TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 5, pp. 1681-1705, October 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

AN ITERATIVE METHOD FOR GENERALIZED MIXED VECTOR EQUILIBRIUM PROBLEMS AND FIXED POINT OF NONEXPANSIVE MAPPINGS AND VARIATIONAL INEQUALITIES

Shu-qiang Shan and Nan-jing Huang*

Abstract. In this paper, we study the problem of finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the generalized mixed vector equilibrium problem and the solution set of a variational inequality problem with a monotone Lipschitz continuous mapping in Hilbert spaces. We first consider an auxiliary problem for the generalized mixed vector equilibrium problem and prove the existence and uniqueness of the solution for the auxiliary problem. We then introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the generalized mixed vector equilibrium problem and the solution set of a variational inequality problem with a monotone Lipschitz continuous mapping. The results presented in this paper can be considered as a generalization of some known results due to Peng and Yao [16, 17].

1. Introduction

Let H be a real Hilbert space with winer product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let X be a nonempty closed convex subset of H. Let $\varphi : X \times X \to \mathbf{R} = (-\infty, +\infty)$ be a bifunction and $\psi : X \to \mathbf{R} \cup \{+\infty\}$ be a function. Let $T : X \to H$ be a nonlinear mapping. The equilibrium problem $EP(\varphi)$ is to find $x \in X$ such that

$$\varphi(x, y) \geq 0, \quad \forall\, y \in X. \tag{1.1}$$

As pointed out by Blum and Oettli [2], $EP(\varphi)$ provides a unified model of several problems, such as the optimization problem, fixed point problem, variational inequality

Received October 17, 2011, accepted October 31, 2011.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 49J40, 47H10, 54H25.
Key words and phrases: Generalized mixed vector equilibrium problem, Auxiliary problem, Variational inequality, Nonexpansive mapping, Iterative scheme.
This work was supported by the Key Program of NSFC (Grant No. 70831005) and the National Natural Science Foundation of China (11171237).
*Corresponding author.

and complementarity problem. Recently, Peng and Yao [16] studied the following generalized mixed equilibrium problem: Find $x \in X$ such that

(1.2) $$\varphi(x, y) + \psi(y) - \psi(x) + \langle Tx, y - x\rangle \geq 0, \ \forall\ y \in X.$$

Problem (1.2) is very general setting and it includes as special cases of Nash-equilibrium problems, complementarity problems, fixed point problems, optimization problems and variational inequalities (see, for example, [16, 17, 7, 10, 23] and the references therein).

It is well known that the vector equilibrium problem provides a unified model of several problems, such as the vector optimization problem, fixed point problem, vector variational inequality and complementarity problem. Let Y be a Hausdorff topological space and C be a proper, closed and convex cone of Y with $intC \neq \emptyset$.

The strong vector equilibrium problem (for short, $SVEP(\varphi)$) is to find $x \in X$ such that

(1.3) $$\varphi(x, y) \in C, \quad \forall\ y \in X$$

and the weak vector equilibrium problem (for short, $WVEP(\varphi)$) is to find $x \in X$ such that

(1.4) $$\varphi(x, y) \notin -intC, \quad \forall y \in X.$$

A mapping $S : X \to H$ is called nonexpansive, if

(1.5) $$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x,\ y \in X.$$

We denote the set of all fixed points of S by $F(S)$, that is, $F(S) = \{x \in X : x = Sx\}$.

It is well known that if $X \subset H$ is bounded, closed, convex and S is a nonexpansive mapping of X onto itself, then $F(S)$ is nonempty (see [24]).

Let $A : X \to H$ be a mapping. The classic variational inequality problem is to find $x \in X$ such that

(1.6) $$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in X.$$

The solution of variational inequality problem is denoted by $VI(A, X)$.

In this paper, we consider the following generalized mixed vector equilibrium problem (for short, $GMVEP(\varphi, \psi, T)$): find $x \in X$ such that

(1.7) $$\varphi(x, y) + \psi(y) - \psi(x) + e\langle Tx, y - x\rangle \in C, \quad \forall y \in X,$$

where $e \in intC$. We denote the set solution of problem (1.7) by

$$\mathbb{B} = \{x \in X : \varphi(x, y) + \psi(y) - \psi(x) + e\langle Tx, y - x\rangle \in C, \quad \forall y \in X\}.$$

Some special cases of problem (1.7) are as follows:

(1) If $Y = \mathbf{R}$, $C = \mathbf{R}^+$ and $e = 1$, then $GMVEP$ (1.7) reduces to the generalized mixed equilibrium problem (1.2);

(2) If $\psi = 0$ and $T = 0$, then $GMVEP$ (1.7) reduces to the classic vector equilibrium problem (1.3).

It is well known that the vector equilibrium problem provides a unified model of several problems, for example, vector optimization, vector variational inequality, vector complementarity problem, and vector saddle point problem ([4, 5, 13]). In recent years, the vector equilibrium problem has been intensively studied by many authors (see, for example, [1, 4, 5, 10, 11, 13, 6] and the references therein).

In 2002, Mondafi [14] introduced an iterative scheme of finding the solution of nonexpansive mappings and proved a strong convergence theorem. In 2003, Insem and Sosa [9] introduced some iterative algorithms for solving equilibrium problem in finite-dimensional space and established the convergence of the algorithms. Recently, Huang et al. [7] introduced the approximate method for solving the equilibrium problem and proved the strong convergence theorem.

On the other hand, Takahashi and Toyodu [25] introduced a new iterative scheme and proved a weak convergence theorem for finding an element of $F(S) \cap VI(A, X)$. Takahashi and Takahashi [23] introduced an iterative scheme for finding a common element of $F(S)$ and $EP(\varphi)$. Recently, Peng and Yao [17] introduced an iterative scheme for finding a common element of $F(S) \cap VI(A, X)$ and the set of solutions of problem (1.2). For finding a common element of $F(S)$ and $SVEP(\varphi)$, Li and Wang [11] introduced an iterative scheme and obtained a strong convergence theorem. In recent years, many authors have intensively studied different types of iterative schemes for finding an element of $F(S) \cap EP(\varphi) \cap VT(A, X)$ (see, for example, [21, 18, 8, 28, 12] and the references therein). However, to the best of our knowledge, there are no results concerned with the problems of finding a common element of the set of fixed points of infinitely many nonexpansive mappings, the set of generalized mixed vector equilibrium problem and the solution set of the variational inequality problem in finite or infinite dimensional spaces.

Inspired and motivated by the works mentioned above, in this paper, we consider the auxiliary problem of $GMVEP(\varphi, \psi)$ and prove the existence and uniqueness of the solutions of auxiliary problem of $GMVEP(\varphi, \psi)$ under some proper conditions. By using the result for the auxiliary problem, we introduce an interactive scheme for finding a common element of the set of fixed points of infinitely many nonexpansive mappings, the set of generalized mixed vector equilibrium problem and the solution set of the variational inequality problem with an α-inverse-strongly monotone mapping in a real Hilbert space. The results presented in this paper improve and generalize some known results of Peng and Yao [16, 17].

2. Preliminaries

A mapping

$$P_X(x) = \{z \in X : \|x - z\| = \inf_{y \in X} \|x - y\|\}$$

is called the metric projection of H onto X. It is well known that, for all $x \in H$, there exists a unique nearest point in X, and P_X is a nonexpansive mapping. Furthermore, for all $x \in H$ and $y \in X$, P_X satisfies that

$$\langle P_X(x) - P_X(y), x - y\rangle \geq \|P_X(x) - P_X(y)\|^2 \tag{2.1}$$

and

$$\langle P_X(x) - x, P_X(y) - y\rangle \leq 0 \tag{2.2}$$

with

$$\|x - P_X(x)\|^2 + \|y - P_X(y)\|^2 \leq \|x - y\|^2. \tag{2.3}$$

A mapping $A : H \to X$ is said to be inverse strongly monotone with a modulus α (in short, α-inverse-strongly monotone) if A satisfies

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in X, \tag{2.4}$$

where α is a positive real number (see [3]).

A mapping $A : H \to X$ is called a k-contraction mapping if there exists a positive real number $k \in [0, 1)$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in X. \tag{2.5}$$

It is well known that

$$\mu \in VI(A, X) \Longleftrightarrow \mu = P_X(\mu - \lambda A\mu), \quad \forall x, y \in X. \tag{2.6}$$

A set-value mapping $\Phi : H \to 2^H$ is called monotone if

$$\langle f - g, x - y\rangle \geq 0, \quad \forall x, y \in H, f \in \Phi(x), g \in \Phi(y). \tag{2.7}$$

The mapping Φ is called maximal monotone if the graph $G(\Phi)$ is not properly contained in the graph of any other monotone mappings.

Let $A : X \to H$ be a monotone $k-$contraction mapping and $N_X\mu$ be the normal cone in X, that is,

$$N_X\mu = \{\omega \in H : \langle \mu - \nu, \omega\rangle \geq 0, \forall \nu \in X\}.$$

Define

$$\Phi\mu = \begin{cases} A\mu + N_X\mu, & \mu \in X, \\ \emptyset, & \mu \notin X. \end{cases}$$

It is well known that Φ is the maximal monotone and $0 \in \Phi\mu$ if and only if $\mu \in VI(A, X)$ (see, for example, [19]).

Let $\{B_n\}_{n=1}^{\infty}$ be a infinite family of nonexpansive mappings of X into itself and let $\{k_n\}_{n=1}^{\infty}$ be real numbers in $[0, 1]$. For any $n = 0, 1, 2, \cdots$, define a mapping S_n of X into itself as follows:

$$\begin{aligned}
&\pi_{n,n+1} = I, \\
&\pi_{n,n} = k_n B_n \pi_{n,n+1} + (1 + k_n)I, \\
&\pi_{n,n-1} = k_{n-1} B_{n-1} \pi_{n,n} + (1 + k_{n-1})I, \\
&\quad\vdots \\
&\pi_{n,m} = k_m B_m \pi_{n,m+1} + (1 + k_m)I, \\
&\pi_{n,m-1} = k_{m-1} B_{m-1} \pi_{n,m} + (1 + k_{m-1})I, \\
&\quad\vdots \\
&\pi_{n,2} = k_2 B_2 \pi_{n,3} + (1 + k_2)I, \\
&S_n = \pi_{1,1} = k_1 B_1 \pi_{n,2} + (1 + k_1)I.
\end{aligned}$$

Such a mapping S_n is called the $S-$mapping generated by $B_n, B_{n-1}, \cdots, B_1$ and $k_n, k_{n-1} \cdots, k_1$ (see [20]).

Remark 2.1. It is easy to see that S_n is nonexpansive, $F(B_n) \subseteq F(S_n)$ and $F(B_n) \subseteq F(\pi_{n,k})$ for every $k \geq 1$.

Lemma 2.1 ([20]). Let X be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of X into itself such that $\cap_{i=1}^{\infty} F(B_n) \neq \emptyset$, and let $\{k_n\}_{n=1}^{\infty}$ be a real numbers such that $k_n \in [0, \delta]$ for some $\delta \in (0, 1)$. Then for every $x \in X$, and $m \in N = \{1, 2, \cdots\}$, the limit $\lim_{n\to\infty} \pi_{n,m}x$ exists.

Remark 2.2. We denote $\pi_m x = \lim_{n\to\infty} \pi_{n,m}x$. Using Lemma 2.1, one can define a mapping S of X into itself as follows:

$$Sx = \lim_{n\to\infty} S_n x = \lim_{n\to\infty} \pi_{n,1}x$$

for every $x \in X$. Such a mapping S is called the S-mapping generated by $B_1, B_2, \cdots$ and $k_1, k_2, \cdots$. It is obvious that $S : X \to X$ is also nonexpansive.

If $\{x_n\}$ is a bounded sequence in X, we put $C = \{x_n : n = 1, 2, \cdots\}$. It follows from Lemma 2.1 that, for any given $\epsilon \geq 0$, there exists $N_0 \in N$ such that, for every

$n > N_0$,

$$\sup_{x\in C} \|\pi_{n,m}x - \pi_m x\| \le \epsilon$$

and

$$\|S_n x_n - Sx_n\| = \|\pi_{n,1}x_n - \pi_1 x_n\| \le \sup_{x\in C} \|\pi_{n,1}x - \pi_1 x\| \le \epsilon.$$

It follows that

$$\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0, \quad \lim_{n\to\infty} \|S_{n+1}x_n - S_n x_n\| = 0.$$

Lemma 2.2 ([20]). Let X be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of X into itself such that $\cap_{i=1}^{\infty} F(B_n) \neq \emptyset$, and let $\{k_n\}_{n=1}^{\infty}$ be a real numbers such that $k_n \in [0, \delta]$ for some $\delta \in (0, 1)$. Then $F(S) = \cap_{i=1}^{\infty} F(B_n)$.

Definition 2.1 ([15]). Let E be a Banach space. We say that E satisfies the *Opial condition* if, for any $\{x_n\} \subset E$ with $x_n \rightharpoonup x \in E$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E, \ y \neq x.$$

We know that Hilbert space H satisfies *Opial condition* (see, for example, [15]).

Definition 2.2 ([26, 13]). Let X and Y be two Hausdorff topological spaces, and let E be a nonempty, convex subset of X and C be a proper, closed, convex cone of Y with $intC \neq \emptyset$. Let 0 be the zero point of Y, $\mathbb{U}(0)$ be the neighborhood set of 0, $\mathbb{U}(x_0)$ be the neighborhood set of x_0, and $f : E \to Y$ be a mapping.

(1) If for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + C, \forall x \in U \bigcap E,$$

then f is called $upper\ C-continuous$ on x_0. If f is $upper\ C-continuous$ for all $x \in E$, then f is called $upper\ C-continue$ on E.

(2) If for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - C, \ \forall x \in U \bigcap E,$$

then f is called $lower\ C-continuous$ on x_0. If f is $lower\ C-continuous$ for all $x \in E$, then f is called $lower\ C-continuous$ on E.

(3) If for any $x, y \in E$ and $t \in [0, 1]$, the mapping f satisfies

$$f(x) \in f(tx + (1-t)y) + C \ or \ f(y) \in f(tx + (1-t)y) + C,$$

then f is called $proper\ C-quasiconvex$.

(4) If for any $x_1, x_2 \in E$ and $t \in [0,1]$, the mapping f satisfies

$$tf(x_1) + (1-t)f(x_2) \in f(tx_1 - (1-t)x_2) + C,$$

then f is called $C-convex$.

Lemma 2.3 ([6]). Let X and Y be two real Hausdorff topological spaces, E is a nonempty, compact, convex subset of X, and C is a proper, closed, and convex cone of Y. Assume that $f: E \times E \to Y$ and $\Phi: E \to Y$ are two vector mappings. Suppose that f and Φ satisfy

(1) $f(x,x) \in C$, for all $x \in E$;

(2) Φ is $upper\ C-continuous$ on E;

(3) $f(\cdot, y)$ is $lower\ C-continuous$ for all $y \in E$;

(4) $f(x,\cdot) + \Phi(\cdot)$ is $proper\ C-quasiconvex$ for all $x \in E$.

Then there exists a point $x \in E$ satisfies

$$F(x,y) \in C \setminus \{0\}, \quad \forall y \in E,$$

where

$$F(x,y) = f(x,y) + \Phi(y) - \Phi(x), \quad \forall x, y \in E.$$

Lemma 2.4 ([22]). Let $\{x_n\}$ and $\{\omega_n\}$ be bounded sequence in Banach space E, and let $\{\beta_n\}$ be a sequence in $[0,1]$. Let $x_{n+1} = (1-\beta_n)\omega_n + \beta_n x_n$ for all integers $n \geq 1$. If

$$\limsup_{n\to\infty}(\|\omega_{n+1} - \omega_n\| - \|x_{n+1} - x_n\|) \leq 0$$

and

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

then $\lim_{n\to\infty} \|\omega_n - x_n\| = 0$.

Lemma 2.5 ([27]). Suppose that $\alpha_n \subseteq [0,\infty]$ and $b_n, c_n \subseteq \mathbf{R}$ satisfy the following conditions:

(1) $\alpha_{n+1} \leq (1-c_n)\alpha_n + c_n b_n$ for all $n = 0, 1, 2, \cdots$;

(2) $c_n \subseteq [0,1]$ and $\sum_{n=1}^{\infty} c_n = \infty$;

(3) $\limsup_{n\to\infty} b_n \leq 0$ or $\sum_{n=1}^{+\infty} b_n c_n$ is convergence.

Then

$$\lim_{n\to\infty} \alpha_n = 0.$$

For solving the generalized mixed vector equilibrium problem, we give the following assumptions. Let H be a real Hilbert space with winner $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Assume that $X \subseteq H$ is nonempty, compact, convex subset and Y is a real Hausdorff topological space, $C \subseteq Y$ is a proper, closed, and convex cone. Let $\varphi : X \times X \to Y$ and $\psi : X \to Y$ be two mappings. For any $z \in H$, define a mapping $\phi_z : X \times X \to Y$ as follows:

$$\phi_z(x, y) = \varphi(x, y) + \psi(y) - \psi(x) + \frac{e}{r}\langle y - x, x - z\rangle,$$

where r is a positive number in $\mathbf{R}$ and $e \in C$. Let ϕ_z, φ, ψ satisfy the following conditions:

(A_1) for all $x \in X$, $\varphi(x, x) = 0$;

(A_2) φ is monotone, that is, $\varphi(x, y) + \varphi(y, x) \in -C$ for all $x,\ y \in X$;

(A_3) $\varphi(\cdot, y)$ is continuous for all $y \in X$;

(A_4) $\varphi(x, \cdot)$ is weakly continuous and $C-convex$, that is,

$$t\varphi(x, y_1)+(1-t)\varphi(x, y_2) \in \varphi(x, ty_1+(1-t)y_2)+C, \quad \forall x,\ y_1,\ y_2 \in X, \forall t \in [0, 1];$$

(A_5) $\phi_z(\cdot, y)$ is $lower\ C-continuous$ for all $y \in X$ and $z \in H$;

(A_6) $\psi(\cdot)$ is $C-convex$ and weakly continuous;

(A_7) $\phi_z(x, \cdot)$ is $proper\ C-quasiconvex$ for all $x \in X$ and $z \in H$.

Remark 2.3. Let $Y = \mathbf{R}$, $C = \mathbf{R}_+$ and $e = 1$. For any given $y \in X$, if $\varphi(\cdot, y)$ is upper semicontinuous and $\psi(\cdot)$ is proper lower semicontinuous, then $\Phi_z(\cdot, y)$ is lower $C-$continuous. In fact, since $\varphi(\cdot, y)$ is upper semicontinuous and $\psi(\cdot)$ is proper lower semicontinuous, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in T = \{x \in X, \|x - x_0\| < \delta\}$, we have

$$\Phi_z(x, y) < \Phi_z(x_0, y) + \varepsilon,$$

where x_0 is a point in X. This means $\Phi_z(\cdot, y)$ is lower $C-$continuous.

Remark 2.4. Let $Y = \mathbf{R}$, $C = \mathbf{R}_+$ and $e = 1$. Assume that $\varphi(x, \cdot)$ and $\psi(\cdot)$ are two convex mappings. Then for any $y_1,\ y_2 \in X$ and $t \in [0, 1]$, we have

$$
\begin{aligned}
&\Phi_z(x, ty_1+(1-t)y_2)\\
&= \varphi(x, ty_1+(1-t)y_2)+\psi(ty_1+(1-t)y_2)-\psi(x)\\
&\quad +\frac{1}{r}\langle ty_1+(1-t)y_2-x, x-z\rangle\\
&\leq t\varphi(x,y_1)+(1-t)\varphi(x,y_2)+t\psi(y_1)+(1-t)\psi(y_2)-t\psi(x)-(1-t)\psi(x)\\
&\quad +\frac{t}{r}\langle y_1-x, x-z\rangle+\frac{1-t}{r}\langle y_2-x, x-z\rangle\\
&= t(\varphi(x,y_1)+\psi(y_1)-\psi(x)+\frac{1}{r}\langle y_1-x, x-z\rangle)+(1-t)(\varphi(x,y_2)\\
&\quad +(1-t)\psi(y_2)-\psi(x)+\frac{1}{r}\langle y_2-x, x-z\rangle)\\
&= t\Phi_z(x,y_1)+(1-t)\Phi_z(x,Y_2)\\
&\leq \max\{\Phi_z(x,y_1), \Phi_z(x,Y_2)\},
\end{aligned}
$$

which implies that $\Phi_z(x,\cdot)$ is proper $C-$quasiconvex.

Theorem 2.1. Let φ, ψ satisfy all the conditions $(A_1)-(A_7)$. Define a mapping $T_r(z): H \to X$ as follows:

$$T_r(z)=\{x\in X: \varphi(x,y)+\psi(y)-\psi(x)+\frac{e}{r}\langle y-x, x-z\rangle \in C, \quad \forall y\in X\}.$$

Then

(1) $T_r(z)\neq\emptyset$ for all $z\in H$;

(2) T_r is single-value;

(3) T_r is firmly nonexpansive and

$$\|T_r(z_1)-T_r(z_2)\|^2 \leq \langle T_r(z_1)-T_r(z_2), z_1-z_2\rangle$$

for all $z_1, z_2\in H$;

(4) $F(T_r)=GMVEP(\varphi,\psi)$;

(5) $GMVEP(\varphi,\psi)$ is closed and convex.

Proof. (1) Let $f(x,y)=\phi_z(x,y)$ and $\Phi(x)=0$ for all $x,y\in X$ and $z\in H$. Then it is easy to check that $f(x,y)$ and $\Phi(y)$ satisfy all the conditions of Lemma 2.1. Thus, there exists a point $x\in E$ such that

$$f(x,y)+\Phi(x)-\Phi(y)\in C, \quad \forall y\in X, z\in H$$

and so $T_r(z)\neq\emptyset$ for all $z\in H$.

(2) For each $z\in H$, $T_r(z)\neq\emptyset$. Let $x_1,x_2\in T_r(z)$. Then

$$\varphi(x_1,y)+\psi(y)-\psi(x_1)+\frac{e}{r}\langle y-x_1, x_1-z\rangle\in C, \quad \forall y\in X \tag{2.8}$$

and

$$\varphi(x_2, y) + \psi(y) - \psi(x_2) + \frac{e}{r}\langle y - x_2, x_2 - z\rangle \in C, \quad \forall y \in X. \tag{2.9}$$

Letting $y = x_2$ in (2.8) and $y = x_1$ in (2.9), adding (2.8) and (2.9), we have

$$\varphi(x_2, x_1) + \varphi(x_1, x_2) + \frac{e}{r}\langle x_2 - x_1, x_1 - x_2\rangle \in C.$$

By the monotonicity of φ and the property of C, we know that $x_1 = x_2$ and so $T_r(z)$ is single-value.

(3) For any $z_1, z_2 \in H$, let $x_1 = T_r(z_1)$ and $x_2 = T_r(z_2)$. Then

$$\varphi(x_1, y) + \psi(y) - \psi(x_1) + \frac{e}{r}\langle y - x_1, x_1 - z_1\rangle \in C, \quad \forall y \in X \tag{2.10}$$

and

$$\varphi(x_2, y) + \psi(y) - \psi(x_2) + \frac{e}{r}\langle y - x_2, x_2 - z_2\rangle \in C, \quad \forall y \in X. \tag{2.11}$$

Letting $y = x_2$ in (2.10) and $y = x_1$ in (2.11), adding (2.10) and (2.11), we obtain

$$\varphi(x_2, x_1) + \varphi(x_1, x_2) + \frac{e}{r}\langle x_2 - x_1, x_1 - x_2 - (z_1 - z_2)\rangle \in C$$

Since φ is monotone and C is a closed convex cone, we get

$$\langle x_2 - x_1, z_2 - z_1\rangle \geq \langle x_2 - x_1, x_2 - x_1\rangle$$

and so

$$\|T_r(z_1) - T_r(z_2)\| \leq \langle T_r(z_1) - T_r(z_2), z_1 - z_2\rangle,$$

which shows that T_r is firmly nonexpansive.

(4) Let $x \in F(T_r)$. Then

$$\varphi(x, y) + \psi(y) - \psi(x) + \frac{e}{r}\langle y - x, x - x\rangle \in C, \quad \forall y \in X$$

and so

$$\varphi(x, y) + \psi(y) - \psi(x) \in C, \quad \forall y \in X.$$

It follows that $x \in GMVEP(\varphi, \psi)$.

Conversely, if $x \in GMVEP(\varphi, \psi)$, then

$$\varphi(x, y) + \psi(y) - \psi(x) \in C, \quad \forall y \in X$$

and so

$$\varphi(x, y) + \psi(y) - \psi(x) + \frac{e}{r}\langle y - x, x - x\rangle \in C, \quad \forall y \in X,$$

which shows that $x \in F(T_r)$.

(5) For any $x_1, x_2 \in F(T_r)$ and $t \in [0,1]$, we have

$$\varphi(x_1, y) + \psi(y) - \psi(x_1) \in C \tag{2.12}$$

and

$$\varphi(x_2, y) + \psi(y) - \psi(x_2) \in C \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$(1-t)\psi(y) \in C + (1-t)\psi(x_1) - (1-t)\varphi(x_1, y) \tag{2.14}$$

and

$$t\psi(y) \in C + t\psi(x_2) - t\varphi(x_2, y). \tag{2.15}$$

Since φ is monotone and ψ is $C-convex$, adding (2.14) and (2.15), we have

$$\begin{aligned} \psi(y) &\in C + (1-t)\psi(x_1) - (1-t)\varphi(x_1, y) + t\psi(x_2) - t\varphi(x_2, y) \\ &\subset C + \psi((1-t)x_1 + tx_2) + \varphi(y, (1-t)x_1 + tx_2). \end{aligned} \tag{2.16}$$

Thus, for any $y \in X$, there exists a point $e_1(y) \in C$ such that

$$-\varphi(y, (1-t)x_1 + tx_2) = e_1(y) - \psi(y) + \psi((1-t)x_1 + tx_2). \tag{2.17}$$

Since φ is monotone, we get

$$-\varphi(y, (1-t)x_1 + tx_2) \in C + \varphi((1-t)x_1 + tx_2, y).$$

Therefore, for any $y \in X$, there exists a point $e_2(y) \in C$ such that

$$-\varphi(y, (1-t)x_1 + tx_2) = e_2(y) + \varphi((1-t)x_1 + tx_2, y). \tag{2.18}$$

Now (2.17) and (2.18) imply that

$$e_1(y) - e_2(y) = \varphi((1-t)x_1 + tx_2, y) + \psi(y) - \psi((1-t)x_1 + tx_2).$$

Since $e_1(y), e_2(y) \in C$, we have

$$\varphi((1-t)x_1 + tx_2, y) + \psi(y) - \psi((1-t)x_1 + tx_2) \in C,$$

which shows that $F(T_r)$ is convex.

Let $\{x_n\} \subseteq F(T_r)$ with $x_n \to x_0$. Then

$$x_n = T_r(x_n), \quad n = 1, 2, \cdots.$$

Since T_r is firmly nonexpansive, we obtain $x_0 = T_r(x_0)$ and so $GMVEP(\varphi, \psi)$ is closed and convex. This completes the proof. ■

Remark 2.5. Theorem 2.1 improves and extends Lemma 2.3 of Peng and Yao [16]. In fact, letting $Y = \mathbf{R}$, $C = \mathbf{R}_+$ and $e = 1$, we can get the same conclusions of Lemma 2.3 in [16].

3. Main Results

In this section, we will introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of $GMVEP(\varphi, \psi, T)$ and the solution set of the variational inequality problem for an $\alpha-$inverse-strongly monotone mapping in a real Hilbert space H.

Theorem 3.1. Let X be a nonempty, compact, convex subset of a real Hilbert space H. Assume that C is a closed, convex cone of a real Hausdorff topological space Y and $e \in C$ is a fixed point. Let $\varphi : X \times X \to Y$ and $\psi : X \to Y$ satisfy $(A_1) - (A_7)$. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings from X into H. Let $T : X \to H$ be a m-inverse-strongly monotone mapping, $A : X \to H$ be a α-inverse-strongly monotone mapping, where constants $\alpha, m \in (0, \infty)$. Let $f : X \to X$ be a contraction mapping with constant $k \in [0, 1)$ and for any $n \in N$, Let S_n be the $S-$mapping from X into itself generated by $B_n, B_{n-1}, \cdots, B_1$ and $k_n, k_{n-1}, \cdots, k_1$ with $\cap_{n=1}^{\infty} F(B_n) \cap GMVEP(\varphi, \psi, T) \cap VI(A, X) \neq \emptyset$. For $x_0 \in X$, suppose that $\{x_n\}, \{\nu_n\}, \{\mu_n\}$ are generated by

$$
\begin{cases}
\mu_n = T_{r_n}(x_n - r_n T x_n) \\
\nu_n = P_X(\mu_n - \lambda_n A \mu_n) \\
\omega_n = \dfrac{\alpha_n}{1 - \beta_n} f(x_n) + \dfrac{\gamma_n}{1 - \beta_n} S_n y_n \\
x_{n+1} = \beta_n x_n + (1 - \beta_n)\omega_n
\end{cases} \tag{3.1}
$$

for all $n = 0, 1, 2, \cdots$, where $\alpha_n + \beta_n + \gamma_n = 1$ with $\alpha_n > 0$, $\beta_n > 0$, $\gamma_n > 0$, $\{\lambda_n\}$ is a sequence in $[0, b]$ and $\{r_n\}$ is a sequence in $[0, d]$, where $b \in [0, 2\alpha)$ and $d \in [0, 2m)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\liminf_{n\to\infty} \lambda_n > 0$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(3) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(4) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$ converge strongly to the point $z^* \in \cap_{n=1}^{\infty} F(B_n) \cap GMVEP(\varphi, \psi, T) \cap VI(A, X)$, where

$$z^* = P_{\cap_{n=1}^{\infty} F(B_n) \cap GMVEP(\varphi,\psi,T) \cap VI(A,X)} f(z^*).$$

Proof. Since the projection mapping P is nonexpasive and f is contractive, it is easy to see that $P_{\cap_{n=1}^{\infty}F(B_n)\cap GMVEP(\varphi,\psi,T)\cap VI(A,X)}f$ is a contraction mapping and so there exists $z^* \in X$ satisfies

$$z^* = P_{\cap_{n=1}^{\infty}F(B_n)\cap GMVEP(\varphi,\psi,T)\cap VI(A,X)}f(z^*).$$

Let $\nu \in \cap_{n=1}^{\infty}F(B_n)\cap GMVEP(\varphi,\psi,T)\cap VI(A,X)$. For each n, we have

$$\nu = T_{r_n}(\nu - r_n T\nu) = S_n\nu$$

and so

$$\begin{aligned} \|\mu_n - \nu\|^2 &= \|T_{r_n}(x_n - r_nTx_n) - T_{r_n}(\nu - r_nT\nu)\|^2 \\ &\leq \|x_n - r_nTx_n - \nu + r_nT\nu\|^2 \\ &= \|x_n - \nu\|^2 - 2r_n\langle x_n - \nu, Tx_n - T\nu\rangle + r_n^2\|Tx_n - T\nu\|^2 \\ &\leq \|x_n - \nu\|^2 - 2mr_n\|Tx_n - T\nu\|^2 + r_n^2\|Tx_n - T\nu\|^2 \\ &= \|x_n - \nu\|^2 + r_n(r_n - 2m)\|Tx_n - T\nu\|^2 \\ &\leq \|x_n - \nu\|^2. \end{aligned} \tag{3.2}$$

It follows from (2.6) that $\nu = P_X(\nu - \lambda_n A\nu)$. Since A is a α-inverse-strongly monotone mapping and P_X is a nonexpansive mapping, we get

$$\begin{aligned} \|\nu_n - \nu\|^2 &= \|P_X(\mu_n - \lambda_nA\mu_n) - P_X(\nu - \lambda_nA\nu)\|^2 \\ &\leq \|\mu_n - \lambda_nA\mu_n - \nu + \lambda_nA\nu\|^2 \\ &\leq \|\mu_n - \nu\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A\mu_n - A\nu\|^2 \\ &\leq \|\mu_n - \nu\|^2. \end{aligned} \tag{3.3}$$

Thus, (3.2) and (3.3) imply that

$$\|\nu_n - \nu\| \leq \|\mu_n - \nu\| \leq \|x_n - \nu\|.$$

Let

$$M = \max\{\|x_n - \nu\|, \frac{1}{1+k}\|f(\nu) - \nu\|\}.$$

Obviously, we know that $\|x_1 - \nu\| \leq M$. Suppose $\|x_n - \nu\| \leq M$. Then

$$
\begin{aligned}
&\|x_{n+1}-\nu\| \\
&= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n\nu_n-\nu\| \\
&\leq \alpha_n\|f(x_n)-\nu\|+\beta_n\|x_n-\nu\|+\gamma_n\|S_n\nu_n-\nu\| \\
&\leq \alpha_n\|f(x_n)-f(\nu)\|+\alpha_n\|f(\nu)-\nu\|+\beta_n\|x_n-\nu\|+\gamma_n\|S_n\nu_n-\nu\| \\
&\leq \alpha_n k\|x_n-\nu\|+\alpha_n\|f(\nu)-\nu\|+\beta_n\|x_n-\nu\|+\gamma_n\|\nu_n-\nu\| \\
&\leq (\alpha_n k+\beta_n)\|x_n-\nu\|+\alpha_n\|f(\nu)-\nu\|+\gamma_n\|x_n-\nu\| \\
&= (\alpha_n k+\beta_n+\gamma_n)\|x_n-\nu\|+\alpha_n\|f(\nu)-\nu\| \\
&= (1-\alpha_n(1-k))\|x_n-\nu\|+\alpha_n(1-k)\frac{\|f(\nu)-\nu\|}{1-k} \\
&\leq (1-\alpha_n(1-k))M+\alpha_n(1-k)M \\
&= M
\end{aligned} \tag{3.4}
$$

and so $\|x_{n+1}-\nu\| \leq M$ for all $n \in N$. Furthermore, we know that $\{x_n\}$ is bounded and so do $\{\mu_n\}$, $\{\nu_n\}$, $\{fx_n\}$, $\{A\mu_n\}$, $\{S_n\nu_n\}$, $\{\omega_n\}$ and $\{Tx_n\}$.

Next we prove that $\|x_{n+1}-x_n\| \to 0$. In fact, it follows from

$$\mu_n = T_{r_n}(x_n - r_nTx_n), \quad \mu_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1}Tx_{n+1})$$

that, for any $y \in X$,

$$\varphi(\mu_n, y) + \psi(y) - \psi(\mu_n) + e\langle Tx_n, y-\mu_n\rangle + \frac{e}{r_n}\langle y-\mu_n, \mu_n - x_n\rangle \in C \tag{3.5}$$

and

$$
\begin{aligned}
&\varphi(\mu_{n+1}, y) + \psi(y) - \psi(\mu_{n+1}) + e\langle Tx_{n+1}, y-\mu_{n+1}\rangle \\
&+\frac{e}{r_{n+1}}\langle y-\mu_{n+1}, \mu_{n+1}-x_{n+1}\rangle \in C.
\end{aligned} \tag{3.6}
$$

Letting $y = \mu_{n+1}$ in (3.5) and $y = \mu_n$ in (3.6) and adding the above two inequalities, we get

$$e\langle \mu_{n+1}-\mu_n, \frac{1}{r_n}(\mu_n - x_n + r_nTx_n) - \frac{1}{r_{n+1}}(\mu_{n+1}-x_{n+1}+r_{n+1}Tx_{n+1})\rangle \in C. \tag{3.7}$$

Since $e \in C$ and C is a closed convex cone, we obtain

$$\langle \mu_{n+1}-\mu_n, \mu_n - x_n + r_nTx_n - \frac{r_n}{r_{n+1}}(\mu_{n+1}-x_{n+1}+r_{n+1}Tx_{n+1})\rangle \geq 0, \tag{3.8}$$

which implies that

$$
\begin{aligned}
&\langle \mu_{n+1}-\mu_n, \mu_n-\mu_{n+1}+\mu_{n+1}-x_n+r_nTx_n \\
&-\frac{r_n}{r_{n+1}}(\mu_{n+1}-x_{n+1}+r_{n+1}Tx_{n+1})\rangle \geq 0
\end{aligned} \tag{3.9}
$$

and so

$$
\begin{aligned}
(3.10)\qquad \|\mu_{n+1}-\mu_n\|^2 &\le \langle \mu_{n+1}-\mu_n, x_{n+1}-x_n \\
&\quad +r_nTx_n - r_nTx_{n+1} + (1-\frac{r_n}{r_{n+1}})(\mu_{n+1}-x_{n+1})\rangle \\
&\le \|\mu_{n+1}-\mu_n\|\|(x_{n+1}-r_nTx_{n+1}) \\
&\quad -(x_n-r_nTx_n)\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}R,
\end{aligned}
$$

where

$$R = \sup\{\|\mu_{n+1}\| + \|x_{n+1}\| + \|Tx_{n+1}\|, \quad n=0,1,2,\cdots\}.$$

For any $x,\ y \in X$ and $r \in (0,2d)$, we have

$$
\begin{aligned}
(3.11)\qquad &\|(I-rT)x-(I-rT)y\|^2 \\
&= \|(x-y)-r(Tx-Ty)\|^2 \\
&= \|x-y\|^2 - 2r\langle x-y, Tx-Ty\rangle + r^2\|Tx-Ty\|^2 \\
&\le \|x-y\|^2 - 2rm\|Tx-Ty\|^2 + r^2\|Tx-Ty\|^2 \\
&\le \|x-y\|^2 + r(r-2m)\|Tx-Ty\|^2 \\
&\le \|x-y\|^2.
\end{aligned}
$$

From (3.10) and (3.11), we get

$$\|\mu_{n+1}-\mu_n\| \le \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}R.$$

On the other hand, it follows that

$$
\begin{aligned}
(3.12)\qquad &\|\nu_{n+1}-\nu_n\| \\
&= \|P_X(\mu_{n+1}-\lambda_{n+1}A\mu_{n+1}) - P_X(\mu_n-\lambda_nA\mu_n)\| \\
&\le \|\mu_{n+1}-\lambda_{n+1}A\mu_{n+1}-\mu_n+\lambda_nA\mu_n\| \\
&= \|\mu_{n+1}-\lambda_{n+1}A\mu_{n+1}-(\mu_n-\lambda_{n+1}A\mu_n)-(\lambda_{n+1}A\mu_n-\lambda_nA\mu_n)\| \\
&\le \|(\mu_{n+1}-\lambda_{n+1}A\mu_{n+1})-(\mu_n-\lambda_{n+1}A\mu_n)\| + |\lambda_{n+1}-\lambda_n|\|A\mu_n\| \\
&\le \|\mu_{n+1}-\mu_n\| + |\lambda_{n+1}-\lambda_n|\|A\mu_n\| \\
&\le \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}R + K|\lambda_{n+1}-\lambda_n|.
\end{aligned}
$$

where

$$K = \sup\{\|f(x_n)\| + \|S_n\nu_n\| + \|x_n\| + \|A\mu_n\|, n \in N\}.$$

Obviously, we get that

$$
\begin{aligned}
&\|\omega_{n+1}-\omega_n\| \\
&= \left\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})+\frac{\gamma_{n+1}}{1-\beta_{n+1}}S_{n+1}\nu_{n+1}-\frac{\alpha_n}{1-\beta_n}f(x_n)-\frac{\gamma_n}{1-\beta_n}S_n\nu_n\right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|+\|f(x_n)\|\cdot\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right| \\
&\quad +\|S_{n+1}\nu_{n+1}\|\cdot\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|+\frac{\gamma_{n+1}}{1-\gamma_{n+1}}\|S_{n+1}\nu_{n+1}-S_n\nu_n\| \\
&\le \frac{k\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+2K\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right| \\
&\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}(\|\nu_{n+1}-\nu_n\|+\|S_{n+1}\nu_n-S_n\nu_n\|) \\
&\le \frac{k\alpha_{n+1}+\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+2K\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right| \\
&\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(\frac{|r_{n+1}-r_n|}{r_{n+1}}R+K|\lambda_{n+1}-\lambda_n|+\|S_{n+1}\nu_n-S_n\nu_n\|\right)
\end{aligned} \tag{3.13}
$$

and so

$$
\begin{aligned}
&\|\omega_{n+1}-\omega_n\|-\|x_{n+1}-x_n\| \\
&\le \frac{\alpha_{n+1}(k-1)}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+2K\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right| \\
&\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(\frac{|r_{n+1}-r_n|}{r_{n+1}}R+K|\lambda_{n+1}-\lambda_n|+\|S_{n+1}\nu_n-S_n\nu_n\|\right)
\end{aligned} \tag{3.14}
$$

It is easy to check that

$$\lim_{n\to\infty}\frac{\alpha_{n+1}(k-1)}{1-\beta_{n+1}}=0,\quad \lim_{n\to\infty}|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|=0$$

and

$$\lim_{n\to\infty}\frac{|r_{n+1}-r_n|}{r_{n+1}}R+K|\lambda_{n+1}-\lambda_n|+\|S_{n+1}\nu_n-S_n\nu_n\|=0.$$

It follows from (3.14) that

$$\limsup_{n\to\infty}(\|\omega_{n+1}-\omega_n\|-\|x_{n+1}-x_n\|)\le 0$$

From Lemma 2.4, we have

$$\lim_{n\to\infty}\|\omega_n-x_n\|=0$$

and so

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0,\quad \lim_{n\to\infty}\|\nu_{n+1}-\nu_n\|=0,\quad \lim_{n\to\infty}\|\mu_{n+1}-\mu_n\|=0.$$

Now, we prove that $\|S_n\nu_n-\nu_n\|\to 0$. In fact, it follows that

$$\begin{aligned}
\|x_n-S_n\nu_n\| &\le \|x_n-S_{n-1}\nu_{n-1}\|+\|S_n\nu_n-S_{n-1}\nu_{n-1}\|\\
&= \|\alpha_{n-1}f(x_{n-1})+\beta_{n-1}x_{n-1}+\gamma_{n-1}S_{n-1}\nu_{n-1}\\
&\quad -S_{n-1}\nu_{n-1}\|+\|S_n\nu_n-S_n\nu_{n-1}\|\\
&\le |\beta_{n-1}|\|x_{n-1}-S\nu_{n-1}\|+|\alpha_{n-1}|\|f(x_{n-1})-S_{n-1}\nu_{n-1}\|\\
&\quad +\|\nu_n-\nu_{n-1}\|+\|S_n\nu_{n-1}-S_{n-1}\nu_{n-1}\|\\
&= \left(1-\frac{1-|\beta_{n-1}|}{1-K}(1-K)\right)\|x_{n-1}-S_{n-1}\nu_{n-1}\|\\
&\quad +|\alpha_{n-1}|\|f(x_{n-1})-S_{n-1}\nu_{n-1}\|\\
&\quad +\|\nu_n-\nu_{n-1}\|+\|S_n\nu_{n-1}-S_{n-1}\nu_{n-1}\|.
\end{aligned}\tag{3.15}$$

From the conditions of Theorem 3.1, we get

$$1-\frac{1-|\beta_{n-1}|}{1-K}(1-K)\in[0,1],\quad \Sigma_{n=1}^{\infty}(1-|\beta_{n-1}|)=\infty$$

and

$$\lim_{n\to\infty}\|\nu_n-\nu_{n-1}\|=0,\quad \lim_{n\to\infty}|\alpha_{n-1}|\|f(x_{n-1})-S\nu_{n-1}\|=0.$$

Thus, we have

$$\lim_{n\to\infty}\frac{\|\nu_n-\nu_{n-1}\|+|\alpha_{n-1}|\|f(x_{n-1})-S\nu_{n-1}\|+\|S_n\nu_{n-1}-S_{n-1}\nu_{n-1}\|}{1-|\beta_{n-1}|}=0.$$

From Lemma 2.5, we know that $\|x_n-S_n\nu_n\|\to 0$. It follows from (3.3) that

$$\begin{aligned}
&\|x_{n+1}-\nu\|^2\\
&= \|\alpha_nf(x_n)+\beta_nx_n+\gamma_nS_n\nu_n-\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|S_n\nu_n-\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|\nu_n-\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\{\|\mu_n-\nu\|^2\\
&\quad +\lambda_n(\lambda_n-2\alpha)\|A\mu_n-A\nu\|^2\}\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-\nu\|^2+\gamma_n\lambda_n(\lambda_n-2\alpha)\|A\mu_n-A\nu\|^2,
\end{aligned}\tag{3.16}$$

which implies that

$$
\begin{aligned}
&\gamma_n\lambda_n(2\alpha-\lambda_n)\|A\mu_n-A\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-\nu\|^2-\|x_{n+1}-\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\|x_{n+1}-x_n\|(\|x_n-\nu\|-\|x_{n+1}-\nu\|)
\end{aligned}
\tag{3.17}
$$

for $n=0,1,2,\cdots$. Since

$$\liminf_{n\to\infty}\gamma_n>0,\quad \liminf_{n\to\infty}\lambda_n>0,\quad \|x_{n+1}-x_n\|\to 0,\quad \alpha_n\to 0,\quad 2\alpha-\lambda_n>0$$

and $\{x_n\}$ is bounded, it follows from (3.17) that $\|A\mu_n-A\nu\|\to 0$. Since $\nu\in VI(A,X)$, we have

$$
\begin{aligned}
\|\nu_n-\nu\|^2 &= \|P_X(\mu_n-\lambda_nA\mu_n)-P_X(\nu-\lambda_nA\nu)\|^2\\
&\le \langle(\mu_n-\lambda_nA\mu_n)-(\nu-\lambda_nA\nu),\nu_n-\nu\rangle\\
&= \frac{1}{2}\{\|(\mu_n-\lambda_nA\mu_n)-(\nu-\lambda_nA\nu)\|^2+\|\nu_n-\nu\|^2\\
&\quad -\|(\mu_n-\nu_n)-\lambda_n(A\mu_n-A\nu)\|^2\}\\
&\le \frac{1}{2}\{\|\mu_n-\nu\|^2+\|\nu_n-\nu\|^2-(\|\mu_n-\nu_n\|^2\\
&\quad -2\lambda_n\langle\mu_n-\nu_n,A\mu_n-A\nu\rangle+\lambda_n^2\|A\mu_n-A\nu\|^2)\}
\end{aligned}
\tag{3.18}
$$

and so

$$
\begin{aligned}
\|\nu_n-\nu\|^2 \le\ & \|\mu_n-\nu\|^2-\|\mu_n-\nu_n\|^2\\
&+2\lambda_n\langle\mu_n-\nu_n,A\mu_n-A\nu\rangle-\lambda_n^2\|A\mu_n-A\nu\|^2
\end{aligned}
\tag{3.19}
$$

From (3.19), we obtain

$$
\begin{aligned}
&\|x_{n+1}-\nu\|^2\\
&= \|\alpha_nf(x_n)+\beta_nx_n+\gamma_nS_n\nu_n-\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|S_n\nu_n-\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|\nu_n-\nu\|^2\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\{\|\mu_n-\nu\|^2-\|\mu_n-\nu_n\|^2\\
&\quad +2\lambda_n\langle\mu_n-\nu_n,A\mu_n-A\nu\rangle-\lambda_n^2\|A\mu_n-A\nu\|^2\}\\
&\le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-\nu\|^2-\gamma_n\|\mu_n-\nu_n\|^2\\
&\quad +2\gamma_n\lambda_n\langle\mu_n-\nu_n,A\mu_n-A\nu\rangle-\gamma_n\lambda_n^2\|A\mu_n-A\nu\|^2,
\end{aligned}
\tag{3.20}
$$

which implies that

$$\gamma_n\|\mu_n-\nu_n\|^2 \le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-x_{n+1}\|(\|x_n-\nu\|+\|x_{n+1}-\nu\|) \\ +2\gamma_n\lambda_n\|A\mu_n-A\nu\|\|\mu_n-\nu_n\|. \tag{3.21}$$

Since $\liminf_{n\to\infty}\gamma_n>0$, $\{x_n\}$, $\{\mu_n\}$ and $\{\nu_n\}$ are all bounded, $\alpha_n\to 0$, $\|A\mu_n - A\nu\|\to 0$ and $\|x_n-x_{n+1}\|\to 0$, it follows from (3.21) that $\|\mu_n-\nu_n\|\to 0$.

On the other hand, by the definition of $\{x_n\}$, we have

$$\begin{aligned} &\|x_{n+1}-\nu\|^2 \\ &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n\nu_n-\nu\|^2 \\ &\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|S_n\nu_n-\nu\|^2 \\ &\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|\nu_n-\nu\|^2 \\ &\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|\mu_n-\nu\|^2 \\ &\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2 \\ &\quad +\gamma_n\{\|x_n-\nu\|^2+\lambda_n(r_n-2m)\|Tx_n-T\nu\|^2\} \\ &\le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-\nu\|^2-\gamma_n r_n(2m-r_n)\|Tx_n-T\nu\|^2 \end{aligned} \tag{3.22}$$

and so

$$\begin{aligned} &\gamma_n r_n(2m-r_n)\|Tx_n-T\nu\|^2 \\ &\le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-\nu\|^2-\|x_{n+1}-\nu\|^2 \\ &\le \alpha_n\|f(x_n)-\nu\|^2+\|x_{n+1}-x_n\|(\|x_n-\nu\|-\|x_{n+1}-\nu\|) \end{aligned} \tag{3.23}$$

Since $\liminf_{n\to\infty}\gamma_n>0$, $\liminf_{n\to\infty}r_n>0$, $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $2m-r_n>0$, both $\|f(x_n)-\nu\|$ and $\|x_n-\nu\|-\|x_{n+1}-\nu\|$ are bounded, we get $\|Tx_n-T\nu\|\to 0$. By $\nu=T_{r_n}(\nu-r_nT\nu)$, we have

$$\begin{aligned} \|\mu_n-\nu\|^2 &= \|T_{r_n}(x_n-r_nTx_n)-T_{r_n}(\nu-r_nT\nu)\|^2 \\ &\le \langle (x_n-r_nTx_n)-(\nu-r_nT\nu), \mu_n-\nu\rangle \\ &= \frac{1}{2}\{\|\mu_n-\nu\|^2+\|(x_n-r_nTx_n)-(\nu-r_nT\nu)\|^2 \\ &\quad -\|x_n-\mu_n-r_n(Tx_n-T\nu)\|^2\} \\ &\le \frac{1}{2}\{\|x_n-\nu\|^2+\|\mu_n-\nu\|^2-(\|x_n-\mu_n\|^2 \\ &\quad -2r_n\langle x_n-\mu_n, Tx_n-T\nu\rangle+r_n^2\|Tx_n-T\nu\|^2)\} \end{aligned} \tag{3.24}$$

and so

$$\begin{aligned} \|\mu_n-\nu\|^2 \le\ & \|x_n-\nu\|^2-\|x_n-\mu_n\|^2 \\ & +2r_n\langle x_n-\mu_n, Tx_n-T\nu\rangle-r_n^2\|Tx_n-T\nu\|^2. \end{aligned} \tag{3.25}$$

Now (3.4) and (3.25) imply that

$$
\begin{aligned}
&\|x_{n+1}-\nu\|^2 \\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|\nu_n-\nu\|^2 \\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\|\mu_n-\nu\|^2 \\
&\le \alpha_n\|f(x_n)-\nu\|^2+\beta_n\|x_n-\nu\|^2+\gamma_n\{\|x_n-\nu\|^2-\|\mu_n-x_n\|^2 \\
&\quad +2r_n\langle x_n-\mu_n, Tx_n-T\nu\rangle-r_n^2\|Tx_n-T\nu\|^2\} \\
&\le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-\nu\|^2-\gamma_n\|\mu_n-x_n\|^2 \\
&\quad +2\gamma_n r_n\langle x_n-\mu_n, Tx_n-T\nu\rangle
\end{aligned}
\tag{3.26}
$$

and so

$$
\begin{aligned}
\gamma_n\|\mu_n-x_n\|^2 &\le \alpha_n\|f(x_n)-\nu\|^2+\|x_n-\nu\|^2-\|x_{n+1}-\nu\| \\
&\quad +2\gamma_n r_n\langle x_n-\mu_n, Tx_n-T\nu\rangle \\
&\le \alpha_n\|f(x_n)-\nu\|^2+2\gamma_n r_n\langle x_n-\mu_n, Tx_n-T\nu\rangle \\
&\quad +\|x_{n+1}-x_n\|(\|x_n-\nu\|-\|x_{n+1}-\nu\|).
\end{aligned}
\tag{3.27}
$$

Since $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$, we know that $\|\mu_n-x_n\|\to 0$. It follows from

$$\|S_n\nu_n-\nu_n\|\le\|S_n\nu_n-x_n\|+\|x_n-\mu_n\|+\|\mu_n-\nu_n\|$$

that $\lim_{n\to\infty}\|S_n\nu_n-\nu_n\|=0$ and so $\lim_{n\to\infty}\|S\nu_n-\nu_n\|=0$.

Next we show that

$$\limsup_{n\to\infty}\langle f(z^*)-z^*, x_n-z^*\rangle\le 0,$$

where z^* is a fixed point of $P_{\cap_{n=1}^{\infty}F(B_n)\cap GMVEP(\varphi,\psi,T)\cap VI(A,X)}f$. We choose a subsequence $\{\nu_{n_i}\}$ of $\{\nu_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z^*)-z^*, S\nu_n-z^*\rangle=\lim_{i\to\infty}\langle f(z^*)-z^*, \nu_{n_i}-z^*\rangle.$$

Since $\{\nu_n\}$ is bounded, there exists a weakly convergent subsequence $\nu_{n_{i_j}}$ of ν_{n_i} such that $\nu_{n_{i_j}}\rightharpoonup z_0$. Without loss of generality, we can assume that $\nu_{n_i}\rightharpoonup z_0$. By the facts that

$$\lim_{n\to\infty}\|S\nu_n-\nu_n\|=0,\quad \lim_{n\to\infty}\|\nu_n-\mu_n\|=0,\quad \lim_{n\to\infty}\|x_n-\mu_n\|=0,$$

we get

$$S\nu_{n_i}\rightharpoonup z_0,\quad \mu_{n_i}\rightharpoonup z_0,\quad x_{n_i}\rightharpoonup z_0.$$

Now we show that $z_0 \in \cap_{n=1}^{\infty} F(B_n) \cap GMVEP(\varphi, \psi, T) \cap VI(A, X)$. We first prove that $z_0 \in \cap_{n=1}^{\infty} F(B_n)$. Assume that $z_0 \notin \cap_{n=1}^{\infty} F(B_n) = F(S)$, that is, $z_0 \neq S(z_0)$. Since $x_{n_i} \rightharpoonup z_0$ and H satisfies the Opial condition, we get

$$\begin{aligned}
&\liminf_{i\to\infty} \|x_{n_i} - z_0\| \\
&< \liminf_{i\to\infty} \|x_{n_i} - Sz_0\| \\
&\leq \liminf_{i\to\infty}(\|x_{n_i} - Sx_{n_i}\| + \|Sx_{n_i} - Sz_0\|) \\
&\leq \liminf_{i\to\infty}(\|x_{n_i} - S\nu_{n_i}\| + \|Sx_{n_i} - S\nu_{n_i}\| + \|Sx_{n_i} - Sz_0\|) \\
&\leq \liminf_{i\to\infty}(\|x_{n_i} - S_{n_i}\nu_{n_i}\| + \|S_{n_i}\nu_{n_i} - S\nu_{n_i}\| + \|x_{n_i} - \nu_{n_i}\| + \|Sx_{n_i} - Sz_0\|) \\
&\leq \liminf_{i\to\infty} |x_{n_i} - z_0\|,
\end{aligned}$$

which is a contradiction. Thus, $z_0 \in \cap_{n=1}^{\infty} F(B_n)$. Next, we show that $z_0 \in GMVEP(\varphi, \psi, T)$. In fact, since $\mu_n = T_{r_n}(x_n - r_n T x_n)$, we have

$$\begin{aligned}
&\varphi(\mu_n, y) + \psi(y) - \psi(\mu_n) + e\langle Tx_n, y - \mu_n\rangle \\
&\quad + \frac{e}{r_n}\langle y - \mu_n, \mu_n - x_n\rangle \in C, \quad \forall y \in X,
\end{aligned} \tag{3.28}$$

which implies that

$$\begin{aligned}
&0 \in \varphi(y, \mu_n) - \{\psi(y) - \psi(\mu_n) + e\langle Tx_n, y - \mu_n\rangle \\
&\quad + \frac{e}{r_n}\langle y - \mu_n, \mu_n - x_n\rangle\} + C, \quad \forall y \in X.
\end{aligned} \tag{3.29}$$

Let $y_t = (1-t)z_0 + ty$ for all $t \in (0, 1]$. Since $y \in X$ and $z_0 \in X$, we get $y_t \in X$ and now (3.28) shows that

$$\begin{aligned}
&e\langle y_t - \mu_{n_i}, Ty_t\rangle \in \varphi(y_t, \mu_{n_i}) - (\psi(y_t) - \psi(\mu_{n_i})) - e\langle Tx_{n_i}, y_t - \mu_{n_i}\rangle \\
&\quad - \frac{e}{r_{n_i}}\langle y_t - \mu_{n_i}, \mu_{n_i} - x_{n_i}\rangle + e\langle y_t - \mu_{n_i}, Ty_t\rangle + C \\
&= \varphi(y_t, \mu_{n_i}) + e\langle y_t - \mu_{n_i}, Ty_t - T\mu_i\rangle + e\langle y_t - \mu_{n_i}, T\mu_{n_i} - Tx_{n_i}\rangle \\
&\quad - (\psi(y_t) - \psi(\mu_{n_i})) - e\langle y_t - \mu_{n_i}, \frac{\mu_{n_i} - x_{n_i}}{r_{n_i}}\rangle + C.
\end{aligned} \tag{3.30}$$

By the fact $\|\mu_n - x_n\| \to 0$ and the properties of T and φ, we have

$$\|T\mu_{n_i} - Tx_{n_i}\| \to 0, \quad \frac{\mu_{n_i} - x_{n_i}}{r_{n_i}} \to 0, \quad \langle y_t - \mu_{n_i}, T\mu_{n_i} - Tx_{n_i}\rangle \to 0.$$

and so

$$e\langle y_t - z_0, Ty_t\rangle \in \varphi(y_t, z_0) - (\psi(y_t) - \psi(z_0)) + C. \tag{3.31}$$

It follows from (A_1), (A_4) and (A_6) that

$$\begin{aligned} & t\varphi(y_t, y) + (1-t)\varphi(y_t, z_0) + t\psi(y) + (1-t)\psi(z_0) - \psi(y_t) \\ \in\ & \varphi(y_t, y_t) + \psi(y_t) - \psi(y_t) + C \\ =\ & C, \end{aligned}$$

which implies that

$$-t[\varphi(y_t, y) + \psi(y) - \psi(y_t)] - (1-t)[\varphi(y_t, z_0) + \psi(z_0) - \psi(y_t)] \in -C. \tag{3.32}$$

From (3.30) and (3.31), we get

$$\begin{aligned} -t[\varphi(y_t, y) + \psi(y) - \psi(y_t)] &\in (1-t)[\varphi(y_t, z_0) + \psi(z_0) - \psi(y_t)] - C \\ &\in (1-t)e\langle y_t - z_0, Ty_t\rangle - C \end{aligned}$$

and so

$$-t[\varphi(y_t, y) + \psi(y) - \psi(y_t)] - e(1-t)t\langle y - z_0, Ty_t\rangle \in -C.$$

It follows that

$$\varphi(y_t, y) + \psi(y) - \psi(y_t) + e(1-t)\langle y_t - z_0, Ty_t\rangle \in C.$$

Letting $t \to 0$, we obtain

$$\varphi(z_0, y) + \psi(y) - \psi(z_0) + e\langle Tz_0, y - z_0\rangle \in C, \quad \forall y \in X$$

and so $z_0 \in GMVEP(\varphi, \psi, T)$. Now, we show that $z_0 \in VI(A, X)$. Define

$$\Phi\mu = \begin{cases} A\mu + N_X\mu, & \mu \in X, \\ \emptyset, & \mu \notin X. \end{cases}$$

Then we know that Φ is a maximal monotone mapping (see [19]). Letting $(\mu, \omega) \in G(\Phi)$, we have

$$\langle \mu - \nu_n, \omega\rangle \geq 0$$

and so

$$\langle \mu - z_0, \omega\rangle \geq 0.$$

Since $\nu_n = P_X(\mu_n - \lambda_n A\mu_n)$ and $\omega - A\mu \in N_X\mu$, we have $\langle \mu - \nu_n, \omega - A\mu\rangle \geq 0$. Furthermore,

$$\begin{aligned} \langle \mu - \nu_{n_i}, \omega\rangle &\geq \langle \mu - \nu_{n_i}, A\mu\rangle \\ &\geq \langle \mu - \nu_{n_i}, \frac{\nu_{n_i} - \mu_{n_i}}{\lambda_{n_i}} + A\mu_{n_i}\rangle \\ &= \langle \mu - \nu_{n_i}, A\mu - A\nu_{n_i}\rangle + \langle \mu - \nu_{n_i}, A\nu_{n_i} - A\mu_{n_i}\rangle - \langle \mu - \nu_{n_i}, \frac{\nu_{n_i} - \mu_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle \mu - \nu_{n_i}, A\nu_{n_i} - A\mu_{n_i}\rangle - \langle \mu - \nu_{n_i}, \frac{\nu_{n_i} - \mu_{n_i}}{\lambda_{n_i}}\rangle. \end{aligned}$$

Since A is a Lipschitz continuous mapping and $\|\mu_n - \nu_n\| \to 0$, we get

$$\langle \mu - z_0, \omega \rangle \geq 0$$

and so $z_0 \in VI(A, X)$. From (3.28), we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(z^*) - z^*, x_n - z^* \rangle &= \limsup_{n\to\infty} \langle f(z^*) - z^*, S_n\nu_n - z^* \rangle \\ &= \lim_{i\to\infty} \langle f(z^*) - z^*, Sx_{n_i} - z^* \rangle \\ &= \langle f(z^*) - z^*, z_0 - z^* \rangle \leq 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_{n+1} - z^*\|^2 \\ =\ &\langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n\nu_n - z^*, x_{n+1} - z^* \rangle \\ \leq\ &\alpha_n \langle f(x_n) - f(z^*), x_{n+1} - z^* \rangle + \alpha_n \langle f(z^*) - z^*, x_{n+1} - z^* \rangle \\ &+\frac{1}{2}\beta_n(\|x_{n+1} - z^*\|^2 + \|x_n - z^*\|^2) + \frac{1}{2}\gamma_n(\|S_n\nu_n - z^*\|^2 + \|x_{n+1} - z^*\|^2) \\ \leq\ &\frac{1}{2}\alpha_n(\|f(x_n) - f(z^*)\|^2 + \|x_{n+1} - z^*\|^2) + \alpha_n\langle f(z^*) - z^*, x_{n+1} - z_0 \rangle \\ &+\frac{1}{2}(1 - \alpha_n)(\|x_{n+1} - z^*\|^2 + \|x_n - z^*\|^2) \\ \leq\ &\frac{1}{2}\|x_{n+1} - z^*\|^2 + \frac{1}{2}[1 - \alpha_n(1 - k^2)]\|x_n - z^*\|^2 + \alpha_n\langle f(z^*) - z^*, x_{n+1} - z^* \rangle,\end{aligned}$$

which implies that

$$\|x_{n+1} - z^*\|^2 \leq [1 - \alpha_n(1 - k^2)]\|x_n - z^*\|^2 + 2\alpha_n\langle f(z^*) - z^*, x_{n+1} - z^* \rangle.$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$ and

$$\limsup_{n\to\infty} \langle f(z^*) - z^*, x_n - z^* \rangle \leq 0,$$

It follows from Lemma 2.5 that

$$\|x_n - z^*\| \to 0$$

and so

$$\|\mu_n - z^*\| \to 0, \quad \|\nu_n - z^*\| \to 0.$$

This completes the proof. ■

Remark 3.2. Theorem 3.1 extends and improves Theorem 3.1 of Peng and Yao [17].

ACKNOWLEDGMENTS

The authors are grateful to the editor and referee for their valuable comments and suggestions.

REFERENCES

1. Q. H. Ansari and F. Flores-Bazan, Recession methods for generalized vector equilibrium problems, *J. Math. Anal. Appl.*, **321** (2006), 132-146.
2. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63** (1994), 123-145.
3. F. E. Brower and W. V. Petryshyn, Construction of fixe points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **20** (1967), 197-228.
4. G. Chen, X. Huang and X. Yang, Vector Optimization: Set-Valued and Variational Analysis, Vol. 541, Lecture Notes in Economics and Mathematical Systems, Springer, Berlin, Germany, 2005.
5. F. Giannessi, ed., Vector Variational Inequalities and Vector Equilibrium, Vol. 38, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
6. X. H. Gong and H. M. Yue, Existence of efficient solutions and strong solutions for vector equilibrium problems, *J. Nanchang Univ.*, **32** (2008), 1-5.
7. N. J. Huang, H. Y. Lan and K. L. Teo, On the exisitence and convergence of approximate solutions for equilibrium problems in Banach spaces, *J. Inequal. Appl.*, 2007 (2007), Article ID 17294, 14 pages.
8. H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.*, **61** (2005), 341-350.
9. A. N. Iusem and W. Sosa, Iterative algorithms for equilibrium problems, *Optimization*, **52** (2003), 301-306.
10. J. Li, N. J. Huang and J. K. Kim, On implicit vector equilibrium problems, *J. Math. Anal. Appl.*, **283** (2003), 501-502.
11. Q. Y. Li and S. H. Wang, Viscosity approximation methods for strong vector equilibrium problems and fixed point problems, *J. Nanchang Univ.*, **26** (2003), 1-5.
12. Q. Y. Liu, W. Y. Zeng and N. J. Huang, An iterative method for generalized equilibrium problems, fixed point problems and variational inequality problems, *Fixed Point Theory Appl.*, 2009 (2009), Article ID 531308, 20 pages.
13. D. T. Luc, Theory of Vector Opimization *Lecture Notes in economics and Mathematical Systems*, Vol. 319, Springer-Verlag, Berlin, 1989.
14. A. Moudafi, Viscosity approximation methods fixed-point problems, *J. Math. Anal. Appl.*, **241** (2000), 46-55.

15. Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings *Bull. Amer. Math. Soc.*, **73** (1967), 561-597.

16. J. W. Peng and J. C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems, *Taiwan. J. Math.*, **12** (2008), 1401-1432.

17. J. W. Peng and J. C. Yao, Two extragradient methods for generalized mixed equilibrium problems, nonexpansive mappings and monotone mappings, *Comput. Math. Appl.*, **58** (2009), 1287-1301.

18. S. Plubtieng and R. Punpaeng, A new iterative Method for equilibrium problems and fixed point problems of nonlinear mappings and monotone mappings, *Appl. Math. Comput.*, **197** (2008), 548-558.

19. R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.*, **149** (1970), 75-88.

20. S. Shimoji and W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, *Taiwan. J. Math.*, **5** (2001), 387-404.

21. Y. F. Su, M. J. Shang and X. L. Qin, An iterative Method of solution for equilibrium and optimization problems, *Nonlinear Anal.*, **69** (2008), 2709-2719.

22. T. Suzuki, Strong convergence theorems for infinite families of nonexpansive mappings in general Banach space, *Fixed Point Theory Appl.*, **2005** (2005), 103-123.

23. S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hiblert spaces, *J. Math. Anal. Appl.*, **331** (2007), 506-515.

24. W. Takahashi, *Nonlinear Function Analysis*, Yokohama, 2000.

25. W. Takahashi and M. Toyoda, Weak convergence theorema for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **118** (2003), 417-428.

26. N. X. Tan, On the existence of solutions of quasivariational inclusion problems, *J. Optim. Theory Appl.*, **123** (2004), 619-638.

27. H. K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.*, **116** (2003), 659-678.

28. Y. Yao and J. C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.*, **186** (2007), 1551-1558.

Shu-qiang Shan and Nan-jing Huang
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
P. R. China
E-mail: nanjinghuang@hotmail.com