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SOME REMARKS ON MARCINKIEWICZ INTEGRALS ALONG SUBMANIFOLDS

Wenjuan Li and Kôzô Yabuta

Abstract. We investigate the L^p boundedness for a class of parametric Marcinkiewicz integral operators associated to submanifolds under the $L(\log L)^\alpha(S^{n-1})$ or Block space condition on the kernel functions. Our results improve the recent results by Al-Qassem and Pan in Studia Mathematica.

1. Introduction

The L^p boundedness of Marcinkiewicz integrals has attracted the attention of many authors in the recent years [1-4, 10, 21]. Our main object in this paper is to improve the recent results by Al-Qassem and Pan [4] about the L^p boundedness for a class of parametric Marcinkiewicz integral operators associated to submanifolds.

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma = d\sigma(\cdot)$. For $x \in \mathbb{R}^n \setminus \{0\}$, let $x' = x/|x|$. Let Ω be a function in $L^1(S^{n-1})$ satisfying the cancellation condition

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{1.1}$$

For $1 \leq \gamma \leq \infty$, let $\Delta_\gamma(\mathbb{R}_+)$ denote the collection of all measurable functions h : $[0, \infty) \to \mathbb{C}$ satisfying $\|h\|_{\Delta_\gamma} = \sup_{R>0}\big(R^{-1}\int_0^R |h(t)|^\gamma dt\big)^{1/\gamma} < \infty$. We note that

$$L^\infty(\mathbb{R}_+) \subset \Delta_\beta(\mathbb{R}_+) \subset \Delta_\alpha(\mathbb{R}_+) \quad \text{for } \alpha < \beta,$$
$$L^\gamma(\mathbb{R}_+, dt/t) \subset \Delta_\gamma(\mathbb{R}_+) \quad \text{for } 1 \leq \gamma < \infty,$$

and all these inclusions are proper. Let $L(\log L)^\alpha(S^{n-1})$ (for $\alpha > 0$) denote the class of all measurable functions Ω which satisfy

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$$\|\Omega\|_{L(\log L)^\alpha(S^{n-1})} = \int_{S^{n-1}} |\Omega(y')| \log^\alpha(2 + |\Omega(y')|)\, d\sigma(y') < \infty.$$

For $q \geq 1$, let $B_q^{(0,\gamma)}(S^{n-1})$ denote the block space generated by q-blocks (its precise definition will be given in Section 3).

In this paper, we are concerned with parametric Marcinkiewicz integral operators of the form

$$\mu^\rho_{\Omega,\phi,\psi,h} f(x, x_{n+1})$$
$$= \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|y| \leq t} f\big(x - \phi(|y|)y', x_{n+1} - \psi(|y|)\big) \frac{\Omega(y')}{|y|^{n-\rho}} h(|y|)\, dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

where $\rho > 0$, $(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, ϕ and ψ are suitable real-valued functions defined on $\mathbb{R}_+$, and $f \in \mathcal{S}(\mathbb{R}^{n+1})$, the space of Schwartz functions. (We may take $\rho \in \mathbb{C}$ with $\operatorname{Re} \rho > 0$, but for the simplicity we take only positive one.) We investigate L^p boundedness of $\mu^\rho_{\Omega,\phi,\psi,h} f$ for ϕ and ψ satisfying the following assumptions (A-1) and (A-2).

(A-1) ϕ is a positive $C^1(\mathbb{R}_+)$ function and $\phi(t)/\big(t\phi'(t)\big) \in L^\infty(\mathbb{R}_+)$.

(A-2) ϕ satisfies one of the following conditions:

(i) ϕ is increasing, and $\phi(2t) \leq c_1\phi(t)$.

(ii) ϕ is increasing, and $t\phi'(t)$ is increasing.

(iii) ϕ is decreasing, and $\phi(t) \leq c_2\phi(2t)$.

(iv) ϕ is decreasing and convex.

Remark 1. Under the condition (A-1), if ϕ is increasing and convex, then $t\phi'(t)$ is increasing. And if ϕ is decreasing and $-t\phi'(t)$ is decreasing, then ϕ is convex. We shall discuss these relations in the second section and give several examples in the last section.

Theorem 1. *Let $h \in \Delta_\gamma$ for some $1 < \gamma \leq \infty$. Let ϕ and ψ satisfy the assumptions* (A-1) *and* (A-2).

(a) *If $\Omega \in L(\log L)^{1/2}(S^{n-1})$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2 \leq p < 1/\big(1/2 - \min(1/2, 1/\gamma')\big)$.*

(b) *If $\Omega \in L(\log L)^{1/\gamma'}(S^{n-1})$ with $2 < \gamma \leq \infty$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $\gamma' < p < \infty$.*

(c) *If $\Omega \in L(\log L)^{(2\gamma-1)/(2\gamma)}(S^{n-1})$ with $1 < \gamma \leq 2$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2\gamma/(2\gamma-1) < p < 2$.*

(d) *If $\Omega \in L(\log L)^{(3\gamma-2)/(2\gamma)}(S^{n-1})$ with $1 < \gamma \leq 2$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2\gamma/(3\gamma-2) < p < 2$.*

If h satisfies a more restrictive condition, we have sharper results with respect to the condition on Ω and p.

Theorem 2. *Let $h \in L^{\gamma}(\mathbb{R}_+, dt/t)$ for some $1 < \gamma \leq \infty$. Let ϕ and ψ satisfy the assumptions* (A-1) *and* (A-2).

(a) *If $\Omega \in L(\log L)^{1/\gamma'}(S^{n-1})$ with $1 < \gamma \leq 2$, then $\mu^{\rho}_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $\gamma' \leq p < \infty$.*

(b) *If $\Omega \in L(\log L)^{1/2}(S^{n-1})$ and $2 < \gamma \leq \infty$, then $\mu^{\rho}_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2 \leq p < \infty$.*

(c) *If $\gamma = 1$ and $\Omega \in L(\log L)^1(S^{n-1})$ with $1 < \gamma \leq 2$, then $\mu^{\rho}_{\Omega,\phi,\psi,h}$ is bounded on $L^{\infty}(\mathbb{R}^{n+1})$.*

And we also have the same results for the following maximal operator related to Marcinkiewicz integral, defined by

$$\mathcal{M}^{(\gamma)}_{\Omega,\phi,\psi}f(x, x_{n+1}) = \sup_h |\mu^{\rho}_{\Omega,\phi,\psi,h}f(x, x_{n+1})|, \tag{1.2}$$

where the supremum is taken over all measurable radial functions h with $\|h\|_{L^{\gamma}(\mathbb{R}_+,\, dt/t)} \leq 1$. This is the counterpart of the maximal operator related to homogeneous singular integrals.

Theorem 3. *Let ϕ and ψ satisfy the assumptions* (A-1) *and* (A-2).

(a) *If $\Omega \in L(\log L)^{1/\gamma'}(S^{n-1})$ and $1 < \gamma \leq 2$, then $\mathcal{M}^{(\gamma)}_{\Omega,\phi,\psi}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $\gamma' \leq p < \infty$.*

(b) *If $\Omega \in L(\log L)^{1/2}(S^{n-1})$ and $2 < \gamma \leq \infty$, then $\mathcal{M}^{(\gamma)}_{\Omega,\phi,\psi}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2 \leq p < \infty$.*

(c) *If $\gamma = 1$ and $\Omega \in L(\log L)^1(S^{n-1})$ with $1 < \gamma \leq 2$, then $\mathcal{M}^{(\gamma)}_{\Omega,\phi,\psi}$ is bounded on $L^{\infty}(\mathbb{R}^{n+1})$.*

To understand the relationship in the above results, we remark the following proper inclusion relations:

$$L^q(S^{n-1}) \subset L(\log L)(S^{n-1}) \subset H^1(S^{n-1}) \subset L^1(S^{n-1}) \quad (q > 1), \tag{1.3}$$

$$L(\log L)^{\beta}(S^{n-1}) \subset L(\log L)^{\alpha}(S^{n-1}) \quad \text{if } 0 < \alpha < \beta, \tag{1.4}$$

$$L(\log L)^{\alpha}(S^{n-1}) \subset H^1(S^{n-1}) \quad \text{for all } \alpha \geq 1, \tag{1.5}$$

while

$$L(\log L)^{\alpha}(S^{n-1}) \not\subset H^1(S^{n-1}) \not\subset L(\log L)^{\alpha}(S^{n-1}) \quad \text{for } 0 < \alpha < 1, \tag{1.6}$$

where $H^1(S^{n-1})$ is the Hardy space on the unit sphere.

On one hand, Al-Qassem and Pan [4] showed the above three theorems for ϕ and ψ under the condition that they are positive, convex and $\phi(0)=\psi(0)=0$. On the other hand, when under the condition of (A-1) and (i) or (iii) in (A-2), Ding, Xue, and Yabuta [12] got the weighted L^p boundedness of the Marcinkiewicz integrals $\mu_{\Omega,\rho,\phi,b}(f)(x)$ with rough kernel associated to surfaces, where

$$\mu_{\Omega,\rho,\Phi,b}(f)(x) := \left(\int_0^\infty \Big|\frac{1}{t^\rho}\int_{|y|<t}\frac{b(|y|)\Omega(y')f(x-\Phi(|y|)y')}{|y|^{n-\rho}}\,dy\Big|^2\frac{dt}{t}\right)^{1/2}. \tag{1.7}$$

So, it is natural to ask whether the operators $\mu^\rho_{\Omega,\phi,\psi,h}f(x,x_{n+1})$ can be bounded when ϕ and ψ satisfy (A-1) and (A-2) conditions. As is easily checked, from the conditions in [4] it follows that $\phi(t)/\big(t\phi'(t)\big)$, $\psi(t)/\big(t\psi'(t)\big)\le 1$. Hence, our results are improvements of theirs. In particular, we can cover the case where $\phi(\cdot)$, $\psi(\cdot)$ are positive, increasing and concave, such as $\phi(t)=t^a$ and $\psi(t)=t^b$ $(0<a,b<1)$. We can also cover the case $\phi(t)=t^a$ $(0<t<1)$, $=at^b/b$ $(t\ge 1)$, where $0<a<1<b$.

As related function spaces of $L(\log L)^\alpha(S^{n-1})$, there are block spaces $B_q^{(0,v)}(S^{n-1})$ (see Section 3 about precise definition). In very similar ways, we can get the following results for block space kernels Ω. Note that $L(\log L)^{v+1+\varepsilon}(S^{n-1})$ does not contain $B_q^{(0,v)}(S^{n-1})$ for any $v>-1$ and $\varepsilon>0$.

Theorem 4. *Let $h\in\Delta_\gamma$ for some $1<\gamma\le\infty$. Let ϕ and ψ satisfy the assumptions* (A-1) *and* (A-2)*.*

(a) *If $\Omega\in B_q^{(0,-1/2)}(S^{n-1})$ for some $1<q\le\infty$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2\le p<1/\big(1/2-\min(1/2,1/\gamma')\big)$.*

(b) *If $\Omega\in B_q^{(0,-1/\gamma)}(S^{n-1})$ with $2<\gamma\le\infty$ for some $1<q\le\infty$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $\gamma'<p<\infty$.*

(c) *If $\Omega\in B_q^{(0,-1/(2\gamma))}(S^{n-1})$ with $1<\gamma\le 2$ for some $1<q\le\infty$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2\gamma/(2\gamma-1)<p<2$.*

(d) *If $\Omega\in B_q^{(0,(\gamma-2)/(2\gamma))}(S^{n-1})$ with $1<\gamma\le 2$ for some $1<q\le\infty$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2\gamma/(3\gamma-2)<p<2$.*

Theorem 5. *Let $h\in L^\gamma(\mathbb{R}_+,\,dt/t)$ for some $1<\gamma\le\infty$. Let ϕ and ψ satisfy the assumptions* (A-1) *and* (A-2)*.*

(a) *If $\Omega\in B_q^{(0,-1/\gamma)}(S^{n-1})$ with $1<\gamma\le 2$ for some $1<q\le\infty$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $\gamma'\le p<\infty$.*

(b) *If $\Omega\in B_q^{(0,-1/2)}(S^{n-1})$ with $2<\gamma\le\infty$ for some $1<q\le\infty$, then $\mu^\rho_{\Omega,\phi,\psi,h}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2\le p<\infty$.*

Theorem 6. *Let ϕ and ψ satisfy the assumptions* (A-1) *and* (A-2).

(a) *If $\Omega \in B_q^{(0,-1/\gamma)}(S^{n-1})$ with $1 < \gamma \le 2$ for some $1 < q \le \infty$, then $\mathcal{M}_{\Omega,\phi,\psi}^{(\gamma)}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $\gamma' \le p < \infty$.*

(b) *If $\Omega \in B_q^{(0,-1/2)}(S^{n-1})$ with $2 < \gamma \le \infty$ for some $1 < q \le \infty$, then $\mathcal{M}_{\Omega,\phi,\psi}^{(\gamma)}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $2 \le p < \infty$.*

Unfortunately, we could not get similar results for $H^1(S^{n-1})$ kernels Ω, besides $L^2(\mathbb{R}^{n+1})$ boundedness.

The main tools in this paper come from Al-Qassem and Pan [4]. Our main contributions are two ones. One is about relations between monotonic functions and the directional Hardy-Littlewood maximal function (Lemma 2.7). The other is about behaviors of the Fourier transform of measures arising from our parametric Marcinkiewicz integral operator (Lemma 3.1).

This paper is organized as follows. In Section 2, we investigate some properties of monotone functions satisfying (A-1) and (A-2), and give Lemma 2.7, and that $\{\Phi(a^k)\}_{k\in\mathbb{Z}}$ is a lacunary sequence. We also give Fourier transform estimates of some measures in this section. In Section 3, we prepare necessary lemmas to prove our theorems, in the framework by Al-Qassem and Pan [4], such as Lemma 3.1. In Section 4, we discuss briefly the proofs of Theorems 1, 2 and 3. The proofs of Theorems 4, 5 and 6 are given in Section 5. In the last section, we give several examples of monotone functions satisfying the assumptions (A-1) and (A-2).

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Preliminaries

In this section, we study fundamental properties between monotonic functions and the directional Hardy-Littlewood maximal function. We begin with investigating fundamental properties of positive and monotone C^1 functions $\Phi(t)$ satisfying the condition (A-1), i.e. $\Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$.

Lemma 2.1. *Suppose Φ is positive and increasing. Then $\Phi(t)/(t\Phi'(t)) \le b$ $(t > 0)$, if and only if $\Phi(at)/\Phi(t) \ge a^{1/b}$ for all $a > 1$ and $t > 0$. Hence, if $a > 1$, $\Phi(a^{k+1})/\Phi(a^k) \ge a^{1/b}$ for $k \in \mathbb{Z}$, i.e. $\{\Phi(a^k)\}_{k\in\mathbb{Z}}$ is a lacunary sequence. Moreover,*

$$\Phi(t) \le \Phi(1)t^{1/b} \quad (0 < t \le 1), \qquad \Phi(t) \ge \Phi(1)t^{1/b} \quad (t \ge 1),$$
$$t\Phi'(t) \ge \frac{\Phi(1)}{b}t^{1/b} \quad (t \ge 1),$$

and hence $\lim_{t\to 0}\Phi(t) = 0$, $\lim_{t\to\infty}\Phi(t) = +\infty$. Also, $t\Phi'(t)$ cannot be a decreasing function on $(0,\infty)$.

Proof. From the assumption $\Phi(t)/(t\Phi'(t)) \le b$ we get $\big(\log\Phi(t)\big)' \ge 1/(bt)$, and integrating this inequality from t to at, we obtain $\log\big(\Phi(at)/\Phi(t)\big) \ge (\log a)/b$, i.e. $\Phi(at)/\Phi(t) \ge a^{1/b}$ for any $a > 1$ and $t > 0$. Conversely, for $t, \delta > 0$, taking $a = 1 + \delta/t$, we get from $\Phi(at)/\Phi(t) \ge a^{1/b}$

$$\log\Phi(t+\delta) - \log\Phi(t) \ge \frac{1}{b}\log\Big(1 + \frac{\delta}{t}\Big).$$

Dividing the above inequality by δ, and then letting $\delta \to 0$, we obtain $\frac{\Phi'(t)}{\Phi(t)} \ge \frac{1}{bt}$, which shows $\Phi(t)/(t\Phi'(t)) \le b$.

In $\Phi(at)/\Phi(t) \ge a^{1/b}$, taking $t = a^k$, we get $\Phi(a^{k+1})/\Phi(a^k) \ge a^{1/b}$ for $k \in \mathbb{Z}$. Taking $t = 1$, we can deduce $\Phi(t) \ge \Phi(1)t^{1/b} \quad (t \ge 1)$, and taking $t = 1/a$, we can deduce $\Phi(t) \le \Phi(1)t^{1/b} \quad (0 < t \le 1)$. ■

Similarly, we can show the following case of decreasing Φ.

Lemma 2.2. *Suppose Φ is positive and decreasing. Then $-\Phi(t)/(t\Phi'(t)) \le b$ $(t > 0)$ if and only if $\Phi(t)/\Phi(at) \ge a^{1/b}$ for all $a > 1$ and $t > 0$. Hence, if $a > 1$, $\Phi(a^{-(k+1)})/\Phi(a^{-k}) \ge a^{1/b}$ for $k \in \mathbb{Z}$, i.e. $\{\Phi(a^{-k})\}_{k\in\mathbb{Z}}$ is a lacunary sequence. Moreover,*

$$\Phi(t) \ge \Phi(1)t^{-1/b} \quad (0 < t \le 1), \qquad \Phi(t) \le \Phi(1)t^{-1/b} \quad (t \ge 1),$$
$$-t\Phi'(t) \ge \frac{\Phi(1)}{b}t^{-1/b} \quad (0 < t \le 1),$$

and hence $\lim_{t\to 0}\Phi(t) = +\infty$, $\lim_{t\to\infty}\Phi(t) = 0$. Also, $-t\Phi'(t)$ cannot be an increasing function on $(0,\infty)$.

Now we investigate several properties between monotonic functions and the one dimensional Hardy-Littlewood maximal function.

Lemma 2.3. *Suppose Φ is a positive and increasing $C^1(0,\infty)$ function, satisfying $\Phi(t)/(t\Phi'(t)) \le b$ and $\Phi(2t) \le c_1\Phi(t)$ for some $b > 0$, $c_1 > 1$. Then*

$$\left|\int_{t/2}^{t} g\big(x - \Phi(s)\big)\frac{ds}{s}\right| \le 2c_1 b\, Mg(x),$$

where Mg is the one dimensional Hardy-Littlewood maximal function of $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, i.e. $Mg(x) = \sup_{r>0} 1/(2r)\int_{-r}^{r}|g(x+s)|\,ds$.

Proof. By a change of variable $r = \Phi(s)$, we have

$$\begin{aligned}\left|\int_{t/2}^{t} g(x - \Phi(s))\frac{ds}{s}\right| &\le \int_{\Phi(t/2)}^{\Phi(t)} |g(x-r)|\frac{r}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\frac{dr}{r}\\ &\le b\int_{\Phi(t/2)}^{\Phi(t)} |g(x-r)|\frac{dr}{r}\end{aligned}$$

$$\le \frac{b}{\Phi(t/2)} \int_{\Phi(t/2)}^{\Phi(t)} |g(x-r)|dr$$
$$\le \frac{bc_1}{\Phi(t)} \int_0^{\Phi(t)} |g(x-r)|dr \le 2c_1 b\, Mg(x). \qquad \blacksquare$$

Lemma 2.4. *Suppose* Φ *is a positive and increasing* $C^1(0,\infty)$ *function, satisfying* $\Phi(t)/(t\Phi'(t)) \le b$ *for some* $b > 0$ *and* $t\Phi'(t)$ *is increasing. Then*

$$\left| \int_{t/2}^{t} g(x-\Phi(s)) \frac{ds}{s} \right| \le (b + \log 2)\, Mg(x).$$

Proof. By a change of variable $r = \Phi(s)$, we have

$$\left| \int_{t/2}^{t} g(x-\Phi(s)) \frac{ds}{s} \right| \le \int_{\Phi(t/2)}^{\Phi(t)} |g(x-r)| \frac{1}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\, dr. \tag{2.1}$$

We set

$$a_t(r) = \begin{cases} \frac{1}{(t/2)\Phi'(t/2)}, & 0 < r < \Phi(t/2), \\ \frac{1}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}, & \Phi(t/2) \le r < \Phi(t), \\ 0, & r \ge \Phi(t). \end{cases}$$

Then

$$\int_0^\infty a_t(r)\, dr = \frac{1}{(t/2)\Phi'(t/2)} \times \Phi(t/2) + \int_{\Phi(t/2)}^{\Phi(t)} \frac{1}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))} dr$$
$$\le b + \int_{t/2}^{t} \frac{ds}{s} = b + \log 2.$$

Since $1/(t\Phi'(t))$ is decreasing, it follows that $a_t(r)$ is nonnegative, decreasing and integrable on $(0,\infty)$. Hence, we have by (2.1)

$$\left| \int_{t/2}^{t} g(x-\Phi(s)) \frac{ds}{s} \right| \le \int_0^\infty |g(x-r)| a_t(r)\, dr \le (b+\log 2) Mg(x). \qquad \blacksquare$$

Lemma 2.5. *Suppose* Φ *is a positive and decreasing* $C^1(0,\infty)$ *function, satisfying* $-\Phi(t)/(t\Phi'(t)) \le b$ *and* $\Phi(t) \le c_2\Phi(2t)$ *for some* $b > 0, c_2 > 1$. *Then*

$$\left| \int_{t/2}^{t} g(x-\Phi(s)) \frac{ds}{s} \right| \le 2c_2 b\, Mg(x).$$

Proof. By a change of variable $r = \Phi(s)$, we have

$$\begin{aligned}\left|\int_{t/2}^{t} g(x-\Phi(s))\frac{ds}{s}\right| &\leq \int_{\Phi(t)}^{\Phi(t/2)} |g(x-r)| \frac{-r}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\frac{dr}{r} \\ &\leq b\int_{\Phi(t)}^{\Phi(t/2)} |g(x-r)|\frac{dr}{r} \\ &\leq \frac{b}{\Phi(t)}\int_{\Phi(t)}^{\Phi(t/2)} |g(x-r)|dr \\ &\leq \frac{bc_2}{\Phi(t/2)}\int_{0}^{\Phi(t/2)} |g(x-r)|dr \leq 2c_2 b\, Mg(x). \qquad \blacksquare\end{aligned}$$

Lemma 2.6. *Suppose Φ is a positive, decreasing and convex $C^1(0,\infty)$ function, satisfying $-\Phi(t)/(t\Phi'(t)) \leq b$ for some $b > 0$. Then*

$$\left|\int_{t/2}^{t} g(x-\Phi(s))\frac{ds}{s}\right| \leq (2b+1)\, Mg(x).$$

Proof. By a change of variable $r = \Phi(s)$, we have

$$\left|\int_{t/2}^{t} g(x-\Phi(s))\frac{ds}{s}\right| \leq \int_{\Phi(t)}^{\Phi(t/2)} |g(x-r)| \frac{1}{-\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\, dr. \tag{2.2}$$

Since $\Phi(t)$ is positive and decreasing, we see that $\Phi^{-1}(t)$ is also decreasing, and hence $1/\Phi^{-1}(t)$ is increasing. Hence we get

$$\left|\int_{t/2}^{t} g(x-\Phi(s))\frac{ds}{s}\right| \leq \frac{2}{t}\int_{\Phi(t)}^{\Phi(t/2)} |g(x-r)| \frac{1}{-\Phi'(\Phi^{-1}(r))}\, dr. \tag{2.3}$$

We set

$$a_t(r) = \begin{cases} -\frac{2}{t\Phi'(t)}, & 0 < r < \Phi(t), \\ -\frac{2}{t\Phi'(\Phi^{-1}(r))}, & \Phi(t) \leq r < \Phi(t/2), \\ 0, & r \geq \Phi(t/2). \end{cases}$$

Then

$$\begin{aligned}\int_0^\infty a_t(r)\, dr &= -\frac{2}{t\Phi'(t)} \times \Phi(t) + \int_{\Phi(t)}^{\Phi(t/2)} \frac{-2}{t\Phi'(\Phi^{-1}(r))}dr \\ &\leq 2b + \frac{2}{t}\int_{t/2}^{t} ds = 2b+1.\end{aligned}$$

Furthermore, because of the convexity of $\Phi(t)$ it follows that $-\Phi'(\Phi^{-1}(t))$ is increasing. So, we see that $a_t(r)$ is nonnegative, decreasing and integrable on $(0,\infty)$. Hence, we have by (2.3)

$$\left|\int_{t/2}^{t} g(x-\Phi(s))\frac{ds}{s}\right| \le \int_0^\infty |g(x-r)|a_t(r)\,dr \le (2b+1)Mg(x). \qquad \blacksquare$$

We formulate the above considerations to the n-dimensional case, and we have the following directional result.

Lemma 2.7. *Let $\Omega \in L^1(S^{n-1})$. Suppose Φ is a positive function on $(0,\infty)$ satisfying $|\Phi(t)/(t\Phi'(t))| \le b$ and satisfies one of the following conditions:*

(i) *Φ is increasing, and $\Phi(2t) \le c_1\Phi(t)$.*

(ii) *Φ is increasing, and $t\Phi'(t)$ is increasing.*

(iii) *Φ is decreasing, and $\Phi(t) \le c_2\Phi(2t)$.*

(iv) *Φ is decreasing and convex.*

Then

$$\left|\int_{t/2<|y|<t} \frac{\Omega(y')f\big(x-\Phi(|y|)y'\big)}{|y|^n}\,dy\right| \le C_j\int_{S^{n-1}} |\Omega(y')|M_{y'}f(x)\,d\sigma(y'),$$

where $M_{y'}f(x)$ is the directional Hardy-Littlewood maximal function of f, defined by

$$\sup_{r>0}\frac{1}{2r}\int_{|t|<r}|f(x-ty')|\,dt,$$

and

$$C_1 \le 2c_1b,\ C_2 \le b+\log 2,\ C_3 \le 2c_2b,\ C_4 \le 2b+1.$$

Remark 2. (i) If Φ is positive, increasing, and $\Phi(t)/(t\Phi'(t))$ is decreasing, then $t\Phi'(t)$ is increasing on $(0,\infty)$.

(ii) If Φ is positive, increasing and convex, then $t\Phi'(t)$ is increasing on $(0,\infty)$.

(iii) If Φ is positive, decreasing, and $-t\Phi'(t)$ is decreasing on $(0,\infty)$, then $\Phi(t)$ is convex.

(iv) If Φ is positive, decreasing, and $-\Phi(t)/(t\Phi'(t))$ is increasing, then $-t\Phi'(t)$ is decreasing, and hence $\Phi(t)$ is convex.

In fact, (i) is obvious. (iv) follows clearly from (iii). Next, let Φ is positive, increasing, and convex on $(0,\infty)$. From the assumption we get $\Phi'(t_2) \ge \Phi'(t_1)$ and $\Phi'(t_2) \ge 0$ for $t_2 > t_1 > 0$. Hence we get

$$t_2\Phi'(t_2) - t_1\Phi'(t_1) = (t_2-t_1)\Phi'(t_2) + t_1(\Phi'(t_2)-\Phi'(t_1)) \ge 0.$$

This means that $t\Phi'(t)$ is increasing, which shows (ii).

As for (iii), from the assumption we get $-t_2\Phi'(t_2) \le -t_1\Phi'(t_1)$ and $\Phi'(t_1) \le 0$ for $t_2 > t_1 > 0$. Hence we get

$$\Phi'(t_2) - \Phi'(t_1) = \frac{t_2\Phi'(t_2) - t_2\Phi'(t_1)}{t_2} \ge \frac{t_1\Phi'(t_1) - t_2\Phi'(t_1)}{t_2} = \frac{(t_1 - t_2)\Phi'(t_1)}{t_2} \ge 0.$$

This means that $\Phi(t)$ is convex.

Next, we prepare the following estimates about Fourier transforms of some measures on $\mathbb{R}^{n+1}$. In the case Φ is positive and increasing, we have the following

Lemma 2.8. *Let* $1 < q \le \infty$, $\Omega \in L^q(S^{n-1})$ *and* ψ *be a real valued function on* $(0,\infty)$. *If* Φ *is positive, increasing,* $\Phi(2t) \le c_1\Phi(t)$, *and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^{t} \left| \int_{S^{n-1}} \Omega(x') e^{-i(\Phi(s)\xi\cdot x' + \eta\psi(s))} d\sigma(x') \right|^2 \frac{ds}{s} \le \frac{C_\alpha 2^\alpha (\log c_1)^{1-\alpha} \|\varphi\|_\infty \|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t/2)\xi|^\alpha}.$$

Proof. We have

$$\tag{2.4}\begin{aligned}
&\int_{t/2}^{t} \left| \int_{S^{n-1}} \Omega(x') e^{-i(\Phi(s)\xi\cdot x' + \eta\psi(s))} d\sigma(x') \right|^2 \frac{ds}{s} \\
&= \int_{\Phi(t/2)}^{\Phi(t)} \left| \int_{S^{n-1}} \Omega(x') e^{-i(r\xi\cdot x' + \eta\psi(\Phi^{-1}(r)))} d\sigma(x') \right|^2 \frac{r}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))} \frac{dr}{r} \\
&\le \left\| \frac{\Phi(t)}{t\Phi'(t)} \right\|_\infty \int_{\Phi(t/2)}^{\Phi(t)} \left| \int_{S^{n-1}} \Omega(x') e^{-i(r\xi\cdot x' + \eta\psi(\Phi^{-1}(r)))} d\sigma(x') \right|^2 \frac{dr}{r} \\
&= \left\| \frac{\Phi(t)}{t\Phi'(t)} \right\|_\infty \int_{S^{n-1}\times S^{n-1}} \Omega(x')\overline{\Omega(y')} \left(\int_{\Phi(t/2)}^{\Phi(t)} e^{-ir\xi\cdot(x'-y')} \frac{dr}{r} \right) d\sigma(x') d\sigma(y').
\end{aligned}$$

In the second equation, we used the change of variable $r = \Phi(s)$. Clearly we have

$$\left| \int_{\Phi(t/2)}^{\Phi(t)} e^{-ir\xi\cdot(x'-y')} \frac{dr}{r} \right| \le \log \frac{\Phi(t)}{\Phi(t/2)} \le \log c_1$$

and

$$\left| \int_{\Phi(t/2)}^{\Phi(t)} e^{-ir\xi\cdot(x'-y')} \frac{dr}{r} \right| \le \frac{2}{\Phi(t/2)|\xi||\xi'\cdot(x'-y')|},$$

and so we have for any $0 < \alpha \le 1$

$$\left| \int_{\Phi(t/2)}^{\Phi(t)} e^{-ir\xi\cdot(x'-y')} \frac{dr}{r} \right| \le \frac{(\log c_1)^{1-\alpha} 2^\alpha}{|\Phi(t/2)\xi|^\alpha |\xi'\cdot(x'-y')|^\alpha}.$$

This combined with (2.4) yields the desired estimate. ∎

Lemma 2.9. *Let $1 < q \le \infty$, $\Omega \in L^q(S^{n-1})$ and ψ be a real valued function on $(0, \infty)$. If Φ is positive, increasing, $t\Phi'(t)$ is increasing, and $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0, \infty)$, then it holds for any $0 < \alpha < 1/q'$*

$$\int_{t/2}^{t} \left| \int_{S^{n-1}} \Omega(x') e^{-i(\Phi(s)\xi\cdot x' + \eta\psi(s))} d\sigma(x') \right|^2 \frac{ds}{s} \le \frac{C_\alpha 4^\alpha (\log 2)^{1-\alpha} \|\varphi\|_\infty^\alpha \|\Omega\|_{L^q(S^{n-1})}^2}{|\Phi(t/2)\xi|^\alpha}.$$

Proof. We have

$$\begin{aligned} &\int_{t/2}^{t} \left| \int_{S^{n-1}} \Omega(x') e^{-i(\Phi(s)\xi\cdot x' + \eta\psi(s))} d\sigma(x') \right|^2 \frac{ds}{s} \\ &= \int_{S^{n-1}\times S^{n-1}} \Omega(x')\overline{\Omega(y')} \left(\int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')} \frac{ds}{s} \right) d\sigma(x') d\sigma(y'). \end{aligned} \tag{2.5}$$

Clearly we have

$$\left| \int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')} \frac{ds}{s} \right| \le \log 2. \tag{2.6}$$

Applying the change of variable $r = \Phi(s)$, we have

$$\int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')} \frac{ds}{s} = \int_{\Phi(t/2)}^{\Phi(t)} e^{-ir\xi\cdot(x'-y')} \frac{1}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))} dr.$$

Since Φ is positive and increasing, and $t\Phi'(t)$ is increasing, we see that $\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))$ is increasing. Hence we obtain

$$\begin{aligned} &\left| \int_{\Phi(t/2)}^{\Phi(t)} \cos(-r\xi\cdot(x'-y')) \frac{dr}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))} \right| \\ &\qquad \le \frac{1}{t/2\Phi'(t/2)} \frac{2}{|\xi\cdot(x'-y')|} \le \frac{\Phi(t/2)}{t/2\Phi'(t/2)} \frac{2}{\Phi(t/2)|\xi\cdot(x'-y')|}. \end{aligned}$$

We get a similar estimate for sin part, and hence we obtain

$$\left| \int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')} \frac{ds}{s} \right| \le \left\| \frac{\Phi(s)}{s\Phi'(s)} \right\|_\infty \frac{4}{\Phi(t/2)|\xi\cdot(x'-y')|}.$$

Thus, combining this with (2.6) we have for any $0 < \alpha \le 1$

$$\left| \int_{t/2}^{t} e^{-i\Phi(s)\xi\cdot(x'-y')} \frac{ds}{s} \right| \le \left\| \frac{\Phi(t)}{t\Phi'(t)} \right\|_\infty^\alpha \frac{(\log 2)^{1-\alpha} 4^\alpha}{|\Phi(t/2)\xi|^\alpha |\xi'\cdot(x'-y')|^\alpha}.$$

This combined with (2.5) yields the desired estimate. ■

In the case where Φ is positive and decreasing, we have the following

Lemma 2.10. *Let* $1 < q \le \infty$, $\Omega \in L^q(S^{n-1})$ *and* ψ *be a real valued function on* $(0, \infty)$. *If* Φ *is positive, decreasing,* $\Phi(t) \le c_2\Phi(2t)$, *and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0, \infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^{t}\Big|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\Big|^2\frac{ds}{s} \le \frac{C_\alpha 2^\alpha(\log c_2)^{1-\alpha}\|\varphi\|_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t)\xi|^\alpha}.$$

Proof. We have

$$\begin{aligned}
&\int_{t/2}^{t}\Big|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\Big|^2\frac{ds}{s}\\
&= \int_{\Phi(t)}^{\Phi(t/2)}\Big|\int_{S^{n-1}} \Omega(x')e^{-i(r\xi\cdot x'+\eta\psi(\Phi^{-1}(r)))}d\sigma(x')\Big|^2\frac{r}{-\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\frac{dr}{r}\\
&\le \Big\|\frac{\Phi(t)}{t\Phi'(t)}\Big\|_\infty\int_{\Phi(t)}^{\Phi(t/2)}\Big|\int_{S^{n-1}} \Omega(x')e^{-i(r\xi\cdot x'+\eta\psi(\Phi^{-1}(r)))}d\sigma(x')\Big|^2\frac{dr}{r}\\
&= \Big\|\frac{\Phi(t)}{t\Phi'(t)}\Big\|_\infty\int_{S^{n-1}\times S^{n-1}} \Omega(x')\overline{\Omega(y')}\Big(\int_{\Phi(t)}^{\Phi(t/2)} e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\Big)d\sigma(x')d\sigma(y').
\end{aligned}\tag{2.7}$$

In the second equation, we used the change of variable $r = \Phi(s)$. Clearly we have

$$\Big|\int_{\Phi(t)}^{\Phi(t/2)} e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\Big| \le \log\frac{\Phi(t/2)}{\Phi(t)} \le \log c_2$$

and

$$\Big|\int_{\Phi(t)}^{\Phi(t/2)} e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\Big| \le \frac{2}{\Phi(t)|\xi||\xi'\cdot(x'-y')|},$$

and so we have for any $0 < \alpha \le 1$

$$\Big|\int_{\Phi(t)}^{\Phi(t/2)} e^{-ir\xi\cdot(x'-y')}\frac{dr}{r}\Big| \le \frac{(\log c_2)^{1-\alpha}2^\alpha}{|\Phi(t)\xi|^\alpha|\xi'\cdot(x'-y')|^\alpha}.$$

This combined with (2.7) yields the desired estimate. ■

Lemma 2.11. *Let* $1 < q \le \infty$, $\Omega \in L^q(S^{n-1})$ *and* ψ *be a real valued function on* $(0, \infty)$. *If* Φ *is positive, decreasing and convex, and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0, \infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^{t}\Big|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\Big|^2\frac{ds}{s} \le \frac{C_\alpha 8^\alpha(\log 2)^{1-\alpha}\|\varphi\|^\alpha_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t)\xi|^\alpha}.$$

Proof. We have

$$\begin{aligned}
&\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\right|^2\frac{ds}{s}\\
&=\int_{S^{n-1}\times S^{n-1}}\Omega(x')\overline{\Omega(y')}\left(\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right)d\sigma(x')d\sigma(y').
\end{aligned}\tag{2.8}$$

Clearly we have

$$\left|\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right|\le\log 2.\tag{2.9}$$

Applying the change of variable $r=\Phi(s)$, we have

$$\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}=\int_{\Phi(t)}^{\Phi(t/2)}e^{-ir\xi\cdot(x'-y')}\frac{1}{-\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\,dr.$$

Since Φ is positive, decreasing and convex, we see that $-\Phi'(t)$ is decreasing, and hence $-\Phi'(\Phi^{-1}(r))$ is positive and increasing. Hence we see by the second mean value theorem that there exists c with $\Phi(t)\le c\le\Phi(t/2)$ such that

$$\begin{aligned}
&\int_{\Phi(t)}^{\Phi(t/2)}\cos(-r\xi\cdot(x'-y'))\frac{dr}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\\
&=\frac{1}{-\Phi'(t)}\int_{\Phi(t)}^{c}\cos(-r\xi\cdot(x'-y'))\frac{dr}{\Phi^{-1}(r)}.
\end{aligned}$$

Since Φ is positive and decreasing, we see that $\Phi^{-1}(r)$ is also positive and decreasing. Hence we have

$$\begin{aligned}
&\left|\int_{\Phi(t)}^{\Phi(t/2)}\cos(-r\xi\cdot(x'-y'))\frac{dr}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\right|\\
&\le\frac{1}{-\Phi'(t)}\frac{1}{\Phi^{-1}(c)}\frac{2}{|\xi\cdot(x'-y')|}\le\frac{1}{-\Phi'(t)}\frac{1}{t/2}\frac{2}{|\xi||\xi'\cdot(x'-y')|}\\
&\le\left\|\frac{\Phi(s)}{s\Phi'(s)}\right\|_\infty\frac{4}{\Phi(t)|\xi\cdot(x'-y')|}.
\end{aligned}$$

We get a similar estimate for $\sin$ part, and hence we obtain

$$\left|\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right|\le\left\|\frac{\Phi(s)}{s\Phi'(s)}\right\|_\infty\frac{8}{\Phi(t)|\xi\cdot(x'-y')|}.$$

Thus, combining this with (2.9) we have for any $0<\alpha\le 1$

$$\left|\int_{t/2}^{t}e^{-i\Phi(s)\xi\cdot(x'-y')}\frac{ds}{s}\right|\le\left\|\frac{\Phi(s)}{s\Phi'(s)}\right\|_\infty^\alpha\frac{(\log 2)^{1-\alpha}8^\alpha}{|\Phi(t)\xi|^\alpha|\xi'\cdot(x'-y')|^\alpha}.$$

Combining this with (2.8) yields the desired estimate. ■

Finally in this section, we will give a remark on the Littlewood-Paley operator for a lacunary sequence.

Let $\{a_j\}_{j\in\mathbb{Z}}$ be a lacunary sequence of positive numbers satisfying

$$\frac{a_{j+1}}{a_j} \geq a > 1, \quad j \in \mathbb{Z}.$$

Take a nonincreasing $C^\infty\big([0,\infty)\big)$ function $\varphi(t)$ such that

$$0 \leq \varphi(t) \leq 1 \ \big(t \in [0,\infty)\big), \quad \varphi(t) = 1 \ (0 \leq t \leq 1), \quad \varphi(t) = 0 \ (t \geq a).$$

We define functions ψ_j on $(0, \infty)$ by

$$\psi_j(t) = \varphi\Big(\frac{t}{a_{j+1}}\Big) - \varphi\Big(\frac{t}{a_j}\Big).$$

Then

$$\psi_j(t) = \begin{cases} 0, & 0 \leq t \leq a_j,\ t \geq a\, a_{j+1}, \\ 1, & a\, a_j \leq t \leq a_{j+1}, \end{cases} \tag{2.10}$$

and

$$\operatorname{supp} \psi_j \cap \operatorname{supp} \psi_{j+1} \subset \{a_{j+1} \leq t \leq a\, a_{j+1}\},$$
$$\operatorname{supp} \psi_j \cap \operatorname{supp} \psi_\ell = \emptyset, \ \text{for } |j-\ell| \geq 2.$$

We have for $t > 0$

$$\sum_{j=-\infty}^{\infty} \psi_j(t) = 1, \quad t > 0, \tag{2.11}$$

and

$$|\partial^\alpha \psi_j(|\xi|)| \leq C_\alpha \Big(\frac{a}{a-1}\Big)^{|\alpha|} \frac{1}{|\xi|^{|\alpha|}}. \tag{2.12}$$

We set

$$\Psi_j(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi_j(|\xi|) e^{ix\cdot\xi} d\xi. \tag{2.13}$$

Then we can use the Littlewood-Paley theory and get

Lemma 2.12. *Let $\alpha_0 > 1$ and $1 < p < \infty$. Let Ψ_j be as above. Then there exists a positive constant C_p such that*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\Psi_j * f(x)|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p, \quad f \in L^p(\mathbb{R}^n), \tag{2.14}$$

where C_p is independent of $a \ge \alpha_0$.

This can be checked by estimating the kernel of the above operator as a vector valued singular integral (cf. [25]). For a precise proof see, for example, [29, pp. 312-316].

3. Some Definitions and Lemmas

In this section, we give some definitions and prepare some lemmas to prove our theorems.

The block spaces originated in the work of Taibleson and Weiss on the convergence of the Fourier series in connection with the developments of the real Hardy spaces. We will recall the definition of block spaces on S^{n-1}. For further information about the theory of spaces generated by blocks and its applications to harmonic analysis, see the book [24] and a survey article [22].

Definition 7. A q-block on S^{n-1} is an $L^q(S^{n-1})$ $(1 < q \le \infty)$ function $b(x)$ that satisfies

$$\begin{aligned} &\text{(i)} \quad \operatorname{supp} b \subset I; \\ &\text{(ii)} \quad \|b\|_q \le |I|^{-1/q'}, \end{aligned} \tag{3.1}$$

where $|I| = \sigma(I)$, and $I = B(x_0', \theta_0) \cap S^{n-1}$ is a cap on S^{n-1} for some $x_0' \in S^{n-1}$ and $\theta_0 \in (0, 1]$.

Jiang and Lu [18] introduced the class of block spaces $B_q^{(0,v)}(S^{n-1})$ $(v > -1)$ concerning the study of homogeneous singular integral operators.

Definition 8. For $1 < q \le \infty$ and $v > -1$, the block space $B_q^{(0,v)}(S^{n-1})$ is defined by

$$B_q^{(0,v)}(S^{n-1}) = \Big\{ \Omega \in L^1(S^{n-1});\, \Omega = \sum_{j=1}^{\infty} \lambda_j b_j,\, M_q^{(0,v)}(\{\lambda_j\}) < \infty \Big\}, \tag{3.2}$$

where each λ_j is a complex number, each b_j is a q-block supported on a cap I_j on S^{n-1}, and

$$M_q^{(0,v)}(\{\lambda_j\}) = \sum_{j=1}^{\infty} |\lambda_j| \big\{ 1 + \log^{(v+1)}\big(|I_j|^{-1}\big) \big\}. \tag{3.3}$$

Let $\|\Omega\|_{B_q^{(0,v)}(S^{n-1})} = \inf\{M_q^{(0,v)}(\{\lambda_j\}); \Omega = \sum_{j=1}^{\infty} \lambda_j b_j$ and each b_j is a q-block supported on a cap I_j on $S^{n-1}\}$. Then $\|\cdot\|_{B_q^{(0,v)}(S^{n-1})}$ is a norm on the space $B_q^{(0,v)}(S^{n-1})$, and $\left(B_q^{(0,v)}(S^{n-1}), \|\cdot\|_{B_q^{(0,v)}(S^{n-1})}\right)$ is a Banach space.

The following inclusion relations are known.

$$
\begin{aligned}
&B_q^{(0,v_1)}(S^{n-1}) \subset B_q^{(0,v_2)}(S^{n-1}) \quad \text{if } v_1 > v_2 > -1;\\
&B_{q_1}^{(0,v)}(S^{n-1}) \subset B_{q_2}^{(0,v)}(S^{n-1}) \quad \text{if } 1 < q_2 < q_1 \text{ for any } v > -1;\\
&\bigcup_{p>1} L^p(S^{n-1}) \subset B_q^{(0,v)}(S^{n-1}) \quad \text{for any } q > 1,\ v > -1;\\
&\bigcup_{q>1} B_q^{(0,v)}(S^{n-1}) \not\subset \bigcup_{q>1} L^q(S^{n-1}) \quad \text{for any } v > -1;\\
&B_q^{(0,v)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log L)^{1+v}(S^{n-1}) \quad \text{for any } q > 1,\ v > -1.
\end{aligned} \tag{3.4}
$$

Definition 9. Let $\rho > 0$. For arbitrary real-valued functions $\phi(\cdot)$ and $\psi(\cdot)$ on $(0,\infty)$, a measurable function $h : (0,\infty) \to \mathbb{C}$ and $\Omega : S^{n-1} \to \mathbb{C}$, we define the family $\{\sigma_{t,h};\, t \in (0,\infty)\}$ of measures and the maximal operator σ_h^* on $\mathbb{R}^{n+1}$ by

$$
\begin{aligned}
\int_{\mathbb{R}^{n+1}} f\, d\sigma_{t,h} &= \frac{1}{t^\rho} \int_{t/2<|y|<t} f(\phi(|y|)y', \psi(|y|)) h(|y|) \frac{\Omega(y')}{|y|^{n-\rho}}\, dy,\\
\sigma_h^* f(x, x_{n+1}) &= \sup_{t>0} \big| |\sigma_{t,h}| * f(x, x_{n+1}) \big|,
\end{aligned}
$$

where $|\sigma_{t,h}|$ is defined in the same way as $\sigma_{t,h}$, but with Ω replaced by $|\Omega|$ and h by $|h|$.

Lemma 3.1. *Let $1 < q \le +\infty$, $m \in \mathbb{N}$, and $\Omega \in L^q(S^{n-1})$ with $\|\Omega\|_{L^1(S^{n-1})} \le 1$, $\|\Omega\|_{L^q(S^{n-1})} \le 2^m$, satisfying the cancellation condition $\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0$. Let $\psi(\cdot)$ be an arbitrary real-valued function on $(0,\infty)$, and $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Assume that ϕ is a positive $C^1(0,\infty)$ function satisfying the assumptions* (A-1) *and* (A-2).

Then there exist positive constants C and $\alpha < 1/q'$ such that in the case of increasing ϕ

$$
|\widehat{\sigma}_{t,h}(\xi,\eta)| \le C\|h\|_{\Delta_1}, \tag{3.5}
$$

$$
|\widehat{\sigma}_{t,h}(\xi,\eta)| \le \frac{C\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)}{|\phi(t/2)\xi|^{\alpha/m}}, \tag{3.6}
$$

$$
|\widehat{\sigma}_{t,h}(\xi,\eta)| \le C\|h\|_{\Delta_1} |\phi(t)\xi|^{\alpha/m}, \tag{3.7}
$$

and in the case of decreasing ϕ, $\phi(t/2)$ is replaced by $\phi(t)$ in (3.6) *and $\phi(t)$ is replaced by $\phi(t/2)$ in* (3.7).

Proof. From the definition we have

$$|\widehat{\sigma}_{t,h}(\xi,\eta)| \le \frac{1}{t^\rho}\int_{t/2}^{t}\frac{|h(r)|}{r^{1-\rho}}dr\int_{S^{n-1}}|\Omega(y')|\,d\sigma(y') \le 2\|h\|_{\Delta_1}\|\Omega\|_{L^1(S^{n-1})} \le 2\|h\|_{\Delta_1}.$$

Next, we show (3.6). In the case $1<\gamma\le 2$, by a change of variable, Hölder's inequality and $\|\Omega\|_{L^1(S^{n-1})}\le 1$ we have

$$\begin{aligned}|\widehat{\sigma}_{t,h}(\xi,\eta)| &\le \frac{1}{t^\rho}\int_{t/2}^{t}|h(r)|r^\rho\left|\int_{S^{n-1}}\Omega(y')e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}d\sigma(y')\right|\frac{dr}{r}\\ &\le 2^{1/\gamma}\|h\|_{\Delta_\gamma}\left(\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(y')e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}d\sigma(y')\right|^{\gamma'}\frac{dr}{r}\right)^{1/\gamma'}\\ &\le 2^{1/\gamma}\|h\|_{\Delta_\gamma}\left(\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(y')e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}d\sigma(y')\right|^{2}\frac{dr}{r}\right)^{1/\gamma'}.\end{aligned}$$

In the last inequality, we used $|\int_{S^{n-1}}\Omega(y')e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}d\sigma(y')|\le\|\Omega\|_{L^1(S^{n-1})}\le 1$. In the case $\gamma>2$, using Cauchy-Schwarz' inequality in place of Hölder's inequality, we get a similar inequality. Together with, we have

$$|\widehat{\sigma}_{t,h}(\xi,\eta)| \le 2\|h\|_{\Delta_\gamma}\left(\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(y')e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}d\sigma(y')\right|^{2}\frac{dr}{r}\right)^{1/\max\{\gamma',2\}}.$$

So, if ϕ satisfies (A-2) (i), by Lemma 2.8 we have for $0<\alpha<1/q'$

$$\begin{aligned}|\widehat{\sigma}_{t,h}(\xi,\eta)| &\le 2\|h\|_{\Delta_\gamma}\left(\frac{C_\alpha\|\varphi\|_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\phi(t/2)\xi|^\alpha}\right)^{1/\max\{\gamma',2\}}\\ &\le C\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)\left(\frac{2^{2m}}{|\phi(t/2)\xi|^\alpha}\right)^{1/\max\{\gamma',2\}}.\end{aligned}$$

From this and (3.5) we obtain

$$|\widehat{\sigma}_{t,h}(\xi,\eta)| \le C\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)\left(\frac{2^{2m}}{|\phi(t/2)\xi|^\alpha}\right)^{1/(m\max\{\gamma',2\})} \le \frac{C\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)}{|\phi(t/2)\xi|^{\alpha/(m\max\{\gamma',2\})}}.$$

Taking $\alpha/(\max\{\gamma',2\})$ newly as α, we get (3.6). The other three cases can be proved in a similar way, using Lemmas 2.9, 2.10 and 2.11, respectively.

Finally we prove (3.7). Using the cancellation property of Ω and the monotonicity of ϕ, we have

$$\begin{aligned}|\widehat{\sigma}_{t,h}(\xi,\eta)| &\le \frac{1}{t^\rho}\int_{t/2}^{t}|h(r)|r^\rho\left|\int_{S^{n-1}}\Omega(y')\big(e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}-e^{-i\psi(r)\eta}\big)d\sigma(y')\right|\frac{dr}{r}\\ &\le C\|h\|_{\Delta_1}\max\{|\phi(t)\xi|,|\phi(t/2)\xi|\}\|\Omega\|_{L^1(S^{n-1})}.\end{aligned}$$

Combining this with (3.5) yields the desired estimate (3.7). ∎

By a similar argument we have

Lemma 3.2. *Let $1 < q \le +\infty$, $m \in \mathbb{N}$, and Ω, ϕ and ψ be as in Lemma 3.1. For $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}$ let*

$$I_t(\xi,\eta) = \left(\int_{t/2}^{t}\left|\int_{S^{n-1}} \Omega(y')e^{-i(\phi(s)\xi\cdot y'+\eta\psi(s))}d\sigma(y')\right|^2 \frac{ds}{s}\right)^{1/2}.$$

Then $|I_t|$ satisfies the same estimates in (3.5), (3.6), (3.7).

We next state a variant of the Lemma 3.4 in [4].

Lemma 3.3. *Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers with*

$$\frac{a_{k+1}}{a_k} \ge a^A \quad \text{for some } a > 1 \text{ and } A > 0.$$

Let $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a sequence of Borel measures on $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that for all $\ell \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$, and some $\alpha > 0$, $C_0 > 0$, ℓ_0, $\ell_1 \in \mathbb{N}\cup\{0\}$, and $p_0 \ge 2$, we have

(*i*) $|\widehat{\sigma_k}(\xi)| \le C_0 \max\{1, (a_{k+\ell_0}|L(\xi)|)^{\alpha/A}, (a_{k-\ell_1}|L(\xi)|)^{-\alpha/A}\}$,

(*ii*) $\left\|\left(\sum_{k\in\mathbb{Z}} |\sigma_k * g_k|^2\right)^{1/2}\right\|_{p_0} \le C_0\left\|(\sum_{k\in\mathbb{Z}}|g_k|^2)^{1/2}\right\|_{p_0}$ *for arbitrary functions g_k on $\mathbb{R}^n$.*

Then for $p_0' < p < p_0$, there exists a positive constant C_p such that

$$\left\|\left(\sum_{k\in\mathbb{Z}} |\sigma_k * f|^2\right)^{1/2}\right\|_p \le C_pC_0\|f\|_p$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_p is independent of A and of the linear transformation L.

In Al-Qassem and Pan [4], this lemma is given in the case $\ell_0 = 1$ and $\ell_1 = 0$, but one can easily check that the above holds.

We introduce a maximal function

$$\lambda^*_{m,h}(f) = \sup_{k\in\mathbb{Z}} \int_{a_m^k}^{a_m^{k+1}} (\mu_{t,h} * f)\frac{dt}{t}, \qquad \mu_{t,h} = |\sigma_{t,h}|,$$

where $a_m = 2^m$, $m \in \mathbb{N}$. If we define the measure $\lambda_{m,k,h}$ by

$$\hat{\lambda}_{m,k,h}(\xi,\eta) = \int_{a_m^k}^{a_m^{k+1}} \hat{\mu}_{t,h}(\xi,\eta)\frac{dt}{t} \qquad \text{for } (\xi,\eta) \in \mathbb{R}^n\times\mathbb{R},$$

then

$$\lambda^*_{m,h}f(x,x_{n+1}) = \sup_{k\in\mathbb{Z}} |\lambda_{m,k,h} * f(x,x_{n+1})|.$$

For this maximal function we can show the following lemma in the same way as in the proof of the corresponding Lemma 3.5 in [4], by using Lemmas 2.7, 3.1 and 3.3.

Lemma 3.4. *Let $1 < q \leq +\infty$, $m \in \mathbb{N}$, $h \in L^{\infty}(\mathbb{R}^n)$ and $\Omega \in L^q(S^{n-1})$ with $\|\Omega\|_{L^1(S^{n-1})} \leq 1$, $\|\Omega\|_{L^q(S^{n-1})} \leq 2^m$. Assume that ϕ and ψ are positive $C^1(0,\infty)$ functions satisfying the assumptions* (A-1) *and* (A-2).

Then for every $1 < p < \infty$, there exists a positive constant C_p independent of m such that

$$\|\lambda^*_{m,h}(f)\|_p \leq C_p m \|f\|_p \tag{3.8}$$

for every $f \in L^p(\mathbb{R}^{n+1})$.

In a similar way we get

Lemma 3.5. *Let m, Ω, ϕ and ψ be as in Lemma 3.4, and $a_m = 2^m$. Then for every $1 < p < \infty$, there exists a positive constant C_p independent of m such that*

$$\|F^*_m(f)\|_p \leq C_p m \|f\|_p$$

for every $f \in L^p(\mathbb{R}^{n+1})$, where

$$F^*_m(f)(x, x_{n+1}) = \sup_{k\in\mathbb{Z}} \left| \int_{a^k_m<|y|\leq a^{k+1}_m} \int_{1/2}^{1} f(x-\phi(|sy|)y', x_{n+1}-\psi(|sy|)) \frac{\Omega(y')}{|y|^n} \frac{ds}{s} dy \right|.$$

The following 5 lemmas are also proved in the same way as in [4]. So, omitting proofs, we only state them.

Lemma 3.6. *Let $h \in \triangle_\gamma$ for some $\gamma > 1$ and let m, Ω, ϕ and ψ be as in Lemma 3.1. Then for $\gamma' < p \leq \infty$, there exists a positive constant C_p independent of m such that*

$$\|\sigma^*_h(f)\|_p \leq C_p m^{1/\gamma'} \|f\|_p$$

for every $f \in L^p(\mathbb{R}^{n+1})$.

Lemma 3.7. *Let $h \in \triangle_\gamma$ for some $\gamma \geq 2$ and $\gamma' < p < \infty$. Let m, Ω, ϕ and ψ be as in Lemma 3.4. Then there exists a positive constant C_p such that*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \int_{a^k_m}^{a^{k+1}_m} |\sigma_{t,h} * g_k|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_p \leq C_p m^{1/\gamma'} \Big\| \big(\sum_{k\in\mathbb{Z}} |g_k|^2 \big)^{1/2} \Big\|_p \tag{3.9}$$

for any sequence $\{g_k\}$ of functions on $\mathbb{R}^{n+1}$.

Lemma 3.8. *Let $h \in \triangle_\gamma$ for some $1 < \gamma \leq 2$ and $2 \leq p < 2\gamma/(2-\gamma)$. Again let m, Ω, ϕ and ψ be as in Lemma 3.4. Then there exists a positive constant C_p such that*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \int_{a^k_m}^{a^{k+1}_m} |\sigma_{t,h} * g_k|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_p \leq C_p m^{1/2} \Big\| \big(\sum_{k\in\mathbb{Z}} |g_k|^2 \big)^{1/2} \Big\|_p \tag{3.10}$$

for any sequence $\{g_k\}$ of functions on $\mathbb{R}^{n+1}$.

Lemma 3.9. *Let $h \in \triangle_\gamma$ for some $1 < \gamma \leq 2$ and $2\gamma/(3\gamma-2) < p < 2$. Again let m, Ω, ϕ and ψ be as in Lemma 3.4. Then there exists a positive constant C_p such that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{a_m^k}^{a_m^{k+1}} |\sigma_{t,h} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \leq C_p m^{(3\gamma-2)/(2\gamma)} \Big\|\big(\sum_{k\in\mathbb{Z}} |g_k|^2\big)^{1/2}\Big\|_p \tag{3.11}$$

for any sequence $\{g_k\}$ of functions on $\mathbb{R}^{n+1}$.

Lemma 3.10. *Let $h \in \triangle_\gamma$ for some $1 < \gamma \leq 2$ and $2\gamma/(2\gamma-1) < p < 2$. Again let m, Ω, ϕ and ψ be as in Lemma 3.4. Then there exists a positive constant C_p such that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{a_m^k}^{a_m^{k+1}} |\sigma_{t,h} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \leq C_p m^{(2\gamma-1)/(2\gamma)} \Big\|\big(\sum_{k\in\mathbb{Z}} |g_k|^2\big)^{1/2}\Big\|_p \tag{3.12}$$

for any sequence $\{g_k\}$ of functions on $\mathbb{R}^{n+1}$.

4. Proofs of Theorems 1, 2 and 3

Once we have gotten Lemmas 3.1, 3.5–3.10, we use these lemmas in the case $q = 2$. Then we can prove our Theorems 1, 2 and 3 in quite similar ways as in Theorems 1.1, 1.2 and 1.3 in [4]. So, the details will be omitted.

5. Proofs of Theorems 4, 5 and 6

Let $v > -1$ and $q > 1$. Then if $\Omega \in B_q^{(0,v)}(S^{n-1})$ and satisfies the cancellation condition, it can be written as $\Omega = \sum_{\ell=1}^{\infty} \lambda_\ell \breve{\Omega}_\ell$, where $\lambda_\ell \in \mathbb{C}$ and $\breve{\Omega}_\ell$ is a q-block supported on a cap $B_\ell = B(x_\ell, \tau_\ell) \cap S^{n-1}$ on S^{n-1} and

$$\sum_{\ell=1}^{\infty} |\lambda_\ell|\big\{1 + \log^{v+1}\big(|B_\ell|^{-1}\big)\big\} < 2\|\Omega\|_{B_q^{(0,v)}(S^{n-1})} < \infty. \tag{5.1}$$

To each block $\breve{\Omega}_\ell$, we define

$$\Omega_\ell(y') = \breve{\Omega}_\ell(y') - \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \breve{\Omega}_\ell(x')\, d\sigma(x').$$

Let $\Lambda = \{\ell \in \mathbb{N};\ |B_\ell| \leq 1/2\}$ and set

$$\Omega_0 = \Omega - \sum_{\ell\in\Lambda} \lambda_\ell \Omega_\ell. \tag{5.2}$$

Then there exists a positive constant C such that the followings hold for all $\ell \in \Lambda$:

$$\int_{S^{n-1}} \Omega_\ell(x')\, d\sigma(x') = 0, \tag{5.3}$$

$$\|\Omega_\ell\|_{L^q(S^{n-1})} \le C|B_\ell|^{-1/q'}, \tag{5.4}$$

$$\|\Omega_\ell\|_{L^1(S^{n-1})} \le 2, \tag{5.5}$$

$$\Omega = \Omega_0 + \sum_{\ell\in\Lambda} \lambda_\ell \Omega_\ell. \tag{5.6}$$

Moreover, from (5.1) and the definition of Ω_ℓ it follows that

$$\|\Omega_0\|_{L^q(S^{n-1})} \le C \sum_{\ell\in\mathbb{N}\setminus\Lambda} 2^{-1/q'}|\lambda_\ell| \le C\|\Omega\|_{B_q^{(0,v)}(S^{n-1})}, \tag{5.7}$$

$$\int_{S^{n-1}} \Omega_0(x')\, d\sigma(x') = 0. \tag{5.8}$$

For $\ell \in \Lambda$, define a family of measures $\sigma^{(\ell)} = \{\sigma_{\ell,t,h}; 0 < t < \infty\}$ on $\mathbb{R}^{n+1}$, as in Definition 9, by

$$\int_{\mathbb{R}^{n+1}} f\, d\sigma_{\ell,t,h} = \frac{1}{t^\rho}\int_{t/2<|y|<t} f\big(\phi(|y|)y', \psi(|y|)\big)h(|y|)\frac{\Omega_\ell(y')}{|y|^{n-\rho}}\, dy.$$

We only discuss the case of increasing ϕ in the proof of Theorems 4, 5, 6, since decreasing case can be proved in the same way.

For $k \in \mathbb{Z}$ and $\ell \in \Lambda \cup 0$, we set $\omega_\ell = 2^{\theta_\ell}$, $\theta_\ell = [\log_2 |B_\ell|^{-1/q'}] + 1$, where $[\cdot]$ denotes the greatest integer function.

From Lemma 3.1, we have the following estimates:

$$\int_{\omega_\ell^k}^{\omega_\ell^{k+1}} |\hat{\sigma}_{\ell,t,h}|^2 \frac{dt}{t} \le C\theta_\ell; \tag{5.9}$$

$$\int_{\omega_\ell^k}^{\omega_\ell^{k+1}} |\hat{\sigma}_{\ell,t,h}|^2 \frac{dt}{t} \le C\theta_\ell |\phi(\omega_\ell^{k-1})\xi|^{-2\alpha/\theta_\ell}; \tag{5.10}$$

$$\int_{\omega_\ell^k}^{\omega_\ell^{k+1}} |\hat{\sigma}_{\ell,t,h}|^2 \frac{dt}{t} \le C\theta_\ell |\phi(\omega_\ell^{k+1})\xi|^{2\alpha/\theta_\ell}. \tag{5.11}$$

Moreover, we can use Lemmas 3.4–3.10, taking $m = \theta_\ell$. Now, we begin to prove Theorem 4 (a). From the definition of $\mu^\rho_{\Omega,\phi,\psi,h} f(x, x_{n+1})$ we get

$$\begin{aligned}\mu^\rho_{\Omega,\phi,\psi,h} f(x, x_{n+1}) = \bigg(\int_0^\infty \Big| \sum_{k=-\infty}^{\infty} \frac{1}{t^\rho}\int_{2^{-k-1}t\le|u|\le 2^{-k}t} \frac{\Omega(u')}{|u|^{n-\rho}} h(|u|) \\ \times f(x - \phi(|u|)u', x_{n+1} - \psi(|u|))du\Big|^2 \frac{dt}{t}\bigg)^{1/2}\end{aligned}$$

$$\begin{aligned}&\le \sum_{k=-\infty}^{\infty}\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{2^{-k-1}t\le|u|\le 2^{-k}t}\frac{\Omega(u')}{|u|^{n-\rho}}h(|u|)\\ &\qquad\times f(x-\phi(|u|)u',x_{n+1}-\psi(|u|))du\Big|^2\frac{dt}{t}\Big)^{1/2}\\ &\le C\widetilde{\mu}^\rho_{\Omega,\phi,\psi,h}f(x,x_{n+1}),\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mu}^\rho_{\Omega,\phi,\psi,h}f(x,x_{n+1})=\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{t/2\le|u|\le t}\frac{\Omega(u')}{|u|^{n-\rho}}h(|u|)\\ \times f(x-\phi(|u|)u',x_{n+1}-\psi(|u|))du\Big|^2\frac{dt}{t}\Big)^{1/2}.\end{aligned}$$

By (5.6), we have

$$\widetilde{\mu}^\rho_{\Omega,\phi,\psi,h}f(x,x_{n+1})\le\sum_{\ell\in\Lambda\cup 0}|\lambda_\ell|\widetilde{\mu}^\rho_{\ell,\phi,\psi,h}f(x,x_{n+1}), \tag{5.12}$$

where

$$\begin{aligned}\widetilde{\mu}^\rho_{\ell,\phi,\psi,h}f(x,x_{n+1})=\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{t/2\le|u|\le t}\frac{\Omega_\ell(u')}{|u|^{n-\rho}}h(|u|)\\ \times f(x-\phi(|u|)u',x_{n+1}-\psi(|u|))du\Big|^2\frac{dt}{t}\Big)^{1/2},\end{aligned}$$

so we have only to show the boundedness of $\widetilde{\mu}^\rho_{\ell,\phi,\psi,h}f$.

Since $\Delta_\gamma\subseteq\Delta_2$ for $\gamma\ge 2$, we may assume that $1<\gamma\le 2$ and $2\le p<\frac{2\gamma}{2-\gamma}$. For $\ell\in\mathbb{Z}$, let $\theta_{\ell,j}=\phi(\omega_\ell^j)$. From Lemma 2.1, we easily see that $\{\theta_{\ell,j}, j\in\mathbb{Z}\}$ is a lacunary sequence with $\theta_{\ell,j+1}/\theta_{\ell,j}\ge\omega_\ell^{1/b}>1$. Let $\{\widehat{\Psi}_{\ell,j}, j\in\mathbb{Z}\}$ be a smooth partition of unity in $(0,\infty)$, defined in Lemma 2.12, and set $\widehat{(T_{\ell,j}f)}(\xi,\eta)=\widehat{\Psi}_{\ell,j}(|\xi|)\hat{f}(\xi,\eta)$, $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}$. Then

$$\begin{aligned}&\widetilde{\mu}^\rho_{\ell,\phi,\psi,h}f(x,x_{n+1})\\ &\le\Big(\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}}\Big|\sum_{j\in\mathbb{Z}}(\Psi_{\ell,j+k}\otimes\delta_{\{0\}})*\sigma_{\ell,t,h}*f(x)\Big|^2\frac{dt}{t}\Big)^{1/2}\\ &\le\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}}|(\Psi_{\ell,j+k}\otimes\delta_{\{0\}})*\sigma_{\ell,t,h}*f(x)|^2\frac{dt}{t}\Big)^{1/2}\\ &=\sum_{j\in\mathbb{Z}}Q_{\ell,j}f(x,x_{n+1})\end{aligned} \tag{5.13}$$

where $\delta_{\{0\}}$ is the Dirac's delta at the origin in the x_{n+1} space, and $Q_{\ell,j}f(x,x_{n+1}) = \big(\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}} |(\Psi_{\ell,j+k}\otimes\delta_{\{0\}})*\sigma_{\ell,t,h}*f(x)|^2\frac{dt}{t}\big)^{1/2}$.

First, we compute L^2 norm of $Q_{\ell,j}f$. By Plancherel's theorem, Fubini's theorem and (5.10), (5.11), we obtain

$$\begin{aligned}\|Q_{\ell,j}f\|_2^2 &= \int_{\mathbb{R}^{n+1}}\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}} |(\Psi_{\ell,j+k}\otimes\delta_{\{0\}})*\sigma_{\ell,t,h}*f(x)|^2\frac{dt}{t}dx\,dx_{n+1}\\ &\le \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}} |\hat\sigma_{\ell,t,h}(\xi,\eta)|^2\frac{dt}{t}|\hat f(\xi,\eta)|^2d\xi d\eta.\end{aligned}$$

For $j\le -2$ and $\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}$ we get, using (5.10),

$$\|Q_{\ell,j}f\|_2 \le C\theta_\ell^{1/2}\omega_\ell^{j\alpha/(b\theta_\ell)} \le C(\log|B_\ell|^{-1})^{1/2}2^{j\alpha/b}\|f\|_2.$$

For $j\ge 2$ and $\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}$ we get, using (5.11),

$$\|Q_{\ell,j}f\|_2 \le C\theta_\ell^{1/2}\omega_\ell^{-j\alpha/(b\theta_\ell)} \le C(\log|B_\ell|^{-1})^{1/2}2^{-j\alpha/b}\|f\|_2.$$

For $-1\le j\le 1$ and $\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}$ we get, using (5.9),

$$\|Q_{\ell,j}f\|_2 \le C\theta_\ell^{1/2} \le C(\log|B_\ell|^{-1})^{1/2}.$$

Hence, we obtain

$$\|Q_{\ell,j}f\|_2 \le C(\log|B_\ell|^{-1})^{1/2}2^{-|j|\alpha/b}\|f\|_2. \tag{5.14}$$

Next, by using Lemma 3.8 and Lemma 2.12, we have

$$\|Q_{\ell,j}f\|_p \le C(\log|B_\ell|^{-1})^{1/2}\|f\|_p, \qquad \text{for } 2\le p<\frac{2\gamma}{2-\gamma}. \tag{5.15}$$

Interpolating between (5.14) and (5.15), we can find a number $0<\theta<1$ such that

$$\|Q_{\ell,j}f\|_p \le C(\log|B_\ell|^{-1})^{1/2}2^{-|j|\theta\alpha/b}\|f\|_p, \qquad \text{for } 2\le p<\frac{2\gamma}{2-\gamma}. \tag{5.16}$$

Hence, combining (5.12), (5.13) and (5.16) completes the proof of Theorem 4 (a).

Now, applying respectively Lemmas 3.7, 3.9, 3.10 in place of Lemma 3.8, we can obtain similar L^p estimates for $Q_{\ell,j}f$ as in (5.16), and prove Theorem 4 (b), (c) and (d).

Let us next to turn to prove Theorem 6. First, as before, we have only to show that

$$\|\widetilde{\mathcal{M}}^{(\gamma)}_{\phi,\psi,\ell}f\|_{L^p} \le C_p(\log|B_\ell|^{-1})^{1/\gamma'}\|f\|_{L^p}, \tag{5.17}$$

where

$$\widetilde{\mathcal{M}}^{(\gamma)}_{\phi,\psi,\ell}f(x,x_{n+1}) = \sup_{\|h\|_{L^\gamma(\mathbb{R}_+,dt/t)}\leq 1} |\widetilde{\mu}^{\rho}_{\ell,\phi,\psi,h}|.$$

Meanwhile, we know from [4] that when $\gamma = 1$, $\|\widetilde{\mathcal{M}}^{(1)}_{\phi,\psi,\ell}f\|_{L^\infty} \leq C\|f\|_{L^\infty}$. So we start with the case $\gamma = 2$ and then use a suitable interpolation for $1 < \gamma < 2$.

Let

$$E_\ell f(x,x_{n+1}) = \int_{\mathbb{S}^{n-1}} f(x-\phi(s)u, x_{n+1}-\psi(s))\Omega_\ell(u)d\sigma(u).$$

As in the proof of Theorem 4, for each $\ell \in \mathbb{Z}$, let $\{\widehat{\Psi}_{\ell,j}, j \in \mathbb{Z}\}$ be a smooth partition of unity in $\mathbb{R}^n\backslash 0$. As in Al-Qassem-Pan [4, pp. 92-93], we have by duality and a change of variable

$$\begin{aligned}
\widetilde{\mathcal{M}}^{(2)}_{\ell,\phi,\psi}f(x,x_{n+1}) &\leq \sup_{\|h\|_{L^\gamma(\mathbb{R}_+,dt/t)}\leq 1}\Big(\int_0^\infty\Big(\int_{t/2}^t |h(s)||E_\ell f(x,x_{n+1})|\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{1/2}\\
&\leq \Big(\int_0^\infty\Big(\int_{t/2}^t |E_\ell f(x,x_{n+1})|\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{1/2}\\
&\leq \Big(\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}}\Big(\int_{t/2}^t |E_\ell f(x,x_{n+1})|^2\frac{ds}{s}\Big)\frac{dt}{t}\Big)^{1/2}\\
&\leq \Big(\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}}\Big(\int_{t/2}^t \Big|\sum_{j\in\mathbb{Z}} Y_{\ell,k+j,s}f(x,x_{n+1})\Big|^2\frac{ds}{s}\Big)\frac{dt}{t}\Big)^{1/2}\\
&\leq \sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}}\Big(\int_{t/2}^t |Y_{\ell,k+j,s}f(x,x_{n+1})|^2\frac{ds}{s}\Big)\frac{dt}{t}\Big)^{1/2}\\
&= \sum_{j\in\mathbb{Z}} X_{\ell,j}f(x,x_{n+1}),
\end{aligned}$$

where

$$Y_{\ell,j,s}f(x,x_{n+1}) = \int_{S^{n-1}} (T_{\ell,j}f)(x-\phi(s)u, x_{n+1}-\psi(s))\Omega_\ell(u)d\sigma(u),$$

$$X_{\ell,j}f(x,x_{n+1}) = \Big(\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k}^{\omega_\ell^{k+1}}\Big(\int_{t/2}^t |Y_{\ell,k+j,s}f(x,x_{n+1})|^2\frac{ds}{s}\Big)\frac{dt}{t}\Big)^{1/2}.$$

Thus, we have only to prove the L^p boundedness of $X_{\ell,j}f$. We start by proving it in the case of $p = 2$. By employing Plancherel's theorem, Fubini's theorem, Lemma 3.2 and (5.10), (5.11), we obtain as in the proof of Theorem 4

$$
\begin{aligned}
\|X_{\ell,j}f\|_2^2 &= \int_{\mathbb{R}^{n+1}} \sum_{k\in\mathbb{Z}} \int_{\omega_\ell^k}^{\omega_\ell^{k+1}} \left(\int_{t/2}^{t} |Y_{\ell,k+j,s}f(x,x_{n+1})|^2 \frac{ds}{s} \right) \frac{dt}{t} dx dx_{n+1} \\
&\le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \int_{\theta_{\ell,k+j+1}\le|\xi|\le\theta_{\ell,k+j-1}} \int_{\omega_\ell^k}^{\omega_\ell^{k+1}} \left(\int_{t/2}^{t} |\hat{f}(\xi,\eta)|^2 \right. \\
&\quad \times \left. \left| \int_{\mathbb{S}^{n-1}} \Omega_\ell(x) e^{-i(\phi(s)\xi\cdot x+\eta\psi(s))} d\sigma(x) \right|^2 \frac{ds}{s} \right) \frac{dt}{t} d\xi d\eta \\
&\le C \log|B_\ell|^{-1} 2^{-\alpha|j|/(2b)} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \int_{\theta_{\ell,k+j+1}\le|\xi|\le\theta_{\ell,k+j-1}} |\hat{f}(\xi,\eta)|^2 d\xi d\eta \\
&\le C \log|B_\ell|^{-1} 2^{-\alpha|j|/(2b)} \|f\|_2^2,
\end{aligned}
$$

and hence

$$
\|X_{\ell,j}f\|_2 \le C(\log|B_\ell|^{-1})^{1/2} 2^{-\alpha|j|/b} \|f\|_2. \tag{5.18}
$$

Next, we consider the case $p > 2$. Choose g in $L^{(p/2)'}$ with $\|g\|_{(p/2)'} \le 1$. Then by a similar argument in [4, p. 94], we have $\|X_{\ell,j}f\|_p^2 \le \| \sum_{k\in\mathbb{Z}} |T_{\ell,k+j}f|^2\|_{(p/2)} \|F_\ell^*(\tilde{g})\|_{(p/2)'}$, where

$$
F_\ell^*(f)(x,x_{n+1}) = \sup_{k\in\mathbb{Z}} \left(\int_{\omega_\ell^k<|y|\le\omega_\ell^{k+1}} \int_{1/2}^{1} f(x-\phi(|sy|)y', x_{n+1}-\psi(|sy|)) \frac{\Omega_\ell(y')}{|y|^n} \frac{ds}{s} dy \right).
$$

By using Lemma 3.5 and Lemma 2.14, we have

$$
\|X_{\ell,j}f\|_p \le C_p(\log|B_\ell|^{-1})^{1/2} \|f\|_p \qquad \text{for } 2 \le p < \infty. \tag{5.19}
$$

By interpolation between (5.18) and (5.19), there exists a constant $0 < \theta < 1$ such that

$$
\|X_{\ell,j}f\|_p \le C_p(\log|B_\ell|^{-1})^{1/2} 2^{-\theta\alpha|j|/b} \|f\|_p \qquad \text{for } 2 \le p < \infty. \tag{5.20}
$$

which ends the proof of the desired (5.17).
We use the same step in [4] for interpolation between $\gamma = 1$ and $\gamma = 2$, and obtain (5.17) for $1 < \gamma < 2$. The proof of Theorem 6 (a) is completed.

As for Theorem 6 (b), adapting a similar argument employed in the proof of Theorem 6 (a), we get

$$
\|\mathcal{M}^{(\gamma)}_{\phi,\psi,\ell} f\|_{L^p} \le C_p(\log|B_\ell|^{-1})^{1/2} \|f\|_{L^p},
$$

which shows the conclusion of Theorem 6 (b).

Finally, let's prove Theorem 5. Notice that

$$
\mu^\rho_{\ell,\phi,\psi,h} f(x,x_{n+1}) \le \mathcal{M}^{(\gamma)}_{\ell,\phi,\psi,h} f(x,x_{n+1}) = \sup_{\|h\|_{L^\gamma(\mathbb{R}_+,dt/t)}} |\mu^\rho_{\ell,\phi,\psi,h} f(x,x_{n+1})|
$$

and apply Theorem 6.

6. Appendix

In this section, we give several examples of monotonic functions satisfying our assumptions (A-1) and (A-2).

Example (1). For $0 < \alpha < 1$, set

$$\Phi(t) = t^{\alpha} e^{t}.$$

Then Φ is nonconvex, positive, increasing, and $\Phi(t)/(t\Phi'(t))$ is strictly decreasing and bounded. $t\Phi'(t)$ is strictly increasing. Furthermore, there exists no $C > 1$ such that $\Phi(2t) \le C\Phi(t)$ $(t > 0)$.

In fact, we have

$$\begin{aligned} \Phi'(t) &= t^{\alpha-1}(t+\alpha)e^{t}, \\ \Phi''(t) &= t^{\alpha-2}\big(t^2 + 2\alpha t + \alpha(\alpha-1)\big)e^{t}. \end{aligned}$$

From this it follows

$$\frac{\Phi(t)}{t\Phi'(t)} = \frac{1}{t+\alpha},$$

and

$$\Phi''(t) = \begin{cases} < 0 & \text{for } 0 < t < \sqrt{\alpha} - \alpha \\ > 0 & \text{for } t > \sqrt{\alpha} - \alpha, \end{cases} \qquad \big(t\Phi'(t)\big)' = t^{\alpha-1}(\alpha^2 + (2\alpha+1)t + t^2)e^{t}.$$

Hence, Φ is nonconvex, positive, increasing and $\Phi(t)/(t\Phi'(t))$ and $1/t\Phi'(t)$ are strictly decreasing. The last claim follows from $\Phi(2t)/\Phi(t) = 2^{\alpha}e^{t}$.

Example (2). Set

$$\Phi(t) = (t^2 - \sin^2 t)e^{at}$$

Then for $0 < a < 2/\pi$, Φ is convex, positive, increasing, and $\Phi(t)/(t\Phi'(t))$ is positive and bounded, but non-monotonic. $t\Phi'(t)$ is increasing.

We check this. Let $g(t) = t^2 - \sin^2 t$. Then

$$g'(t) = 2t - 2\sin t\cos t = 2t - \sin 2t > 0 \text{ for } t > 0,$$

and

$$g''(t) = 2 - 2\cos 2t \ge 0 \text{ for } t > 0.$$

Hence, $g(t)$ is increasing and convex. Since e^{at} is clearly increasing and convex, we see that $\Phi(t)$ is also increasing and convex. And we get

$$\begin{aligned} \Phi'(t) &= \{(2t - \sin 2t) + a(t^2 - \sin^2 t)\}e^{at} \\ \varphi(t) := \frac{\Phi(t)}{t\Phi'(t)} &= \frac{t^2 - \sin^2 t}{t\{(2t - \sin 2t) + a(t^2 - \sin^2 t)\}}. \end{aligned}$$

From this we have

$$\lim_{t\to 0}\varphi(t)=\frac{1}{4},\ \varphi(\pi)=\frac{1}{2+a\pi},\ \lim_{t\to\infty}\varphi(t)=0.$$

Hence, if $2+a\pi<4$, i.e. $0<a<2/\pi$, we see that $\varphi(t)$ is not monotone. From the above, we also see that φ is bounded. Since t, $2t-\sin 2t$, $t^2-\sin^2 t$, e^{at} are increasing, $t\Phi'(t)$ is also increasing.

Example (3). Let $\psi(t)=\begin{cases}1, & 0\le t\le \pi/2\\ \sin t, & t\ge \pi/2,\end{cases}$ and

$$\Phi(t)=2t^2+t\psi(t).$$

Then $\Phi(t)$ is positive and increasing on $(0,\infty)$ and satisfies $\Phi(2t)<7\Phi(t)$, but $\Phi(t)$ is not convex nor $t\Phi'(t)$ is not monotone. Moreover, $|\Phi(t)/(t\Phi'(t))|<1$ and $\Phi(t)/(t\Phi'(t))$ is not monotone.

In fact, since $\Phi(t)=2t^2+t\sin t$ for $t\ge\pi/2$, we have

$$\Phi'(t)=4t+t\cos t+\sin t=2t+t(1+\cos t)+t+\sin t>0,$$

and so we see that $\Phi(t)$ is increasing on $(0,\infty)$. Since $\Phi''(t)=4+2\cos t-t\sin t$ for $t\ge\pi/2$, we see that

$$\Phi''(5\pi/2)=4-5\pi/2<0 \text{ and } \Phi''(7\pi/2)=4+7\pi/2>0.$$

This means that $\Phi(t)$ is not convex. Next, since for $t\ge\pi/2$

$$\big(t\Phi'(t)\big)'=8t+3t\cos t-t^2\sin t+\sin t,$$

we have

$$\big(t\Phi'(t)\big)'\Big|_{t=2\pi}=22\pi>0,\ \text{and}\ \big(t\Phi'(t)\big)'\Big|_{t=9\pi/2}=36\pi-81\pi^2/4+1<0,$$

which implies that $t\Phi'(t)$ is not monotone.

For $0<t<\pi/2$ we get

$$\frac{\Phi(t)}{t\Phi'(t)}=\frac{2t^2+t}{4t^2+t}<1.$$

And for $t>\pi/2$ we get

$$\frac{\Phi(t)}{t\Phi'(t)}=\frac{2t^2+t\sin t}{4t^2+t\sin t+t^2\cos t}\le\frac{1}{2+\frac{\cos t-\frac{\sin t}{t}}{2+\frac{\sin t}{t}}}\le\frac{1}{2-\frac{1+\frac{2}{\pi}}{2-\frac{1}{\pi}}}=\frac{2\pi-1}{3\pi-4}<1.$$

Hence we get $|\Phi(t)/(t\Phi'(t))|<1$. Moreover, we get

$$\frac{\Phi(t)}{t\Phi'(t)}\Big|_{t=\pi}=\frac{2}{3},\ \frac{\Phi(t)}{t\Phi'(t)}\Big|_{t=2\pi}=\frac{2}{5},\ \text{and}\ \frac{\Phi(t)}{t\Phi'(t)}\Big|_{t=3\pi}=\frac{2}{3},$$

which shows that $\Phi(t)/(t\Phi'(t))$ is not monotone.

Finally, for $0<t\leq\pi/4$ we have

$$\frac{\Phi(2t)}{\Phi(t)}=\frac{8t^2+2t}{2t^2+t}<4.$$

For $\pi/4\leq t\leq\pi/2$ we have

$$\frac{\Phi(2t)}{\Phi(t)}=\frac{8t^2+2t\sin 2t}{2t^2+t}<4.$$

And for $t\geq\pi/2$ we have

$$\frac{\Phi(2t)}{\Phi(t)}=\frac{8t^2+2t\sin 2t}{2t^2+t\sin t}\leq\frac{8t+2}{2t-1}=4+\frac{6}{2t-1}\leq 4+\frac{6}{\pi-1}<7.$$

Altogether we have $\Phi(2t)<7\Phi(t)$.

Example (4). Let

$$\Phi(t)=te^{-1/t^2}.$$

Then $\Phi(t)$ is positive, increasing and nonconvex, and both $\Phi(t)/(t\Phi'(t))$ and $t\Phi'(t)$ are increasing.

In fact, we have

$$\Phi'(t)=\Big(1+\frac{2}{t^2}\Big)e^{-1/t^2},\quad \frac{\Phi(t)}{t\Phi'(t)}=\frac{1}{1+\frac{2}{t^2}},\quad t\Phi'(t)=\Big(t+\frac{2}{t}\Big)e^{-1/t^2},$$

$$\Phi''(t)=\frac{2}{t^5}(2-t^2)e^{-1/t^2},\ \text{and}\ \big(t\Phi'(t)\big)'=\Big(1+\frac{4}{t^4}\Big)e^{-1/t^2}.$$

Hence, we see that $\Phi(t)$ is positive, increasing and nonconvex, and both $\Phi(t)/(t\Phi'(t))$ and $t\Phi'(t)$ are increasing.

Next, we state some examples in the case where $\Phi(t)$ is decreasing.

Example (5). Let

$$\Phi(t)=\frac{1}{30t}+\frac{1}{1+t}.$$

Then Φ is a positive, convex, decreasing function on $(0,\infty)$ such that $\Phi(t)/\big(t\Phi'(t)\big)$ is bounded and $\Phi(t)<2\Phi(2t)$, but $-t\Phi'(t)$ is non-monotonic.

In fact, we have

$$\Phi'(t)=-\frac{1}{30t^2}-\frac{1}{(1+t)^2},\ \ -t\Phi'(t)=\frac{1}{30t}+\frac{t}{(1+t)^2},\ \ \Phi''(t)=\frac{2}{30t^3}+\frac{2}{(1+t)^3}>0.$$

And

$$\left|\frac{\Phi(t)}{t\Phi'(t)}\right| = \frac{31t^2+32t+1}{31t^2+2t+1} \geq 1.$$

Since the right side of the above equality tends to 1 as $t \to \infty$, we see that there exists $C > 0$ such that

$$1 \leq \left|\frac{\Phi(t)}{t\Phi'(t)}\right| \leq C, \quad 0 \leq t < \infty.$$

Furthermore, we get

$$(-t\Phi'(t))' = -\frac{1}{30t^2} + \frac{1}{(1+t)^2} - \frac{2t}{(1+t)^3} = -\frac{1}{30t^2} + \frac{1-t}{(1+t)^3},$$

and so

$$(-t\Phi'(t))'\Big|_{t=1/2} = -\frac{4}{30} + \frac{4}{27} > 0.$$

Clearly $(-t\Phi'(t))'$ is negative if t is near 0 or $1 \leq t < \infty$. Thus, we see that $-t\Phi'(t)$ is non-monotonic.

Example (6). Let $\psi(t) = \begin{cases} 1, & 0 \leq t \leq \pi/2 \\ \sin t, & t \geq \pi/2, \end{cases}$ and

$$\Phi(t) = \frac{3}{t} + \frac{1}{t^2}\psi(t).$$

Then $\Phi(t)$ is positive and decreasing on $(0, \infty)$ and satisfies $\Phi(2t) > \frac{1}{4}\Phi(t)$, but $\Phi(t)$ is not convex nor $-t\Phi'(t)$ is not monotone. Moreover, $|\Phi(t)/(t\Phi'(t))| < 1$ and $\Phi(t)/(t\Phi'(t))$ is not monotone.

In fact, since $\Phi(t) = \frac{3}{t} + \frac{1}{t^2}\sin t$ for $t \geq \pi/2$, we have

$$\Phi'(t) = -\frac{3}{t^2} - \frac{2}{t^3}\sin t + \frac{1}{t^2}\cos t = -\frac{2}{t^2}\Big(1 + \frac{1}{t}\sin t\Big) - \frac{1}{t^2}(1-\cos t) < 0,$$

and so we see that $\Phi(t)$ is decreasing on $(0, \infty)$. Since for $t \geq \pi/2$

$$\Phi''(t) = \frac{6}{t^3} + \frac{6}{t^4}\sin t - \frac{4}{t^3}\cos t - \frac{1}{t^2}\sin t,$$

we get

$$\Phi''(5\pi/2) = \frac{6 \cdot \frac{5\pi}{2} + 6 - \frac{25\pi^2}{4}}{\frac{5^4\pi^4}{2^4}} < 0.$$

This means that $\Phi(t)$ is not convex. Next, since for $t \geq \pi/2$

$$\big(-t\Phi'(t)\big)' = -\frac{3}{t^2} - \frac{4}{t^3}\sin t + \frac{3}{t^2}\cos t + \frac{1}{t}\sin t,$$

we have

$$\left(-t\Phi'(t)\right)'\Big|_{t=5\pi/2} = -\frac{12}{25\pi^2} - \frac{32}{125\pi^3} + \frac{2}{5\pi} > 0,$$

and

$$\left(-t\Phi'(t)\right)'\Big|_{t=\pi} = -\frac{6}{\pi^2} < 0,$$

which implies that $-t\Phi'(t)$ is not monotone. Moreover, for $0 < t < \pi/2$ we get

$$-\frac{\Phi(t)}{t\Phi'(t)} = \frac{\frac{3}{t} + \frac{1}{t^2}}{\frac{3}{t} + \frac{2}{t^2}} < 1.$$

And for $t \geq \pi/2$ we get

$$-\frac{\Phi(t)}{t\Phi'(t)} = \frac{\frac{3}{t} + \frac{1}{t^2}\sin t}{\frac{3}{t} + \frac{2}{t^2}\sin t - \frac{1}{t}\cos t} = \frac{1}{1 + \frac{\frac{\sin t}{t} - \cos t}{3 + \frac{\sin t}{t}}} \leq \frac{1}{1 - \frac{\frac{1}{\pi}+1}{3-\frac{1}{\pi}}} = \frac{3\pi - 1}{2\pi - 2} < 2.$$

Hence we get $|\Phi(t)/(t\Phi'(t))| < 2$.

$$-\frac{\Phi(t)}{t\Phi'(t)}\Big|_{t=\pi} = \frac{3}{4}, -\frac{\Phi(t)}{t\Phi'(t)}\Big|_{t=2\pi} = \frac{3}{2}, \text{ and } -\frac{\Phi(t)}{t\Phi'(t)}\Big|_{t=3\pi} = \frac{3}{4},$$

which shows that $-\Phi(t)/(t\Phi'(t))$ is not monotone.

Finally, for $0 < t \leq \pi/4$ we have

$$\frac{\Phi(t)}{\Phi(2t)} \leq \frac{\frac{3}{t} + \frac{1}{t^2}}{\frac{3}{2t} + \frac{1}{4t^2}} < 4.$$

For $\pi/4 \leq t \leq \pi/2$, we have

$$\frac{\Phi(t)}{\Phi(2t)} = \frac{\frac{3}{t} + \frac{1}{t^2}}{\frac{3}{2t} + \frac{1}{4t^2}\sin 2t} \leq 2 + \frac{2}{3t} \leq 2 + \frac{8}{3\pi} < 3.$$

And for $t \geq \pi/2$ we have

$$\frac{\Phi(t)}{\Phi(2t)} = \frac{\frac{3}{t} + \frac{1}{t}\frac{\sin t}{t}}{\frac{3}{2t} + \frac{1}{2t}\frac{\sin 2t}{2t}} < \frac{\frac{3}{t} + \frac{1}{t}}{\frac{3}{2t} - \frac{1}{2t}} = 4.$$

Altogether we have $\Phi(2t) > \frac{1}{4}\Phi(t)$.

Example (7). Let $\Phi(t) = t^{-\alpha}e^{1/t}$, $\alpha > 0$. Then $\Phi(t)$ is positive, decreasing and convex on $(0, \infty)$, and $|\Phi(t)/(t\Phi'(t))| < 1/\alpha$, but $\lim_{t\to 0} \Phi(t)/\Phi(2t) = +\infty$, i.e. there is no positive constant C such that $\Phi(2t) \geq C\Phi(t)$, $t > 0$. $-\Phi(t)/(t\Phi'(t))$ is increasing, and $-t\Phi'(t)$ is decreasing.

In fact, we have

$$\Phi'(t) = -t^{-\alpha-1}\Big(\alpha + \frac{1}{t}\Big)e^{1/t}, \text{ and } -\frac{\Phi(t)}{t\Phi'(t)} = \frac{1}{\alpha + 1/t}.$$

Hence we see that $-t\Phi'(t)$ is decreasing, $|\Phi(t)/(t\Phi'(t))| \leq 1/\alpha$, and $-\Phi(t)/(t\Phi'(t))$ is increasing. Since both $t^{-\alpha}$ and $e^{1/t}$ are positive, decreasing and convex, we see that $\Phi(t)$ is also positive, decreasing and convex. And

$$\lim_{t\to 0}\frac{\Phi(t)}{\Phi(2t)} = \lim_{t\to 0} 2^{\alpha}e^{1/(2t)} = +\infty.$$

Example (8). Let $\Phi(t) = t^{-\alpha}e^{-t}$, $\alpha > 0$. Then $\Phi(t)$ is positive, decreasing and convex on $(0,\infty)$, and $|\Phi(t)/(t\Phi'(t))| < 1/\alpha$, but $\lim_{t\to\infty}\Phi(t)/\Phi(2t) = +\infty$, i.e. there is no positive constant C such that $\Phi(2t) \geq C\Phi(t)$, $t > 0$. $-\Phi(t)/(t\Phi'(t))$ is decreasing, and $-t\Phi'(t)$ is decreasing.

In fact, we have

$$\Phi'(t) = -t^{-\alpha-1}\Big(\alpha + t\Big)e^{-t}, \text{ and } -\frac{\Phi(t)}{t\Phi'(t)} = \frac{1}{\alpha + t}.$$

Hence we see that $-t\Phi'(t)$ is decreasing, $|\Phi(t)/(t\Phi'(t))| \leq 1/\alpha$, and $-\Phi(t)/(t\Phi'(t))$ is decreasing. Since both $t^{-\alpha}$ and e^{-t} are positive, decreasing and convex, we see that $\Phi(t)$ is also positive, decreasing and convex. And

$$\lim_{t\to\infty}\frac{\Phi(t)}{\Phi(2t)} = \lim_{t\to\infty} 2^{\alpha}e^{t} = +\infty.$$

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Wenjuan Li
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing 100875
P. R. China

Current address:
Mathematisches Seminar der Universität Kiel
D-24098, Germany
E-mail: facingworld0718@gmail.com

Kôzô Yabuta
Research Center for Mathematical Sciences
Kwansei Gakuin University
Gakuen 2-1, Sanda 669-1337
Japan
E-mail: kyabuta3@kwansei.ac.jp