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MEASURING THE "NON-STOPPING TIMENESS" OF ENDS OF PREVISIBLE SETS

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Abstract. In this paper, we propose some "measurements" of the "non-stopping timeness" of ends $\mathcal{G}$ of previsible sets, such that $\mathcal{G}$ avoids stopping times, in an ambiant filtration. We then study several explicit examples, involving last passage times of some remarkable martingales.

1. Introduction: About Ends of Previsible Sets

In this paper, we are interested in random times $\mathcal{G}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ as ends of $(\mathcal{F}_t)$-previsible sets Γ, that is,

$$\mathcal{G} \equiv \mathcal{G}_\Gamma = \sup\{t : \ (t, \omega) \in \Gamma\}. \tag{1}$$

For simplicity, we shall make the following assumptions:

(C) All $((\mathcal{F}_t), P)$-martingales are continuous;

(A) For any $(\mathcal{F}_t)$-stopping time T, $P(\mathcal{G} = T) = 0$.

To such a random time, one associates the Azéma supermartingale

$$Z_t^{\mathcal{G}} = P(\mathcal{G} > t | \mathcal{F}_t),$$

which, under (C) and (A), admits a continuous version as shown by the following theorem.

Theorem 1.1. *Under (C) and (A), there exists a unique positive local martingale $(N_t, t \geq 0)$, with $N_0 = 1$, such that*

$$Z_t^{\mathcal{G}} = P(\mathcal{G} > t | \mathcal{F}_t) = \frac{N_t}{S_t},$$

where $S_t := \sup_{s \leq t} N_s$ for $t \geq 0$.

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Proof. See [8]: page 16, Proposition 1.3. ∎

Note that since $\mathcal{G} < \infty$ a.s., $N_t \underset{t\to\infty}{\longrightarrow} 0$ a.s. We note further that $\log(S_\infty)$ is distributed exponentially, since by Doob's maximal identity

$$\log(S_\infty) \overset{\text{(law)}}{=} \log\left(\frac{1}{U}\right),$$

where U is uniform on $[0,1]$. Then, the additive decomposition of the supermartingale N_t/S_t is given by

$$\frac{N_t}{S_t} = 1 + \int_0^t \frac{dN_u}{S_u} - \log(S_t) = E[\log(S_\infty)|\mathcal{F}_t] - \log(S_t). \tag{2}$$

Note that the martingale $E[\log(S_\infty)|\mathcal{F}_t]$ belongs to BMO since from (2),

$$E[\log(S_\infty) - \log S_t|\mathcal{F}_t] \leq 1.$$

In a number of questions, it is very interesting to consider the smallest filtration $(\mathcal{F}'_t)_{t\geq 0}$, which contains $(\mathcal{F}_t)$, and makes $\mathcal{G}$ a stopping time; this filtration is usually denoted $(\mathcal{F}^{\mathcal{G}}_t)_{t\geq 0}$. One of the interests of $(Z^{\mathcal{G}}_t)$ is that it allows to write any $(\mathcal{F}_t)$-martingale as a semimartingale in $(\mathcal{F}^{\mathcal{G}}_t)_{t\geq 0}$; see e.g. [2, 3, 8, 9], for both general formulae and many examples.

Recently, it has been understood that Black-Scholes like formulae are closely related with certain such $\mathcal{G}$'s, thus throwing a new light on a cornerstone of mathematical finance, see, e.g. [6, 7]. In the present paper, with (A) as our essential hypothesis, we would like to measure "how much $\mathcal{G}$ differs from an $(\mathcal{F}_t)$ stopping time". The remainder of this paper consists in two sections. In Section 2, we propose several criterions to measure the NST ($\equiv$ Non Stopping Timeness) of $\mathcal{G}$'s which satisfy (C) and (A). In Section 3, we compute explicitly this function $m_{\mathcal{G}}$ for various examples, where $\mathcal{G}$ is the last passage time at a level of a martingale which converges to 0, as $t \to \infty$.

2. Several Possible "NST" Criterions

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, an $(\mathcal{F}_t)$-previsible set Γ and a random time $\mathcal{G}$ given by (1). Our aim is to discuss the difference between $\mathcal{G}$ and an $(\mathcal{F}_t)$-stopping time. A natural question is to consider the function

$$m_{\mathcal{G}}(t) = E\left[\left(1_{(\mathcal{G}\geq t)} - P(\mathcal{G} > t|\mathcal{F}_t)\right)^2\right].$$

If $\mathcal{G}$ is an $(\mathcal{F}_t)$-stopping time, the Azéma supermartingale $Z^{\mathcal{G}}_t \equiv P(\mathcal{G} \geq t|\mathcal{F}_t)$ is identically equal to $1_{(\mathcal{G}\geq t)}$. Thus, $m_{\mathcal{G}}(t) = 0$ for all t. If $\mathcal{G}$ is not an $(\mathcal{F}_t)$-stopping time, a simple but useful remark is

$$m_{\mathcal{G}}(t) = E\left[Z^{\mathcal{G}}_t\left(1 - Z^{\mathcal{G}}_t\right)\right]. \tag{3}$$

Instead of considering the “full”function $(m_{\mathcal{G}}(t),\ t \geq 0)$, we may consider only:

$$m_{\mathcal{G}}^{*} = \sup_{t\geq 0} m_{\mathcal{G}}(t) \tag{4}$$

as a “global” measurement of the NST of $\mathcal{G}$.

Here are two other, a priori natural, measurements of the NST of $\mathcal{G}$:

$$m_{\mathcal{G}}^{**} = E\left[\sup_{t\geq 0}\left(Z_t^{\mathcal{G}}\left(1 - Z_t^{\mathcal{G}}\right)\right)\right] \tag{5}$$

and

$$\widetilde{m}_{\mathcal{G}} = \sup_{T\geq 0} E\left[Z_T^{\mathcal{G}}\left(1 - Z_T^{\mathcal{G}}\right)\right] \tag{6}$$

where T runs over all $(\mathcal{F}_t)$ stopping times.

However, we cannot expect to learn very much from $m_{\mathcal{G}}^{**}$ and $\widetilde{m}_{\mathcal{G}}$, since it is easily shown the following result.

Lemma 2.2.

$$m_{\mathcal{G}}^{**} = \widetilde{m}_{\mathcal{G}} = \frac{1}{4}. \tag{7}$$

Proof.

(i) The fact that $m_{\mathcal{G}}^{**} = 1/4$ follows immediately from

$$\sup_{x\in[0,1]} x(1-x) = \frac{1}{4},$$

and the fact that, a.s., the range of the process $(Z_t^{\mathcal{G}}, t \geq 0)$ is $[0,1]$ since $Z_0^{\mathcal{G}} = 1$, $Z_\infty^{\mathcal{G}} = 0$, and $(Z_t^{\mathcal{G}}, t \geq 0)$ is continuous.

(ii) Let us consider $T_a = \inf\{t : Z_t^{\mathcal{G}} = a\}$, for $0 < a < 1$. Then

$$Z_t^{\mathcal{G}}(1 - Z_t^{\mathcal{G}})\big|_{t=T_a} = a(1-a).$$

Hence,

$$\sup_{a\in]0,1[} E\left[Z_{T_a}^{\mathcal{G}}\left(1 - Z_{T_a}^{\mathcal{G}}\right)\right] = \sup_{a\in]0,1[} (a(1-a)) = \frac{1}{4}. \qquad \blacksquare$$

An immediate result is that $1/4$ is an upper bound of $m_{\mathcal{G}}$ due to the definition. Moreover, there are some other measurements which have been investigated in a number of literatures.

Remark 2.3.

(1) (The optional stopping time discrepancy $\mu_{\mathcal{G}}$) It has been shown in [4], of stopping times, among random times, as the times τ such that for every bounded martingale $(M_t)_{t\geq 0}$ one has

$$M_\tau = E\left[M_\infty|\mathcal{F}_\tau\right],$$

where, under our hypothesis (C), we may define $\mathcal{F}_\tau = \sigma\{H_\tau;\ H \text{ previsible}\}$. Thus, another measurement of the NST of $\mathcal{G}$ is

$$\mu_{\mathcal{G}} = \sup_{\substack{M_\infty \in L^2(\mathcal{F}_\infty) \\ E(M_\infty^2)\le 1}} E\left[\left(M_{\mathcal{G}} - E\left[M_\infty|\mathcal{F}_{\mathcal{G}}\right]\right)^2\right].$$

(2) (Distance from stopping times) We introduce

$$\nu_{\mathcal{G}} = \inf_{T\ge 0} E|\mathcal{G} - T|,$$

where T runs over all $(\mathcal{F}_t)$ stopping times. However, this quantity may be infinite as $\mathcal{G}$ may have infinite expectation. We note that this distance was precisely computed by du Toit-Peskir-Shiryaev in the example of [1]. A more adequate distance may be:

$$\nu'_{\mathcal{G}} = \inf_{T\ge 0}\left(E\left[\frac{|\mathcal{G}-T|}{1+|\mathcal{G}-T|}\right]\right)$$

In this paper we concentrate uniquely on the study of $(m_{\mathcal{G}}(t),\ t\ge 0)$ using the technique of Azéma supermartingale and enlargement of filtration.

3. A Study of Several Interesting Examples of Functions $m_{\mathcal{G}}(t)$

3.1. Some general formulae

We shall compute $(m_{\mathcal{G}}(t),\ t\ge 0)$ in some particular cases where

$$\mathcal{G} = \mathcal{G}_K = \sup\{t\ge 0 : M_t = K\}, \qquad K \le 1,$$

with $M_0 = 1$, $M_t \ge 0$, a continuous local martingale such that $M_t \underset{t\to\infty}{\longrightarrow} 0$. We recall that (see, e.g. [2, 8]):

$$Z_t = P(\mathcal{G}_K \ge t|\mathcal{F}_t) = 1 \wedge \left(\frac{M_t}{K}\right).$$

Thus

$$m_K(t) = E\left[Z_t\left(1-Z_t\right)\right] = \frac{1}{K^2}\, E\left[M_t\left(K - M_t\right)^+\right]. \tag{8}$$

Consider the particular case $M_t = \mathcal{E}_t = \exp(B_t - t/2)$, with (B_t) a standard Brownian motion, and $\mathcal{G}_K = \sup\{t : \mathcal{E}_t = K\}$ for $K \le 1$.

From formula (8), we deduce:

$$
\begin{aligned}
m_K(t) &= \frac{1}{K^2}\, E\left[\mathcal{E}_t\,(K-\mathcal{E}_t)^+\right] \\
&= \frac{1}{K^2}\, E\left[\left(K-\exp\left(B_t+\frac{t}{2}\right)\right)^+\right] \quad \text{(by Cameron-Martin)} \\
&= \frac{1}{K^2}\left\{KP\left(\exp\left(B_t+\frac{t}{2}\right)<K\right)-E\left[1_{\left(\exp\left(B_t+\frac{t}{2}\right)<K\right)}\exp\left(B_t+\frac{t}{2}\right)\right]\right\}.
\end{aligned}
$$

Set $K=e^l$, we have

$$
\begin{aligned}
m_K(t) &= e^{-l}P\left(B_t+\frac{t}{2}<l\right)-e^t e^{-2l}P\left(B_t+\frac{3t}{2}<l\right) \\
&= \left(e^{-l}-e^{t-2l}\right)P\left(B_1<-\frac{3\sqrt{t}}{2}+\frac{l}{\sqrt{t}}\right) \\
&\quad +e^{-l}P\left(-\frac{3\sqrt{t}}{2}+\frac{l}{\sqrt{t}}<B_1<-\frac{\sqrt{t}}{2}+\frac{l}{\sqrt{t}}\right).
\end{aligned}
$$

In particular,

$$
m_1(t)=\left(1-e^t\right)P\left(B_1<-\frac{3\sqrt{t}}{2}\right)+P\left(-\frac{3\sqrt{t}}{2}<B_1<-\frac{\sqrt{t}}{2}\right).
$$

Figure 1 presents the graphs of $m_K(t)$ for some K's.

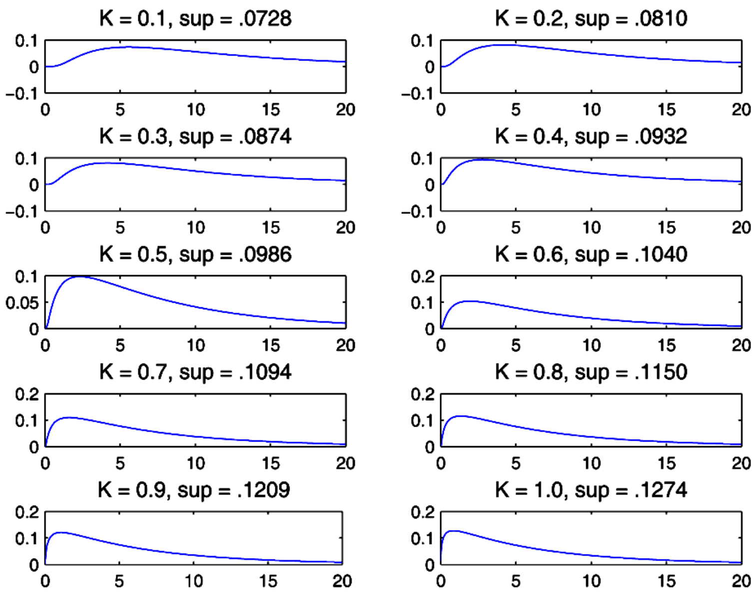

Fig. 1. Graphs of $m_K(t)$, for $K=0.1, 0.2, \ldots 1$.

3.2. The case $\mathcal{G} = \mathcal{G}_{\gamma_T^a} = \sup\{u \leq T : B_u = a\}$

For fixed time T and $a \in \mathbb{R}$, the associated Azéma supermartingale is of the form

$$Z_t = \Phi\left(\frac{|B_t - a|}{\sqrt{T-t}}\right) 1_{\{t<T\}}$$

(see, e.g., Table (1α) of Progressive Enlargements, p.32 of [8]), where $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2}\, du$. Then for $t < T$, using change of variables we have

$$m_{\mathcal{G}}^{a,T}(t) = E\left[\Phi\left(\frac{|\sqrt{t}\, B_1 - a|}{\sqrt{T-t}}\right)\left(1 - \Phi\left(\frac{|\sqrt{t}\, B_1 - a|}{\sqrt{T-t}}\right)\right)\right] = m^{a/\sqrt{T}}\left(\sqrt{\frac{T-t}{t}}\right),$$

where

$$\begin{aligned} m^D(c) : = \frac{c}{\sqrt{2\pi}} \int_0^\infty \Phi(y)(1-\Phi(y)) \Bigg(&\exp\left(-\frac{(cy + D\sqrt{c^2+1})^2}{2}\right) \\ &+ \exp\left(-\frac{(cy - D\sqrt{c^2+1})^2}{2}\right)\Bigg)\, dy. \end{aligned}$$

Hence

$$m_{\mathcal{G}}^{a,T} := \sup_{0 \leq t \leq T} m_{\mathcal{G}}^{a,T}(t) = \sup_{c \geq 0} m^{a/\sqrt{T}}(c).$$

Remark 3.4.

(1) For $a \in \mathbb{R}$, $m_{\mathcal{G}}^{a,T} = m_{\mathcal{G}}^{-a,T}$, since $m^D(c) = m^{-D}(c)$.

(2) $m_{\mathcal{G}}^{0,T}$ is independent of T, since

$$m_{\mathcal{G}}^{0,T} = \sup_{c \geq 0} \frac{2c}{\sqrt{2\pi}} \int_0^\infty \Phi(y)(1-\Phi(y)) \exp\left(-\frac{c^2 y^2}{2}\right) dy$$

is independent of T.

(3) the value of $m_{\mathcal{G}}^{a,T}$ depends only on $D := a/\sqrt{T}$, e.g., $(a,T) = (1,1)$ and $(a,T) = (1/2, 1/4)$ have the same $m_{\mathcal{G}}^{a,T}$ value, since $D = 1$ in both cases.

Remark 3.5. Table 1 gives the values of $m_{\mathcal{G}}^{a,T}$ for some D.

Table 1. The values of $m_{\mathcal{G}}^{a,T}$ for some D

D	0	0.1	0.2	0.3	0.4
$m_{\mathcal{G}}^{a,T}$	0.17548	0.175531	0.173103	0.220612	0.244867
D	0.5	0.6	0.7	1	1.1
$m_{\mathcal{G}}^{a,T}$	0.249704	0.24059	0.218382	0.132556	0.105833
D	1.2	1.5	2	3	5
$m_{\mathcal{G}}^{a,T}$	0.0840563	0.0416004	0.0122678	0.000653202	1.30174×10^{-7}

In fact, if D satisfies $\Phi(D) = \frac{1}{2}$ (i.e., D around 0.47693627), then $m_{\mathcal{G}}^{a,T} = \frac{1}{4}$ and the maximum occurs at $t = 0$. The same as $m_{\mathcal{G}}^{**}$ and $\tilde{m}_{\mathcal{G}}$.

Figures 2–4 present the graphs $m^D(c)$ for some D. The horizontal axis is the value of $c = \sqrt{\frac{T-t}{t}}$ and the vertical axis is the value of $m^D(c)$, and its maximum is exactly $m_{\mathcal{G}}^{a,T}$.

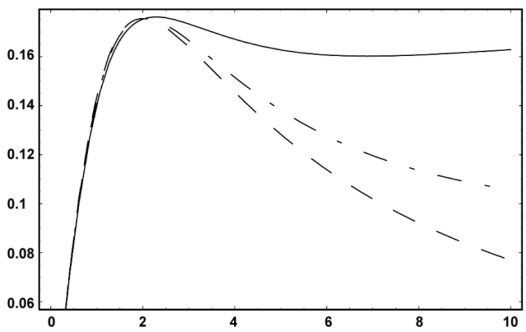

Fig. 2. $D = 0$: — — — ; $D = 0.1$: — - — - ; $D = 0.2$: ————.

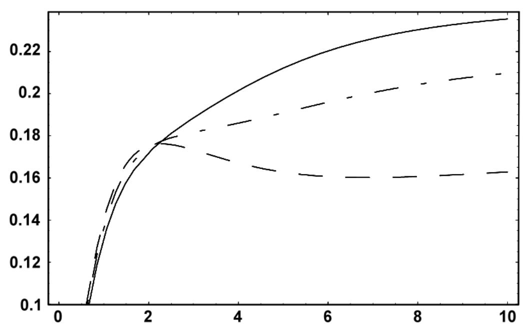

Fig. 3. $D = 0.2$: — — — ; $D = 0.3$: — - — - ; $D = 0.4$: ————.

3.3. The case $\mathcal{G} = \mathcal{G}_{T_a} = \sup\{t < T_a : B_t = 0\}$

Here, we denote $T_a = \inf\{u : B_u = a\}$, for $a > 0$; and $S_t = \sup_{0 \le u \le t} B_u$. The corresponding Azéma supermartingale is given by

$$Z_t = 1 - \frac{1}{a}\, B^+_{t \wedge T_a},$$

see, e.g., Table (1α) of Progressive Enlargements, p. 32 of [8]. Thus, we obtain:

$$\begin{aligned} m_{\mathcal{G}}(t) &= E\left[\left(\frac{1}{a}\, B^{+}_{t\wedge T_a}\right)\left(1-\frac{1}{a}\, B^{+}_{t\wedge T_a}\right)\right] \\ &= \frac{1}{a^2}\, E\left[1_{(t<T_a)}1_{(B_t>0)}B_t(a-B_t)\right] \\ &= \frac{1}{a^2}\, E\left[1_{(S_t<a)}1_{(B_t>0)}B_t(a-B_t)\right]. \end{aligned}$$

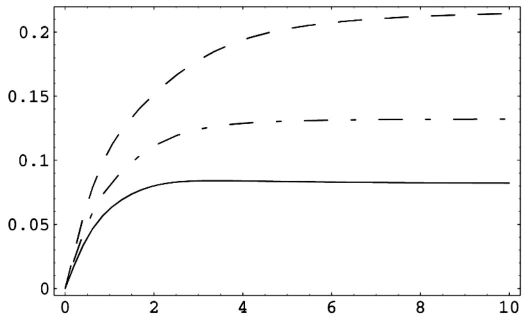

Fig. 4. $D = 0.7$: — — — ; $D = 1$: — - — - ; $D = 1.2$: ————.

Let

$$\varphi(x) = E\left[1_{(S_1<x)}1_{(B_1>0)}B_1(x-B_1)\right],$$

then

$$m_{\mathcal{G}}(t) = \frac{t}{a^2}\,\varphi\left(\frac{a}{\sqrt{t}}\right).$$

Now, it remains to compute the function φ. We note that

$$\varphi(x) = E\left[B_1^+(x-B_1)^+\right] - E\left[1_{(S_1>x)}B_1^+(x-B_1)^+\right].$$

We shall take advantage of the very useful formula:

$$P(S_1 > x|B_1 = a) = \exp(-2x(x-a)), \qquad x \geq a > 0,$$

see, e.g., [5], p.425. Thus, we find

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x dy\; y(x-y)\left(\exp\left(-\frac{y^2}{2}\right) - \exp\left(-\frac{1}{2}(2x-y)^2\right)\right)$$

Thus,

$$\frac{\varphi(x)}{x^2} = \frac{x}{\sqrt{2\pi}}\int_0^1 du\; u(1-u)\left(\exp\left(-\frac{x^2u^2}{2}\right) - \exp\left(-\frac{x^2}{2}(2-u)^2\right)\right).$$

Note that the value of $\sup_{t>0} m_{\mathcal{G}}(t) = \sup_{x>0} \frac{\varphi(x)}{x^2}$ is independent of the value of a, since $m_{\mathcal{G}}(t)$ depends only on the value of $x := a/\sqrt{t}$.

3.4. The case $\mathcal{G} = \mathcal{L}_a = \sup\{u : R_u = a\}$

We have

$$Z_t = 1 \wedge \left(\frac{a}{R_t}\right)^{2\mu},$$

see, e.g., Table (1α) of Progressive Enlargements, p.32 of [8]. Here, (R_u) is a Bessel process of index μ starting at 0, i.e., R is a d-dimensional Bessel process with $d = 2(\mu+1)$. Thus,

$$\begin{aligned} m_{\mathcal{G}}(t) &= E\left[\left(1 \wedge \left(\frac{a}{R_t}\right)^{2\mu}\right)\left(1 - 1 \wedge \left(\frac{a}{R_t}\right)^{2\mu}\right)\right] \\ &= E\left[1_{\left(\frac{a}{\sqrt{t}R_1}<1\right)}\left(\frac{a}{\sqrt{t}R_1}\right)^{2\mu}\left(1 - \left(\frac{a}{\sqrt{t}R_1}\right)^{2\mu}\right)\right]. \end{aligned}$$

Using the fact that $R_1^2 \overset{(\text{law})}{=} 2\gamma_{d/2}$, where $\gamma_{d/2}$ has a gamma law with parameter $(d/2, 1)$, we get

$$m_{\mathcal{G}}(t) = \varphi_\mu\left(\frac{a^2}{2t}\right),$$

where

$$\varphi_\mu(z) = \frac{1}{\Gamma(\mu+1)}\left\{z^\mu e^{-z} - z^{2\mu}\int_z^\infty \frac{du}{u^\mu}e^{-u}\right\}.$$

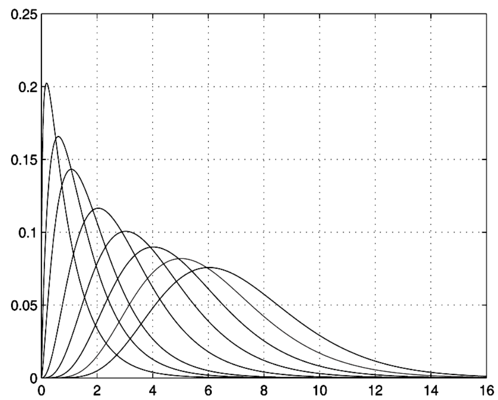

Fig. 5. Graphs of $\varphi_\mu(z)$, for $\mu = 1/2,\ 1,\ 3/2,\ 5/2,\ 7/2,\ 9/2,\ 11/2,\ 13/2$, and that $z_{1/2} = 0.19$, $z_1 = 0.61$, $z_{3/2} = 1.08$, $z_{5/2} = 2.05$, $z_{7/2} = 3.04$, $z_{9/2} = 4.03$, $z_{11/2} = 5.02$, $z_{13/2} = 6.02$.

Figure 5 presents the graphs of φ_μ for $\mu = 1/2, 1, 3/2, 5/2, 7/2, 9/2, 11/2$ and $13/2$. We also approximate z_μ, the unique positive real number which achieves the max of φ_μ. This will give us the value $m_\mu \stackrel{\text{def}}{=} m^*_{\mathcal{G}}$, for these $\mathcal{G} \equiv \mathcal{L}_a$ (note that, for a given μ, the value does not depend on a; this is because of the scaling property).

It is not difficult to show that: z_μ is the unique solution of

$$(E_\mu): \quad \frac{1}{2z} = \int_0^\infty \frac{dh}{(1+h)^\mu}\, e^{-hz}$$

and also

$$m_\mu = \frac{1}{\Gamma(\mu+1)}\, e^{-z_\mu}\, \frac{(z_\mu)^\mu}{2}.$$

Note that

$$m_\mu \le m'_\mu \stackrel{\text{def}}{=} \frac{1}{\Gamma(\mu+1)} \sup_{z\ge 0} \left(e^{-z}\frac{z^\mu}{2}\right).$$

Figure 6 presents the graphs of m_μ and m'_μ.

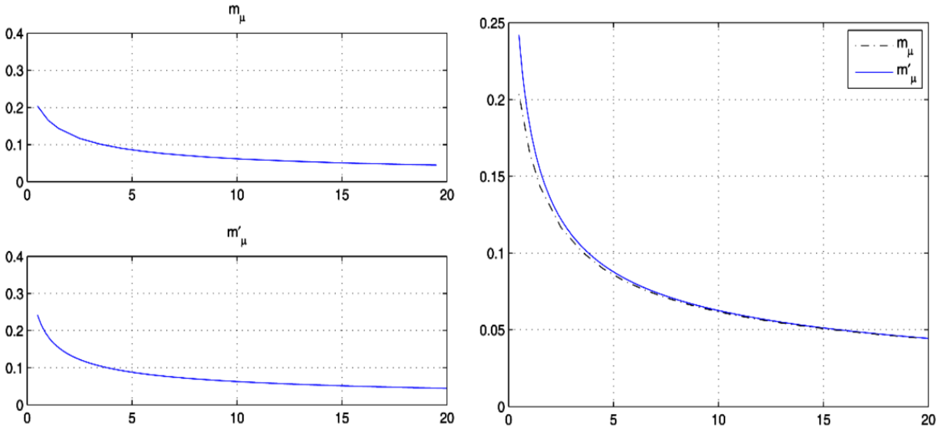

Fig. 6. Graphs of m_μ and m'_μ.

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