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SHEPHARD TYPE PROBLEMS FOR GENERAL L_p-PROJECTION BODIES

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Abstract. Lutwak, Yang and Zhang introduced L_p-projection bodies and Ludwig defined general L_p-projection bodies. In this paper, a solution to the Shephard problem for general L_p-projection bodies is established.

1. Introduction

For the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$, we write $\mathcal{K}^n$. The set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$ we write $\mathcal{K}_o^n$. Denote by $\mathcal{S}_o^n$ the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, $V(K)$ the n-dimensional volume of body K and $\omega_n = V(B)$ the volume of the standard unit ball B in $\mathbb{R}^n$.

If $K \in \mathcal{K}^n$, then its support function h_K is defined by [3]:

$$h_K(x) = h(K, x) = max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The classical projection body was introduced by Minkowski [3, 18] at the turn of the previous century. For each $K \in \mathcal{K}^n$, the classical projection body, ΠK, of K is the origin-symmetric convex body whose support function is given by

$$h_{\Pi K}(u) = \frac{1}{2} \int_{S^{n-1}} | u \cdot v | \, dS(K, v), \tag{1.1}$$

for all $u \in S^{n-1}$. Here $S(K, \cdot)$ denotes the surface area measure of K. The classical projection body is a very important object for study in the Brunn-Mnkowski theory. In particular, Shephard in [19] asked the following question:

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Question 1. *Suppose* $K, L \in \mathcal{K}^n$. *If*

$$\Pi K \subseteq \Pi L,$$

is it true that

$$V(K) \leq V(L)?$$

Question 1 is called the Shephard problem. Since $h_{\Pi K}(u)$ is just the $(n-1)$-dimensional volume of the image of the projection of K on the subspace orthogonal to u, it asks whether convex bodies with smaller projections in all directions must have smaller volume. For centrally symmetric convex bodies K and L, Question 1 was solved independently by Petty [15] and Schneider [17], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. They also proved that the Shephard problem has an affirmative answer if L is the projection body of some convex bodies.

The notion of L_p-projection body was introduced by Lutwak, Yang and Zhang [12]. For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-projection body, $\Pi_p K$, of K is the origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_p K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} \mid u \cdot v \mid^p dS_p(K, v), \tag{1.2}$$

for all $u \in S^{n-1}$, and

$$c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}. \tag{1.3}$$

Here $S_p(K, \cdot)$ denotes the L_p-surface area measure of $K \in \mathcal{K}_o^n$. Lutwak [10] showed that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S(K, \cdot)$ of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h_K^{1-p}. \tag{1.4}$$

The unusual normalization of definition (1.2) is chosen so that for the unit ball, B, we have $\Pi_p B = B$. In particular, for $p = 1$, the convex body $\Pi_1 K$ is a dilate of the classical projection body ΠK of K and $\Pi_1 B = B$.

Whereas classical projection bodies are notion of the Brunn-Minkowski theory, L_p-projection bodies belong to the L_p-Brunn-Minkowski theory and have attracted a lot of attention (see [8, 9, 12, 13, 14, 16, 20, 22, 23, 24]). In particular, Ryabogin and Zvavitch in [16] considered the following Shephard problem for the L_p-projection bodies:

Question 2. *Suppose* $K, L \in \mathcal{K}_o^n$ *and* $p \geq 1$. *If*

$$\Pi_p K \subseteq \Pi_p L,$$

is it true that

$$V(K) \le V(L),\ \ for\ 1 \le p < n,$$

and

$$V(K) \ge V(L),\ \ for\ p > n?$$

For $p = 1$, Question 2 is equivalent to Question 1. For $p > 1$ and $n \ge 2$, it was proved in [16] that the answer is negative. If $K, L \in \mathcal{K}_o^n$ and L is the L_p-projection body of some convex body, Ryabogin and Zvavitch [16] proved that Question 2 has an affirmative answer [16]. Recently, Ma and Wang [14] studied a L_p-affine surface area form of the Shephard problem for the L_p-projection bodies.

Recall that Ludwig [8] (see also [6]) introduced asymmetric L_p-projection bodies. For $K \in \mathcal{K}_o^n$ and $p \ge 1$, the asymmetric L_p-projection body, $\Pi_p^+ K$, of K is defined by

$$h^p_{\Pi_p^+ K}(u) = \alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p dS_p(K, v), \tag{1.5}$$

where $(u \cdot v)_+ = \max\{u \cdot v, 0\}$ and

$$\alpha_{n,p} = \frac{1}{n\omega_n c_{n-2,p}}. \tag{1.6}$$

From (1.6) and (1.5), we see $\Pi_p^+ B = B$. In [6] they also defined

$$\Pi_p^- K = \Pi_p^+(-K). \tag{1.7}$$

Further, authors in [6, 8] introduced a function $\varphi_\tau : \mathbb{R} \longrightarrow [0, +\infty)$ by

$$\varphi_\tau(t) = |t| + \tau t \tag{1.8}$$

for $\tau \in [-1, 1]$, and for $K \in \mathcal{K}_o^n$, $p \ge 1$, let $\Pi_p^\tau K \in \mathcal{K}_o^n$ be the convex body with support function

$$h^p_{\Pi_p^\tau K}(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_p(K, v), \tag{1.9}$$

where

$$\alpha_{n,p}(\tau) = \frac{\alpha_{n,p}}{(1+\tau)^p + (1-\tau)^p}. \tag{1.10}$$

The normalization is chosen such that $\Pi_p^\tau B = B$ for every $\tau \in [-1, 1]$. Here $\Pi_p^\tau K$ may be called general L_p-projection body. Obviously, if $\tau = 0$ then $\Pi_p^\tau K = \Pi_p K$.

From (1.5), (1.7) and (1.9), Haberl and Schuster in [6] showed that for $K \in \mathcal{K}_o^n$, $p \ge 1$ and $\tau \in [-1, 1]$,

$$\Pi_p^\tau K = f_1(\tau) \cdot \Pi_p^+ K +_p f_2(\tau) \cdot \Pi_p^- K,$$

where

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p},$$

and "$+_p$" denotes the Firey L_p-combination of convex bodies.

Associating with asymmetric L_p-projection bodies, Haberl and Schuster in [6] established general L_p-Petty projection inequalities and gave the extremum values of volume for the polar of asymmetric L_p-projection bodies.

In this article, we study the following Shephard type problem for general L_p-projection bodies:

Question 3. *Suppose $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$. If*

$$\Pi_p^\tau K \subseteq \Pi_p^\tau L,$$

is it true that

$$V(K) \leq V(L), \ for \ 1 \leq p < n,$$

and

$$V(K) \geq V(L), \ for \ p > n?$$

Associated with Question 3, we first give the following affirmative answer:

Theorem 1.1. *Let $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$. If $L \in \mathcal{Z}_p^{\tau,n}$ and $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then for $n > p \geq 1$,*

$$V(K) \leq V(L);$$

for $n < p$,

$$V(K) \geq V(L).$$

In each case equality holds for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$.

Here $\mathcal{Z}_p^{\tau,n}$ denotes the set of general L_p-projection bodies of a parameter τ, that is, the set of convex bodies K such that there is a convex body L with $K = \Pi_p^\tau L$.

The original Shephard problem is in a certain sense dual to the famous Busemann-Petty problem (see [3, 7] for the definition and the solution). The (symmetric) L_p version of the Busemann-Petty problem was solved in [4, 26]. General L_p-intersection bodies were introduced in [5]. Theorem 1.1 corresponds to the solution of the general L_p Busemann-Petty problem by Haberl [4].

Let $\mathcal{F}_o^n$ denote the set of convex bodies in $\mathcal{K}_o^n$ with positive continuous curvature function. Further, we get a L_p-affine surface area form of the Shephard type problem for general L_p-projection bodies.

Theorem 1.2. *Let $K \in \mathcal{F}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$. If $L \in \mathcal{W}_p^{\tau,n}$ and $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then*

$$\Omega_p(K) \leq \Omega_p(L),$$

with equality for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$.

Here

$$\mathcal{W}_p^{\tau,n} = \{Q \in \mathcal{F}_o^n : there\ exists\ Z \in \mathcal{Z}_p^{\tau,n}\ with\ f_p(Q, \cdot) = h(Z, \cdot)^{-(n+p)}\},$$

and where $f_p(Q, \cdot)$ is the L_p-curvature function of Q (see Section 2.5).

In Section 3, we shall prove general forms of Theorems 1.1-1.2, respectively.

2. Basic Notions

2.1. Radial Function and Polar Body

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0, +\infty)$, is defined

$$\rho(K, x) = max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is nonempty in $\mathbb{R}^n$, the polar set of K, K^*, is defined by [3]

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

2.2. Firey L_p-Combination and L_p-Harmonic Radial Combination

For $K, L \in \mathcal{K}^n$, and $\lambda, \mu \geq 0$ (not both zero), the Minkowski linear combination, $\lambda K + \mu L \in \mathcal{K}^n$, of K and L is defined by

$$h(\lambda K + \mu L, \cdot\,) = \lambda h(K, \cdot) + \mu h(L, \cdot),$$

where $\lambda K = \{\lambda x : x \in K\}$.

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the Firey L_p-combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined in [2] by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot\,)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p, \tag{2.1}$$

where " $\cdot$ " in $\lambda \cdot K$ denotes the Firey scalar multiplication.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$, of K and L is defined in [11] by

(2.2) $$\rho(\lambda\star K +_{-p} \mu\star L,\cdot)^{-p} = \lambda\rho(K,\cdot)^{-p} + \mu\rho(L,\cdot)^{-p}.$$

Note that for convex bodies, the L_p-harmonic radial combination was investigated by Firey in [1].

2.3. L_p-Mixed Volume

Associated with Firey L_p-combination (2.1), Lutwak in [10] introduced the following: For $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of K and L can be defined by

$$\frac{n}{p}V_p(K,L) = \lim_{\varepsilon\longrightarrow 0^+} \frac{V(K +_p \varepsilon\cdot L) - V(K)}{\varepsilon}.$$

Corresponding to each $K \in \mathcal{K}_o^n$, Lutwak ([10]) proved that there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that

(2.3) $$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(v)dS_p(K,v).$$

for each $L \in \mathcal{K}_o^n$. The measure $S_p(K,\cdot)$ is just the L_p-surface area measure of K.

From formulas (2.3) and (1.4), it follows immediately that for each $K \in \mathcal{K}_o^n$,

(2.4) $$V_p(K,K) = V(K) = \frac{1}{n}\int_{S^{n-1}} h_K(v)dS(K,v).$$

The Minkowski inequality for the L_p-mixed volume is called L_p-Minkowski inequality. The L_p-Minkowski inequality may be stated:

Theorem 2.A. *If $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, then*

(2.5) $$V_p(K,L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}},$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

A simple consequence of Theorem 2.A was established in [11]:

Theorem 2.B. *Let $K, L \in \mathcal{K}_o^n$ and $p \geq 1$. For all $Q \in \mathcal{K}_o^n$,*

$$V_p(K,Q) = V_p(L,Q) \quad or \quad V_p(Q,K) = V_p(Q,L)$$

if and only if K is translation of L for $p = 1$, or $K = L$ for $p > 1$.

2.4. L_p-Dual Mixed Volume

Using the L_p-harmonic radial combination (2.2), Lutwak [11] introduced the notion of L_p-dual mixed volume. For $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L is defined by

$$\frac{n}{-p}\widetilde{V}_{-p}(K,L)=\lim_{\varepsilon\longrightarrow 0^+}\frac{V(K+_{-p}\varepsilon\star L)-V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the L_p-dual mixed volume:

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n+p}(v)\rho_L^{-p}(v)dS(v), \tag{2.6}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From (2.6), it follows that for each $K\in\mathcal{S}_o^n$ and $p\geq 1$,

$$\widetilde{V}_{-p}(K,K)=V(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^n(v)dS(v). \tag{2.7}$$

Lutwak [11] established the L_p-dual Minkowski inequality:

Theorem 2.C. *If $K,L\in\mathcal{S}_o^n$ and $p\geq 1$, then*

$$\widetilde{V}_{-p}(K,L)\geq V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}, \tag{2.8}$$

with equality if and only if K and L are dilates.

A simple consequence of Theorem 2.C was established in [25]:

Theorem 2.D. *Let $K,L\in\mathcal{S}_o^n$ and $p\geq 1$, For all $Q\in\mathcal{S}_o^n$,*

$$\widetilde{V}_{-p}(K,Q)=\widetilde{V}_{-p}(L,Q)\quad or\quad \widetilde{V}_{-p}(Q,K)=\widetilde{V}_{-p}(Q,L)$$

if and only if $K=L$.

2.5. L_p-Affine Surface Area

The notion of L_p-affine surface area was introduced by Lutwak in [11].

A convex body $K\in\mathcal{K}_o^n$ is said to have a L_p- curvature function [11] $f_p(K,\cdot):S^{n-1}\longrightarrow\mathbb{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_p(K,\cdot)}{dS}=f_p(K,\cdot). \tag{2.9}$$

In [11], Lutwak proved that if $K\in\mathcal{F}_o^n$ and $p\geq 1$, then the L_p-affine surface area of K have the integral representation

$$\Omega_p(K)=\int_{S^{n-1}}f_p(K,u)^{\frac{n}{n+p}}dS(u). \tag{2.10}$$

Wang and Leng in [21] defined the ith L_p-mixed affine surface area as follows: For $K, L \in \mathcal{F}_o^n$, $p \geq 1$ and real i, the ith L_p-mixed affine surface area, $\Omega_{p,i}(K, L)$, of K and L is defined by

$$\Omega_{p,i}(K, L) = \int_{S^{n-1}} f_p(K, u)^{\frac{n-i}{n+p}} f_p(L, u)^{\frac{i}{n+p}} dS(u). \tag{2.11}$$

In the case $i = -p$, we write $\Omega_{p,-p}(K, L) = \Omega_{-p}(K, L)$ and see by (2.11) that

$$\Omega_{-p}(K, L) = \int_{S^{n-1}} f_p(K, u) f_p(L, u)^{-\frac{p}{n+p}} dS(u). \tag{2.12}$$

If $p = 1$, then $\Omega_{1,-1}(K, L)$ is just $\Omega_{-1}(K, L)$ (see [9]). Obviously,

$$\Omega_{-p}(K, K) = \Omega_p(K). \tag{2.13}$$

For the ith L_p-mixed affine surface area, Wang and Leng in [21] proved the following Minkowski inequality.

Theorem 2.E. *If $K, L \in \mathcal{F}_o^n$, $p \geq 1$, $i \in \mathbb{R}$, then for $i < 0$ or $i > n$,*

$$\Omega_{p,i}(K, L)^n \geq \Omega_p(K)^{n-i} \Omega_p(L)^i; \tag{2.14}$$

for $0 < i < n$, inequality (2.14) is reversed. In every case, equality holds for $p = 1$ if and only if K and L are homothetic, for $n \neq p > 1$ if and only if K and L are dilates. For $i = 0$ or $i = n$, (2.14) is an identity.

For $i = -p$ in (2.14), we get that *if $K, L \in \mathcal{F}_o^n$, $p \geq 1$, then*

$$\Omega_{-p}(K, L)^n \geq \Omega_p(K)^{n+p} \Omega_p(L)^{-p}, \tag{2.15}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $n \neq p > 1$ if and only if K and L are dilates.

From (2.15), we easily obtain that

Theorem 2.F. *Let $K, L \in \mathcal{F}_o^n$ and $p \geq 1$. For all $Q \in \mathcal{F}_o^n$,*

$$\Omega_{-p}(K, Q) = \Omega_{-p}(L, Q)$$

if and only if K is translation of L for $p = 1$, or if and only if $K = L$ for $p > 1$.

2.6. General L_p-Moment Bodies

Ludwig in [8] (also see [6]) introduced the notion of general L_p-moment body as follows: For $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, the general L_p-moment body, $M_p^\tau K$, of K is the convex body whose support function is given by

$$h^p_{M_p^\tau K}(u) = (n + p)\alpha_{n,p}(\tau) \int_K \varphi_\tau(u \cdot x)^p dx \tag{2.16}$$

for all $u \in S^{n-1}$. Here $\varphi_\tau(u \cdot v)$ and $\alpha_{n,p}(\tau)$ satisfy (1.8) and (1.10), respectively.

Using definitions (1.9) and (2.16), Haberl and Schuster ([6]) proved the following result:

Theorem 2.G. *If $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$V_p(K, M_p^\tau L) = \widetilde{V}_{-p}(L, \Pi_p^{\tau,*} K). \tag{2.17}$$

3. Shephard Type Problems

In the section, we will study Shephard type problems for general L_p-projection bodies. We first give a general version of Theorem 1.1. It may be regarded as an extension of the Shephard type problem to general L_p-projection bodies.

Theorem 3.1. *Let $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$. If $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then for every $Q \in \mathcal{Z}_p^{\tau,n}$,*

$$V_p(K, Q) \leq V_p(L, Q), \tag{3.1}$$

with equality for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$.

Lemma 3.1. If K, $L \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then

$$V_p(K, \Pi_p^\tau L) = V_p(L, \Pi_p^\tau K). \tag{3.2}$$

Proof. From (1.8) and (2.3), we easily obtain

$$\begin{aligned} V_p(L, \Pi_p^\tau K) &= \frac{1}{n} \int_{S^{n-1}} h^p_{\Pi_p^\tau K}(u) dS_p(L, u) \\ &= \frac{1}{n} \int_{S^{n-1}} \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_p(K, v) dS_p(L, u) \\ &= \frac{1}{n} \int_{S^{n-1}} h^p_{\Pi_p^\tau L}(v) dS_p(K, v) \\ &= V_p(K, \Pi_p^\tau L). \end{aligned}$$

□

Proof of Theorem 3.1. Since $Q \in \mathcal{Z}_p^{\tau,n}$, there exists $M \in \mathcal{K}_o^n$ such that $Q = \Pi_p^\tau M$. Thus by (3.2) and (2.3) we get

$$\frac{V_p(L, Q)}{V_p(K, Q)} = \frac{V_p(L, \Pi_p^\tau M)}{V_p(K, \Pi_p^\tau M)} = \frac{V_p(M, \Pi_p^\tau L)}{V_p(M, \Pi_p^\tau K)}$$

$$= \frac{\int_{S^{n-1}} h(\Pi_p^\tau L, u)^p dS_p(M, u)}{\int_{S^{n-1}} h(\Pi_p^\tau K, u)^p dS_p(M, u)}.$$

If $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, this implies (3.1).

According to Theorem 2.B, we know equality holds in (3.1) for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$. Obviously, above the condition of equality implies $\Pi_p^\tau K = \Pi_p^\tau L$. ■

Proof of Theorem 1.1. Since $L \in \mathcal{Z}_p^{\tau,n}$, taking $Q = L$ in Theorem 3.1, and combining with (2.4) and inequality (2.5), we get

$$V(L) \geq V_p(K, L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}.$$

Hence, for $n > p \geq 1$, $V(K) \leq V(L)$; for $n < p$, $V(K) \geq V(L)$. ■

Now, associated with the L_p-affine surface area, we give a general form of Theorem 1.2.

Theorem 3.2. *Let* $K, L \in \mathcal{F}_o^n$, $p \geq 1$ *and* $\tau \in [-1, 1]$. *If* $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, *then for every* $Q \in \mathcal{W}_p^{\tau,n}$,

$$\Omega_{-p}(K, Q) \leq \Omega_{-p}(L, Q), \tag{3.3}$$

with equality for $p = 1$ *if and only if* K *is a translate of* L, *and for* $n \neq p > 1$ *if and only if* $K = L$.

Proof. Since $Q \in \mathcal{W}_p^{\tau,n}$, there exists $Z \in \mathcal{Z}_p^{\tau,n}$ such that

$$f_p(Q, \cdot)^{-\frac{p}{n+p}} = h(Z, \cdot)^p.$$

Moreover, for $Z \in \mathcal{Z}_p^{\tau,n}$, let $Z = \Pi_p^\tau M$ for $M \in \mathcal{K}_o^n$. Hence, using (2.9), (2.12), (2.3) and (3.2), we have

$$\begin{aligned}
\frac{\Omega_{-p}(L, Q)}{\Omega_{-p}(K, Q)} &= \frac{\int_{S^{n-1}} f_p(Q, u)^{-\frac{p}{n+p}} dS_p(L, u)}{\int_{S^{n-1}} f_p(Q, u)^{-\frac{p}{n+p}} dS_p(K, u)} \\
&= \frac{\int_{S^{n-1}} h(Z, u)^p dS_p(L, u)}{\int_{S^{n-1}} h(Z, u)^p dS_p(K, u)} \\
&= \frac{V_p(L, Z)}{V_p(K, Z)} = \frac{V_p(L, \Pi_p^\tau M)}{V_p(K, \Pi_p^\tau M)} = \frac{V_p(M, \Pi_p^\tau L)}{V_p(M, \Pi_p^\tau K)} \\
&= \frac{\int_{S^{n-1}} h(\Pi_p^\tau L, u)^p dS_p(M, u)}{\int_{S^{n-1}} h(\Pi_p^\tau K, u)^p dS_p(M, u)}.
\end{aligned}$$

If $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, this implies (3.3).

According to Theorem 2.F, we know that equality holds in (3.3) for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$. Obviously, above the condition of equality implies $\Pi_p^\tau K = \Pi_p^\tau L$. ■

Note that the case $\tau = 0$ of Theorem 3.1 and Theorem 3.2 can be found in [13].

Proof of Theorem 1.2. Since $L \in \mathcal{W}_p^{\tau,n}$, taking $Q = L$ in Theorem 3.2, and together with (2.13) and inequality (2.15), we get

$$\Omega_p(L) \geq \Omega_{-p}(K, L) \geq \Omega_p(K)^{\frac{n+p}{n}} \Omega_p(L)^{-\frac{p}{n}},$$

i.e.,

$$\Omega_p(K) \leq \Omega_p(L).$$ ■

4. Monotonicity Inequalities

Regarding Theorem 3.1, we can prove the following monotonicity inequalities for the general L_p-projection bodies.

Theorem 4.1. *Let $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$. If*

$$V_p(K, Q) \leq V_p(L, Q), \tag{4.1}$$

then for every $Q \in \mathcal{Z}_p^{\tau,n}$,

$$V(\Pi_p^\tau K) \leq V(\Pi_p^\tau L). \tag{4.2}$$

In every inequality equality holds for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$.

Proof of Theorem 4.1. Since $Q \in \mathcal{Z}_p^{\tau,n}$, we take $Q = \Pi_p^\tau M$ for $M \in \mathcal{K}_o^n$. From this, (4.1) can be written as

$$V_p(K, \Pi_p^\tau M) \leq V_p(L, \Pi_p^\tau M).$$

Together with (3.2), we get

$$V_p(M, \Pi_p^\tau K) \leq V_p(M, \Pi_p^\tau L).$$

Letting $M = \Pi_p^\tau L$, and using (2.4) and inequality (2.5), we have

$$V(\Pi_p^\tau L) \geq V_p(\Pi_p^\tau L, \Pi_p^\tau K) \geq V(\Pi_p^\tau L)^{\frac{n-p}{n}} V(\Pi_p^\tau K)^{\frac{p}{n}},$$

i.e., (4.2) is obtained.

According to Theorem 2.B, we see that the equality condition of (4.1) implies $V(\Pi_p^\tau K) = V(\Pi_p^\tau L)$. Therefore, we know that equalities hold in (4.1) and (4.2) for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$. ■

Theorem 4.2. *Let $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$. If for every general L_p-moment body Q*

$$V_p(K, Q) \leq V_p(L, Q), \tag{4.3}$$

then

$$V(\Pi_p^{\tau,*} K) \geq V(\Pi_p^{\tau,*} L). \tag{4.4}$$

In every inequality equality holds for $p = 1$ if and only if K is a translate of L, and for $p > 1$ if and only if $K = L$.

Proof. Since Q is an general L_p-moment body, we take $Q = M_p^\tau N$ for $N \in \mathcal{S}_o^n$, inequality (4.3) can be written as

$$V_p(K, M_p^\tau N) \leq V_p(L, M_p^\tau N).$$

This together with (2.17) gives

$$\widetilde{V}_{-p}(N, \Pi_p^{\tau,*} K) \leq \widetilde{V}_{-p}(N, \Pi_p^{\tau,*} L)$$

Taking $N = \Pi_p^{\tau,*} L$, using (2.7) and inequality (2.8), we get

$$V(\Pi_p^{\tau,*} L) \geq \widetilde{V}_{-p}(\Pi_p^{\tau,*} L, \Pi_p^{\tau,*} K) \geq V(\Pi_p^{\tau,*} L)^{\frac{n+p}{n}} V(\Pi_p^{\tau,*} K)^{-\frac{p}{n}}.$$

This yields (4.4).

According to Theorem 2.D, we see that the equality condition of (4.3) implies $V(\Pi_p^{\tau,*} K) = V(\Pi_p^{\tau,*} L)$. Therefore, we know that equality hold in (4.3) and (4.4) if and only if $K = L$. ■

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