x)$ and subtract, after integrating on $[0, \pi]$, we obtain

$$(\widetilde{y}y' - y\widetilde{y}')\,|_0^\pi + \int_0^\pi [(q_1 - q) + 2\lambda(p_1 - p)]y\widetilde{y}dx = 0. \tag{3.1}$$

Together with the initial conditions at 0 and given assumption $(q(x),\ p(x)) = (q_1(x), p_1(x))$ on $[\frac{\pi}{2}, \pi]$, then it yields

$$[\widetilde{y}(\pi, \lambda)y'(\pi, \lambda) - y(\pi, \lambda)\widetilde{y}'(\pi, \lambda)] + \int_0^{\frac{\pi}{2}} [(q_1 - q) + 2\lambda(p_1 - p)]y\widetilde{y}dx = 0. \tag{3.2}$$

Denote

$$Q(x) = q_1 - q, \quad P(x) = p_1 - p, \quad H(\lambda) = \int_0^{\frac{\pi}{2}} [Q(x) + 2\lambda P(x)]y\widetilde{y}dx. \tag{3.3}$$

For $\lambda = \lambda_n \overset{def}{=} \lambda_n(p, q, 0, 0)$, from the boundary conditions in (1.2), a direct calculation implies that the first term in (3.2) vanishes and hence

$$H(\lambda_n) = 0, \quad n \in \mathbb{Z}_0. \tag{3.4}$$

Introduce

$$H_1(\lambda) = \int_0^{\frac{\pi}{2}} Q(x)y\widetilde{y}dx, \qquad H_2(\lambda) = \int_0^{\frac{\pi}{2}} P(x)y\widetilde{y}dx, \tag{3.5}$$

then equation (3.4) can be written as

$$H_1(\lambda_n) + 2\lambda_n H_2(\lambda_n) = 0, \quad n \in \mathbb{Z}_0. \tag{3.6}$$

From (2.9) and (3.3), we find that for all complex λ

$$|H(\lambda)| \leq \frac{1}{|\lambda|^2}(C_1 + C_2|\lambda|)e^{\tau\pi} \tag{3.7}$$

for some positive constants C_1 and C_2. Since $\sigma(p, q, \alpha, \beta) = \sigma(p_1, q_1, \alpha_1, \beta)$ we have $y(\pi, \lambda) = \tilde{y}(\pi, \lambda) = \omega(\lambda)$. Thus,

$$H(\lambda) = \int_0^{\frac{\pi}{2}\%} [Q(x) + 2\lambda P(x)]y\widetilde{y}dx = \omega(\lambda)[\tilde{y}'(\pi, \lambda) - y'(\pi, \lambda)],$$

which implies that the multiplicity of zero of $H(\lambda)$ is not less than the multiplicity of zero of $\omega(\lambda)$. Define

$$\Phi(\lambda) = \frac{H(\lambda)}{\omega(\lambda)}, \tag{3.8}$$

which is an entire function from the above argument.

Using a similar method in [20], for fixed $\delta > 0$, the following estimates are valid in the sector $\arg\lambda \in [\delta, 2\pi - \delta]$:

$$\omega(\lambda) \geq \frac{M}{|\lambda|} e^{\tau\pi}, \quad \arg\lambda \in [\delta, 2\pi - \delta]. \tag{3.9}$$

It follows from (.12), (3.7) and (3.9) that

$$\Phi(\lambda) = O(1), \quad \arg\lambda \in [\delta, 2\pi - \delta]$$

for $|\lambda|$ enough large. From this, using the Phragmen-Lindelöf theorem[20] and Liouville's theorem[2], we obtain for all λ

$$\Phi(\lambda) = C,$$

where C is a constant.

Let us show that $C = 0$. We can rewrite the equation $H(\lambda) = C\omega(\lambda)$ in the form

$$\lambda \int_0^{\frac{\pi}{2}} [Q(x) + 2\lambda P(x)] y\widetilde{y} dx = C \sin[\lambda\pi - \alpha(\pi)] + O\left(\frac{e^{\tau\pi}}{\lambda}\right).$$

By use of the Riemann-Lebesgue Lemma, the limit of the left side of the above equality exists as $\lambda \to \infty, \lambda \in \mathbf{R}$. Thus we obtain that $C = 0$. Thus,

$$H(\lambda) = 0 \text{ for all } \lambda. \tag{3.10}$$

Introduce

$$Q_1(t) = Q(t) + \int_t^{\frac{\pi}{2}} Q(x) H_c(x,t) dx, \quad Q_2(t) = \int_t^{\frac{\pi}{2}} Q(x) H_s(x,t) dx. \tag{3.11}$$

Taking (2.9) into account, we get

$$\begin{aligned} 2\lambda^2 H_1(\lambda) = &\int_0^{\frac{\pi}{2}} Q(x) \cos[\alpha(x) - \widetilde{\alpha}(x)] dx - \int_0^{\frac{\pi}{2}} Q_1(t) \cos[2\lambda t - \theta(t)] dt \\ &+ \int_0^{\frac{\pi}{2}} Q_2(t) \sin[2\lambda t - \theta(t)] dt, \end{aligned}$$

which, by changing the order of integration, can be rewritten as

$$\begin{aligned} 2\lambda^2 H_1(\lambda) = &\int_0^{\frac{\pi}{2}} Q(x) \cos[\alpha(x) - \widetilde{\alpha}(x)] dx + \int_0^{\frac{\pi}{2}} R_1(t) e^{2i\lambda t} dt \\ &+ \int_0^{\frac{\pi}{2}} R_2(t) e^{-2i\lambda t} dt, \end{aligned} \tag{3.12}$$

where

$$R_1(t) = -\frac{Q_1(t) + iQ_2(t)}{2} e^{-i\theta(t)}, \quad R_2(t) = -\frac{Q_1(t) - iQ_2(t)}{2} e^{i\theta(t)}.$$

Similarly, we have

$$\begin{aligned} 2\lambda^2 H_2(\lambda) &= \int_0^{\frac{\pi}{2}} P(x)\cos[\alpha(x) - \widetilde{\alpha}(x)]dx + \int_0^{\frac{\pi}{2}} T_1(t)e^{2i\lambda t}dt \\ &\quad + \int_0^{\frac{\pi}{2}} T_2(t)e^{-2i\lambda t}dt, \\ &= \int_0^{\frac{\pi}{2}} P(x)\cos[\alpha(x) - \widetilde{\alpha}(x)]dx + \frac{P(\pi/2)}{2\lambda}\sin[\lambda\pi - \theta(\pi/2)] \\ &\quad - \frac{P_2(0)}{2\lambda} + \frac{i}{2\lambda}\int_0^{\frac{\pi}{2}} T_1'(t)e^{2i\lambda t}dt - \frac{i}{2\lambda}\int_0^{\frac{\pi}{2}} T_2'(t)e^{-2i\lambda t}dt, \end{aligned} \tag{3.13}$$

where

$$T_1(t) = -\frac{P_1(t) + iP_2(t)}{2} e^{-i\theta(t)}, \qquad T_2(t) = -\frac{P_1(t) - iP_2(t)}{2} e^{i\theta(t)}$$

with

$$P_1(t) = P(t) + \int_t^{\frac{\pi}{2}} P(x)H_c(x,t)dx, \quad P_2(t) = \int_t^{\frac{\pi}{2}} P(x)H_s(x,t)dx. \tag{3.14}$$

By (3.12) and (3.13) it follows that

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} Q(x)\cos[\alpha(x) - \widetilde{\alpha}(x)]dx + \int_0^{\frac{\pi}{2}} R_1(t)e^{2i\lambda t}dt + \int_0^{\frac{\pi}{2}} R_2(t)e^{-2i\lambda t}dt \\ &\quad + 2\lambda\int_0^{\frac{\pi}{2}} P(x)\cos[\alpha(x) - \widetilde{\alpha}(x)]dx + P(\pi/2)\sin[\lambda\pi - \theta(\pi/2)] \\ &\quad - P_2(0) + i\int_0^{\frac{\pi}{2}} T_1'(t)e^{2i\lambda t}dt - i\int_0^{\frac{\pi}{2}} T_2'(t)e^{-2i\lambda t}dt = 0. \end{aligned} \tag{3.15}$$

Letting $\lambda \to \infty$ in (3.15) and using the Riemann-Lebesgue Lemma, then it yields

$$\begin{cases} \int_0^{\frac{\pi}{2}} P(x)\cos[\alpha(x) - \widetilde{\alpha}(x)]dx = 0, \\ P(\pi/2) = 0, \\ \int_0^{\frac{\pi}{2}} Q(x)\cos[\alpha(x) - \widetilde{\alpha}(x)]dx = 0. \end{cases}$$

Using these assertions in (3.15), we have for all complex number λ

$$\int_0^{\frac{\pi}{2}} [R_1(t) + iT_1'(t)]e^{2i\lambda t}dt + \int_0^{\frac{\pi}{2}} [R_2(t) - iT_2'(t)]e^{-2i\lambda t}dt = 0.$$

By the system $\{e^{\pm 2i\lambda t} : \lambda \in \mathbf{R}\}$ is complete in $L^2(-\pi/2, \pi/2)$, consequently,

$$R_1(t) + iT_1'(t) = 0 = R_2(t) - iT_2'(t) \quad \text{on } (0, \pi/2).$$

From the definitions of $R_1(t), R_2(t), T_1(t), T_2(t)$ by (3.12) and (3.13), we can infer

$$[Q_1(t) + P_1(t)\theta'(t) - P_2'(t)] + i[Q_2(t) + P_2(t)\theta'(t) + P_1'(t)] = 0$$

and

$$[Q_1(t) + P_1(t)\theta'(t) - P_2'(t)] - i[Q_2(t) + P_2(t)\theta'(t) + P_1'(t)] = 0,$$

and this yields

$$Q_1(t) + P_1(t)\theta'(t) - P_2'(t) = 0 = Q_2(t) + P_2(t)\theta'(t) + P_1'(t). \tag{3.16}$$

Substituting (3.11) and (3.14) into equation (3.16), together with $P(\pi/2) = 0$, it follows that

$$\left\{\begin{aligned} Q(t) &= -\int_t^{\pi/2} [H_c(x,t) + \frac{\partial}{\partial t}H_s(x,t)]Q(x)dx - \int_t^{\pi/2} H_c(x,t)\theta'(t)P(x)dx \\ &\qquad -(\theta'(t) - H_s(t,t))P(t), \\ P'(t) &= -\int_t^{\pi/2} H_s(x,t)Q(x)dx - \int_t^{\pi/2} [H_s(x,t)\theta'(t) + \frac{\partial}{\partial t}H_s(x,t)]P(x)dx \\ &\qquad +H_c(t,t)P(t), \\ P(t) &= -\int_t^{\pi/2} P'(x)dx. \end{aligned}\right. \tag{3.17}$$

Introduce

$$F(t) = (Q(t), P(t), P'(t))^T$$

and

$$K(x,t) = \begin{pmatrix} H_c(x,t) + \frac{\partial}{\partial t}H_s(x,t) & H_c(x,t)\theta'(t) & -\theta'(t) + H_s(t,t) \\ 0 & 0 & 1 \\ H_s(x,t) & H_s(x,t)\theta'(t) + \frac{\partial}{\partial t}H_s(x,t) & H_c(t,t) \end{pmatrix}.$$

Equation (3.17) can readily be reduced to a vector form

$$F(t) + \int_t^{\pi/2} K(x,t)F(x)dx = 0 \quad \text{for} \quad 0 < t < \frac{\pi}{2}.$$

But this equation is a homogeneous Volterra integral equation and has only a zero solution. Thus we have obtained

$$F(t) = \mathbf{0} \quad \text{for} \quad 0 < t < \frac{\pi}{2},$$

which yields that

$$Q(t) = P(t) = 0 \quad \text{for} \quad 0 < t < \frac{\pi}{2}.$$

Therefore, we have proven

$$p(x) = p_1(x) \ \text{ on } \ [0, \pi] \ \text{ and } \ q(x) = q_1(x) \ \text{ a.e. } \ \text{ on } \ [0, \pi).$$

The proof of theorem is complete.

Remark 3. Although we give a detailed proof only for the case when $\alpha = 0 = \beta$ our proof extends readily to the other values of α, β. For instance when $\alpha \in (0, \pi)$, $\alpha_1 \in [0, \pi)$ and we need to prove that $\alpha = \alpha_1$. We proceed as follows: From (3.1) it follows that

$$\widetilde{y}'(0, \lambda) y(0, \lambda) - \widetilde{y}(0, \lambda) y'(0, \lambda) = \sin(\alpha - \alpha_1),$$

where the solution $y(x, \lambda)$ of (1.3) is determined by the initial condition $y(0, \lambda) = \sin\alpha$ and $y'(0, \lambda) = \cos\alpha$, and the solution $\widetilde{y}(x, \lambda)$ of (2.6) is determined by the initial condition $\widetilde{y}(0, \lambda) = \sin\alpha_1$ and $\widetilde{y}'(0, \lambda) = \cos\alpha_1$. Thus we have

$$H(\lambda) = \sin(\alpha - \alpha_1) + \int_0^{\frac{\pi}{2}} [Q(x) + 2\lambda P(x)] y\widetilde{y} dx$$

and

$$H_1(\lambda) = \sin(\alpha - \alpha_1) + \int_0^{\frac{\pi}{2}} Q(x) y\widetilde{y} dx.$$

Letting $\lambda_n \to \infty$ and using the Riemann-Lebesgue Lemma, this yields

$$\cot\alpha_1 - \cot\alpha + \int_0^{\frac{\pi}{2}} Q(x) \cos[\alpha(x) - \widetilde{\alpha}(x)] dx = 0.$$

After $Q(x) = 0$ is obtained, we get

$$\alpha_1 = \alpha.$$

Appendix A: Proof of Lemma 1

The representation (2.2) can be established using the method introduced in [5] (see also [12], [14], [21]) with kernels $A(x, t)$, $B(x, t)$ having continuous partial derivatives up to order two with respect to x and t and with $\alpha(x) = \int_0^x p(t) dt$. From (2.3) it follows that the kernels $A(x, t)$, $B(x, t)$ are uniquely determined.

From (2.2) we get

$$\begin{aligned} &\lambda\, y'(x, \lambda) \\ &= [\lambda - p(x)] \cos[\lambda x - \alpha(x)] + A(x, x) \cos(\lambda x) + B(x, x) \sin(\lambda x) \\ &\quad + \int_0^x \frac{\partial}{\partial x} A(x, t) \cos(\lambda t) dt + \int_0^x \frac{\partial}{\partial x} B(x, t) \sin(\lambda t) dt \end{aligned} \tag{.1}$$

and

$$
\begin{aligned}
(.2)\qquad & \lambda y''(x,\lambda) \\
=& -\lambda^2 \sin[\lambda x-\alpha(x)]+2\lambda p(x)\sin[\lambda x-\alpha(x)]-\lambda A(x,x)\sin(\lambda x) \\
& +\lambda B(x,x)\cos(\lambda x)-p^2(x)\sin[\lambda x-\alpha(x)]-p'(x)\cos[\lambda x-\alpha(x)] \\
& +\frac{d}{dx}A(x,x)\cos(\lambda x)+\frac{\partial}{\partial x}A(x,t)|_{t=x}\cos(\lambda x)+\frac{d}{dx}B(x,x)\sin(\lambda x) \\
& +\frac{\partial}{\partial x}B(x,t)|_{t=x}\sin(\lambda x)+\int_0^x \frac{\partial^2}{\partial x^2}A(x,t)\cos(\lambda t)dt \\
& +\int_0^x \frac{\partial^2}{\partial x^2}B(x,t)\sin(\lambda t)dt.
\end{aligned}
$$

On the other hand, using integration by parts twice, we obtain

$$
\begin{aligned}
(.3)\qquad & \lambda y(x,\lambda) \\
&= \sin[\lambda x-\alpha(x)]+\frac{1}{\lambda}[A(x,x)\sin(\lambda x)-B(x,x)\cos(\lambda x)+B(x,0)] \\
&\quad -\frac{1}{\lambda}\int_0^x \frac{\partial}{\partial t}A(x,t)\sin(\lambda t)dt+\frac{1}{\lambda}\int_0^x \frac{\partial}{\partial t}B(x,t)\cos(\lambda t)dt \\
&= \sin[\lambda x-\alpha(x)]+\frac{1}{\lambda}[A(x,x)\sin(\lambda x)-B(x,x)\cos(\lambda x)+B(x,0)] \\
&\quad +\frac{1}{\lambda^2}[\frac{\partial}{\partial t}A(x,t)|_{t=x}\cos(\lambda x)+\frac{\partial}{\partial t}B(x,t)|_{t=x}\sin(\lambda x)-\frac{\partial}{\partial t}A(x,t)|_{t=0}] \\
&\quad -\frac{1}{\lambda^2}\int_0^x \frac{\partial^2}{\partial t^2}A(x,t)\cos(\lambda t)dt-\frac{1}{\lambda^2}\int_0^x \frac{\partial^2}{\partial t^2}B(x,t)\sin(\lambda t)dt.
\end{aligned}
$$

From (.2) and (.3) we obtain

$$
\begin{aligned}
(.4)\qquad & [\lambda^2-2\lambda p(x)-q(x)]y(x,\lambda) \\
&= [\lambda-2p(x)]\sin[\lambda x-\alpha(x)]+A(x,x)\sin(\lambda x)-B(x,x)\cos(\lambda x)+B(x,0) \\
&\quad -\frac{1}{\lambda}[\frac{\partial}{\partial t}A(x,t)|_{t=0}+2p(x)B(x,0)+\frac{\partial}{\partial t}A(x,t)|_{t=x}\cos(\lambda x)+\frac{\partial}{\partial t}B(x,t)|_{t=x}\sin(\lambda x)] \\
&\quad -\frac{1}{\lambda}\int_0^x \frac{\partial^2}{\partial t^2}A(x,t)\cos(\lambda t)dt-\frac{1}{\lambda}\int_0^x \frac{\partial^2}{\partial t^2}B(x,t)\sin(\lambda t)dt \\
&\quad -\frac{1}{\lambda}[2p(x)A(x,x)\sin(\lambda x)-2p(x)B(x,x)\cos(\lambda x)]+\frac{2p(x)}{\lambda}\int_0^x \frac{\partial}{\partial t}A(x,t)\sin(\lambda t)dt \\
&\quad -\frac{2p(x)}{\lambda}\int_0^x \frac{\partial}{\partial t}B(x,t)\cos(\lambda t)dt-\frac{q(x)}{\lambda}\sin[\lambda x-\alpha(x)] \\
&\quad -\frac{q(x)}{\lambda}\int_0^x A(x,t)\cos(\lambda t)dt-\frac{q(x)}{\lambda}\int_0^x B(x,t)\sin(\lambda t)dt.
\end{aligned}
$$

Let

$$
\begin{aligned}
&I(x,\lambda)\\
&= [-(p^2(x)+q(x))\cos\alpha(x) - p'(x)\sin\alpha(x) + 2\frac{d}{dx}B(x,x) - 2p(x)A(x,x)]\\
&\times \sin(\lambda x) + [(p^2(x)+q(x))\sin\alpha(x) - p'(x)\cos\alpha(x) + 2\frac{d}{dx}A(x,x)\\
&\quad +2p(x)B(x,x)]\cos(\lambda x) - \frac{\partial}{\partial t}A(x,t)|_{t=0}.
\end{aligned}
\tag{.5}
$$

From (.3), (.4) and (.5) we obtain

$$
\begin{aligned}
&y''(x,\lambda) + [\lambda^2 - 2\lambda p(x) - q(x)]\, y(x,\lambda)\\
&= \frac{1}{\lambda}I(x,\lambda) + B(x,0)\\
&\quad +\frac{1}{\lambda}\int_0^x [\frac{\partial^2}{\partial x^2}A(x,t) - \frac{\partial^2}{\partial t^2}A(x,t) - 2p(x)\frac{\partial}{\partial t}B(x,t) - q(x)A(x,t)]\cos(\lambda t)dt\\
&\quad +\frac{1}{\lambda}\int_0^x [\frac{\partial^2}{\partial x^2}B(x,t) - \frac{\partial^2}{\partial t^2}B(x,t) + 2p(x)\frac{\partial}{\partial t}A(x,t) - q(x)B(x,t)]\sin(\lambda t)dt\\
&= 0.
\end{aligned}
\tag{.6}
$$

By the Riemann-Lebesgue Lemma (.6) holds for all real $\lambda \neq 0$ if and only if $B(x,0)=0$, $I(x,\lambda)=0$ and the following two equations are satisfied:

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}A(x,t) - 2p(x)\frac{\partial}{\partial t}B(x,t) - q(x)A(x,t) &= \frac{\partial^2}{\partial t^2}A(x,t),\\
\frac{\partial^2}{\partial x^2}B(x,t) + 2p(x)\frac{\partial}{\partial t}A(x,t) - q(x)B(x,t) &= \frac{\partial^2}{\partial t^2}B(x,t).
\end{aligned}
$$

From $I(x,\lambda)=0$ we obtain

$$
\frac{\partial}{\partial t}A(x,t)|_{t=0} = 0 \tag{.7}
$$

and

$$
[p^2(x)+q(x)]\cos\alpha(x) + p'(x)\sin\alpha(x) - 2\frac{d}{dx}B(x,x) + 2p(x)A(x,x) = 0, \tag{.8}
$$

$$
[p^2(x)+q(x)]\sin\alpha(x) - p'(x)\cos\alpha(x) + 2\frac{d}{dx}A(x,x) + 2p(x)B(x,x) = 0. \tag{.9}
$$

From (.8) and (.9) it follows that

$$
p^2(x)+q(x) = 2\frac{d}{dx}[A(x,x)\sin\alpha(x) + B(x,x)\cos\alpha(x)]. \tag{.10}
$$

Integrating (.10) and taking into account that $\alpha(0) = 0 = B(0,0)$ yields

$$
A(x,x)\sin\alpha(x) + B(x,x)\cos\alpha(x) = \frac{1}{2}\int_0^x \left[q(x)+p^2(x)\right]dx,\ x\in[0,\pi]. \tag{.11}
$$

This completes the proof of Lemma 1.

Appendix B: Proof of Theorem 1

Here we only give the proof for Dirichlet condition (i.e., $\alpha = \beta = 0$). The other case is treated similarly. It is well known that the eigenvalues are the zeros of the characteristic function. Since this function is an entire function it follows that they are isolated and at most countable. From the given asymptotic form it follows that there are an infinite number of eigenvalues and hence they can be indexed by $\mathbb{Z}_0$ as indicated [2]. Thus we only need to establish the indicated asymptotic form of the eigenvalues.

We start with establishing the formula (1.4) under an additional assumption $c_0 = 0$ and then remove it.

Substituting the initial solution $y(x, \lambda)$ into the boundary condition at the point π in (1.2), we obtain characteristic equation for eigenvalues of problem (1.3), (1.2):

$$\omega(\lambda) \stackrel{def}{=} \frac{\sin(\lambda\pi)}{\lambda} + \frac{1}{\lambda}\int_0^\pi A(\pi, t)\cos(\lambda t)dt + \frac{1}{\lambda}\int_0^\pi B(\pi, t)\sin(\lambda t)dt = 0.$$

By integration by parts the characteristic function $\omega(\lambda)$ can be expressed as

$$\omega(\lambda) = \frac{\sin(\lambda\pi)}{\lambda} + \frac{A(\pi,\pi)\sin(\lambda\pi) - B(\pi,\pi)\cos(\lambda\pi)}{\lambda^2} + O\left(\frac{e^{\tau\pi}}{\lambda^3}\right), \tag{.12}$$

where $\tau = |\text{Im } \lambda|$.

Let us introduce the auxiliary function

$$\omega_0(\lambda) = \frac{\sin(\lambda\pi)}{\lambda}, \tag{.13}$$

then we enumerate its zeros $\left\{\lambda_n^{(0)}\right\}_{n\in\mathbb{Z}_0}$ in the following way:

$$\lambda_n^{(0)} = n, \quad n \in \mathbb{Z}_0. \tag{.14}$$

For convenience we denote $\lambda_n \stackrel{def}{=} \lambda_n(p, q, 0, 0)$. Let us denote by $\{\lambda_n\}_{n\in\mathbb{Z}_0}$ the set of zeros of $\omega(\lambda)$. We enumerate the zeros in the following way: 1) $\text{Re}\lambda_{n+1} \geq \text{Re}\lambda_n$, 2) the multiplicities are taken into account.

We claim that zeros of $\omega(\lambda)$ can be enumerated as follows

$$\lambda_n = \lambda_n^{(0)} + o(1), \quad \text{for large } |n|. \tag{.15}$$

The main term of the asymptotics is determined by the term (.13). Suppose that there exists a subsequence $\{\lambda_{k_m}\}_{m=1}^\infty$ of the sequence $\{\lambda_n, n \in \mathbb{Z}_0\}$ such that $\text{Im}\lambda_{k_m} \to +\infty$ as $m \to +\infty$. Using the equality

$$\sin(\lambda_{k_m}\pi) = -\frac{ie^{-i\lambda_{k_m}\pi}}{2} + O\left(\frac{e^{|\text{Im}\lambda_{k_m}|\pi}}{|\lambda_{k_m}|}\right),$$

then (.12) implies

$$\omega(\lambda_{k_m}) + \frac{ie^{-i\lambda_{k_m}\pi}}{2\lambda_{k_m}} = O\left(\frac{e^{|\mathrm{Im}\lambda_{k_m}|\pi}}{|\lambda_{k_m}|^2}\right),$$

which contradicts the identity $\omega(\lambda_{k_m}) \equiv 0$. Hence there exists a number $M_1 > 0$ such that $\mathrm{Im}\lambda_{k_m} \le M_1$. In the same way it can be proved that $\{\mathrm{Im}\lambda_{k_m}\}$ is bounded below. Hence, there exists a constant $M > 0$ such that $|\mathrm{Im}\lambda_{k_m}| \le M$.

Comparing (.12) with (.13) we conclude that there exists a constant $C > 0$ such that

$$|\omega(\lambda) - \omega_0(\lambda)| < \frac{C}{|\lambda|^2}$$

for all $\lambda \in \Pi$, where $\Pi = \{\lambda : |\mathrm{Im}\lambda| < M + \epsilon\}$ and ϵ is an arbitrary positive number. Since the function $\lambda\omega_0(\lambda) = \sin(\lambda\pi)$ is periodic, for every $R \in (0, \epsilon)$ it is possible to find $d > 0$ such that

$$|\lambda\omega_0(\lambda)| > d$$

for all $\lambda \in \Pi \setminus \bigcup_n C_n$, where C_n are disks of radii R with the centers at the points $\lambda_n^{(0)}$. Taking R sufficiently small we obtain $C_n \bigcap C_{n+1} = \emptyset$ for all n. Consequently, for all $\lambda \in \left\{\lambda : \lambda \in \Pi \setminus \bigcup_n C_n, |\lambda| > \frac{C}{d}\right\}$, the following inequalities are valid

$$|\omega_0(\lambda)| > \frac{d}{|\lambda|} > \frac{C}{|\lambda|^2} > |\omega(\lambda) - \omega_0(\lambda)|.$$

Since $R > 0$ can be chosen arbitrary small, applying Rouché's theorem we obtain the assertion of (.15) and for $|n|$ large enough the eigenvalue λ_n is simple algebraically.

It is easy to prove (see [17, 20]): Let $\mathrm{Re}\lambda = n + \frac{1}{4}$, $n \in \mathbb{Z}$; or $|\mathrm{Im}\lambda| \ge 1$, then

$$e^{|\mathrm{Im}\lambda|\pi}|\sin\lambda\pi|^{-1} \le 4.$$

Take a rectangular contour γ_n with the four vertices at $(n \pm \frac{1}{4}) \pm i$. As $\lambda \in \gamma_n$, $|n| \to \infty$:

$$\text{(.16)} \qquad \frac{\omega(\lambda)}{\omega_0(\lambda)} = 1 + \frac{A(\pi,\pi) - B(\pi,\pi)\cot(\lambda\pi)}{\lambda} + O\left(\frac{1}{\lambda^2}\right),$$

which implies that

$$\text{(.17)} \qquad \log\frac{\omega(\lambda)}{\omega_0(\lambda)} = \frac{A(\pi,\pi) - B(\pi,\pi)\cot(\lambda\pi)}{\lambda} + O\left(\frac{1}{\lambda^2}\right).$$

Here we take principal branch of the complex logarithm function.

Now we are interested in asymptotics of eigenvalues of problem (1.3), (1.2), i.e., in asymptotics of the zeros of function $\omega(\lambda)$.

Since $\omega_0(\lambda)$ has exactly a simple zero inside γ_n, $\omega(\lambda)$ has a unique zero inside γ_n for sufficiently large $|n|$ by Rouché's theorem and the eigenvalues are algebraically simple for large $|n|$. Therefore, the zero of $\omega(\lambda)$ inside γ_n is exactly λ_n for sufficiently large $|n|$. Integrating by parts and resorting to $\log \frac{\omega(\lambda)}{\omega_0(\lambda)}$ along γ_n and

$$\oint_{\gamma_n} \frac{\cot(\lambda\pi)}{\lambda} d\lambda = \frac{2i}{n}, \qquad \oint_{\gamma_n} O\left(\frac{1}{\lambda^2}\right) d\lambda = O\left(\frac{1}{n^2}\right),$$

we have

$$\begin{aligned}
\lambda_n - n &= \frac{1}{2\pi i}\oint_{\gamma_n} \lambda \left[\frac{\omega'(\lambda)}{\omega(\lambda)} - \frac{\omega_0'(\lambda)}{\omega_0(\lambda)}\right] d\lambda \\
&= -\frac{1}{2\pi i}\oint_{\gamma_n} \log \frac{\omega(\lambda)}{\omega_0(\lambda)} d\lambda \\
&= -\frac{1}{2\pi i}\oint_{\gamma_n} \left[\frac{A(\pi,\pi) - B(\pi,\pi)\cot(\lambda\pi)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right] d\lambda \\
&= \frac{B(\pi,\pi)}{n\pi} + O\left(\frac{1}{n^2}\right),
\end{aligned}$$

i.e.,

$$\lambda_n = n + \frac{B(\pi,\pi)}{n\pi} + O\left(\frac{1}{n^2}\right). \tag{.18}$$

Since $c_0 = 0$ implies $\alpha(\pi) = 0$, from

$$A(x,x)\sin\alpha(x) + B(x,x)\cos\alpha(x) = \frac{1}{2}\int_0^x \left[q(x) + p^2(x)\right] dx$$

we obtain that

$$B(\pi,\pi) = \frac{1}{2}\int_0^\pi \left[q(x) + p^2(x)\right] dx.$$

Substituting the expression of $B(\pi,\pi)$ into (.18), one has

$$\lambda_n = n + \frac{\int_0^\pi \left[q(x) + p^2(x)\right] dx}{2n\pi} + O\left(\frac{1}{n^2}\right), \ n \in \mathbb{Z}_0.$$

Now we consider the case $c_0 \neq 0$. By a direct calculation we note that equation

$$-y''(x) + [q(x) + 2\lambda p(x)]y(x) = \lambda^2 y(x)$$

is equivalent to

$$-y''(x) + [q(x) + 2pc_0 - c_0^2 + 2(\lambda - c_0)(p(x) - c_0)]y(x) = (\lambda - c_0)^2 y(x). \tag{.19}$$

Let

$$\widehat{\lambda}_n = \lambda_n - c_0, \quad \widehat{q}(x) = q(x) + 2pc_0 - c_0^2$$

and

$$\widehat{p}(x) = p(x) - c_0 = p(x) - \frac{1}{\pi}\int_0^{\pi} p(x)dx,$$

then for the problem with the form (.19) we have $\int_0^{\pi} \widehat{p}(x)dx = 0$ and

$$\widehat{c}_1 = \frac{1}{2\pi}\int_0^{\pi} \left[\widehat{q}(x) + \widehat{p}^2(x)\right] dx = c_1.$$

Thus, we obtain that

$$\lambda_n = n + c_0 + \frac{\int_0^{\pi} \left[q(x) + p^2(x)\right] dx}{2n\pi} + O\left(\frac{1}{n^2}\right),\ n \in \mathbb{Z}_0,$$

the proof of the formula (1.4) is finished.

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