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SOME OPERATORS ACTING ON WEIGHTED SEQUENCE BESOV SPACES AND APPLICATIONS

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Abstract. In this article, we study the boundedness of matrix operators acting on weighted sequence Besov spaces $\dot{b}^{\alpha,q}_{p,w}$. First we obtain the necessary and sufficient condition for the boundedness of diagonal matrices acting on weighted sequence Besov space $\dot{b}^{\alpha,q}_{p,w}$, and investigate the duals of $\dot{b}^{\alpha,q}_{p,w}$, where the weight is non-negative and locally integrable. In particular, when $0 < p < 1$, we find a type of new sequence sapces which characterize the dual space of $\dot{b}^{\alpha,q}_{p,w}$.

We also use the duals of $\dot{b}^{\alpha,q}_{p,w}$ to characterize an algebra of matrix operators acting on weighted sequence Besov spaces $\dot{b}^{\alpha,q}_{p,w}$ and find the necessary and sufficient conditions to such a characterization. Note that we do not require that the given weight satisfies the doubling condition in this situation.

Using these results, we give some applications to characterize the boundedness of Fourier-Haar multipliers and paraproduct operators. In this situation, we need to require that the weight w is an A_p weight.

1. INTRODUCTION

In order to study the boundedned of some kind of linear operators, such as Haar multipliers and paraproduct operators, one can do it by norm equivalence between function spaces and their corresponding sequence spaces. Precisely, for example, if we consider a linear operator T acting on homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha,q}_p$, then one can use a discrete wavelet transform identity or the φ-transform identity introduced by Frazier and Jawerth [5] to deduce a linear operator T to a matrix $A(T) :=$ $\{\langle T\psi_P, \varphi_Q\rangle\}$ and to consider the boundedness of $A(T)$ acting on sequence Triebel-Lizorkin space $\dot{f}^{\alpha,q}_p$. For simplicity, we only work with the φ-transform indetity, but let us emphasize that Meyer's wavelet transform indetity could be used equally well in our development.

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Let us start with recalling some definitions and properties. For some $\nu \in \mathbb{Z}$ and $\mathbf{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$, let $Q = Q_{\nu\mathbf{k}} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : 2^{-\nu}k_i \le x_i < 2^{-\nu}(k_i + 1),\ i = 1, 2, \cdots, n\}$, and $x_Q = 2^{-\nu}\mathbf{k}$ the "lower left corner" of $Q = Q_{\nu\mathbf{k}}$. Also $\mathcal{Q}$ denotes the collection of all dyadic cubes in $\mathbb{R}^n$ and $\mathcal{Q}_\nu$ denotes the subcollection of $\mathcal{Q}$ with side length $2^{-\nu}$ for $\nu \in \mathbb{Z}$. For $P \in \mathcal{Q}$, $\mathcal{Q}_P$ denotes the subcollection of $\mathbb{R}^n$ such that each cube in $\mathcal{Q}_P$ is a subset of P. We choose a function $\varphi \in \mathcal{S}$ satisfying

$$\begin{cases} \text{supp}(\widehat{\varphi}) \subseteq \{\xi : 1/2 \le |\xi| \le 2\}; \\ |\widehat{\varphi}(\xi)| \ge c > 0 \quad \text{if} \quad 3/5 \le |\xi| \le 5/3. \end{cases} \tag{1}$$

Then there exists a function $\psi \in \mathcal{S}$ satisfying the same conditions as (1) such that

$$\sum_{\nu\in\mathbb{Z}} \overline{\widehat{\varphi}}(2^{-\nu}\xi)\widehat{\psi}(2^{-\nu}\xi) = 1 \qquad \text{for} \quad \xi \neq 0.$$

Hence the *φ-transform identity* [5] is given by

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q \rangle \psi_Q, \tag{2}$$

where $g_Q(x) := |Q|^{-1/2} g((x - x_Q)/\ell(Q)) = 2^{\nu n/2} g(2^\nu x - \mathbf{k})$ if $Q = Q_{\nu\mathbf{k}}$ for some $\nu \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$. Here $|Q|$ is the usual Lebesgue measure of Q in $\mathbb{R}^n$.

Let $\mathcal{P}$ denote the class of all polynomials on $\mathbb{R}^n$ and $\mathcal{S}'/\mathcal{P}$ denote the tempered distributions on $\mathbb{R}^n$ modulo polynomials. For $\nu \in \mathbb{Z}$, let $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$. For $\alpha \in \mathbb{R}$, $0 < p,\ q \le +\infty$ and $f \in \mathcal{S}'/\mathcal{P}$, define the *homogeneous Triebel-Lizorkin spaces* $\dot{F}_p^{\alpha,q}$ via the norms

$$\|f\|_{\dot{F}_p^{\alpha,q}} := \begin{cases} \left\| \left\{ \sum_{\nu\in\mathbb{Z}} \left(2^{\nu\alpha}|\varphi_\nu * f|\right)^q \right\}^{1/q} \right\|_{L^p} < \infty & \text{if } 0<p<\infty \\ \sup_{Q\in\mathcal{Q}} \left\{ |Q|^{-1} \int_Q \sum_{\nu=-\log_2 \ell(Q)}^{+\infty} \left(2^{\nu\alpha}|\varphi_\nu * f|\right)^q \right\}^{1/q} < \infty & \text{if } p=\infty \end{cases}.$$

The *homogeneous Besov spaces* $\dot{B}_p^{\alpha,q}$ are defined by

$$\|f\|_{\dot{B}_p^{\alpha,q}} := \left\| \left\{ 2^{\nu\alpha} \|\varphi_\nu * f\|_{L^p} \right\}_{\nu\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} < \infty.$$

Triebel-Lizorkin spaces include many other spaces as special cases; $L^p \approx \dot{F}_p^{0,2}$ for $1 < p < +\infty$, $H^p \approx \dot{F}_p^{0,2}$ for $0 < p \le 1$, and $BMO \approx \dot{F}_\infty^{0,2}$ (see [7, 23] for details).

The corresponding sequence spaces $\dot{f}_p^{\alpha,q}$ and $\dot{b}_p^{\alpha,q}$ can be defined as follows. For $\alpha \in \mathbb{R}$ and $0 < p,\ q \le \infty$, the space $\dot{f}_p^{\alpha,q}$ consists of all sequences $s = \{s_Q\}$ satisfying

$$\|s\|_{\dot{f}_p^{\alpha,q}} := \begin{cases} \left\| \left\{ \sum_{Q\in\mathcal{Q}} \left(|Q|^{-\alpha/n-1/2}|s_Q|\chi_Q\right)^q \right\}^{1/q} \right\|_{L^p} < \infty & \text{if } 0<p<\infty \\ \sup_{P\in\mathcal{Q}} \left\{ |P|^{-1} \int_P \sum_{Q\in\mathcal{Q}_P} \left(|Q|^{-1/2-\alpha/n}|s_Q|\chi_Q\right)^q \right\}^{1/q} < \infty & \text{if } p=\infty \end{cases},$$

where χ_Q denotes the characteristic function of the cube Q. The space $\dot{b}_p^{\alpha,q}$ consists of all sequences $s = \{s_Q\}$ such that

$$\|s\|_{\dot{b}_p^{\alpha,q}} := \begin{cases} \left(\sum_{\nu\in\mathbb{Z}} \left\{ \sum_{Q\in\mathcal{Q}_\nu} \left(|Q|^{-\alpha/n-1/2+1/p}|s_Q|\right)^p \right\}^{q/p} \right)^{1/q} < \infty & \text{if } 0<p<\infty \\ \left(\sum_{\nu\in\mathbb{Z}} \left\{ \sup_{Q\in\mathcal{Q}_\nu} |Q|^{-\alpha/n-1/2}|s_Q| \right\}^q \right)^{1/q} < \infty & \text{if } p=\infty \end{cases}.$$

The function spaces $\dot{F}_p^{\alpha,q}$, $\dot{B}_p^{\alpha,q}$ and the sequence spaces $\dot{f}_p^{\alpha,q}$, $\dot{b}_p^{\alpha,q}$ are equivalent in norms, respectively.

Proposition 1.1. ([4, 5, 7]). *Suppose $\alpha \in \mathbb{R}$, $0 < p,\ q \leq +\infty$, and the functions φ and ψ are given in* (2). *Given $f \in \mathcal{S}'/\mathcal{P}$, there exists a sequence of numbers $\{s_Q = \langle f, \varphi_Q\rangle\}_Q$ such that $f = \sum_Q s_Q\psi_Q$. Furthermore,*

(a) *$f \in \dot{F}_p^{\alpha,q}$ if and only if the sequence $s = \{s_Q\}_Q \in \dot{f}_p^{\alpha,q}$, and $\|f\|_{\dot{F}_p^{\alpha,q}} \approx \|s\|_{\dot{f}_p^{\alpha,q}}$;*

(b) *$f \in \dot{B}_p^{\alpha,q}$ if and only if the sequence $s = \{s_Q\}_Q \in \dot{b}_p^{\alpha,q}$, and $\|f\|_{\dot{B}_p^{\alpha,q}} \approx \|s\|_{\dot{b}_p^{\alpha,q}}$.*

The prototypes of operators in this article are paraproduct operators and Haar-Fourier multipliers, which are defined below.

Definition 1.2. Fix a function Φ in $\mathcal{S}$ such that $\operatorname{supp}(\Phi) \subseteq [0,1)^n = Q_{00}$ and $\int \Phi = 1$. (We will use this Φ in the sequel.) For $\alpha \in \mathbb{R}$ and $g \in \dot{B}_\infty^{\alpha,\infty} = \dot{F}_\infty^{\alpha,\infty}$, the *paraproduct operator* Π_g via the φ-transform identity is defined by

$$\Pi_g(f) := \sum_Q \langle g, \varphi_Q\rangle |Q|^{-1/2}\langle f, \Phi_Q\rangle \psi_Q. \tag{3}$$

Thus, the adjoint operator of Π_g is

$$\Pi_g^* f(x) = \sum_Q \overline{\langle g, \varphi_Q\rangle} |Q|^{-1/2}\langle f, \psi_Q\rangle \Phi_Q(x).$$

Note that $\Pi_g 1 = g$ and $\Pi_g^* 1 = 0$. Also, when $g \in \dot{B}_\infty^{0,\infty}$, Π_g is a singular integral operators (c.f. [25]).

Let $f \in \dot{F}_p^{0,q}$. Plugging (2) into (3), we obtain

$$\begin{aligned} \Pi_g(f) &= \sum_Q \langle g, \varphi_Q\rangle |Q|^{-1/2}\Big\langle \sum_P \langle f, \varphi_P\rangle \psi_P, \Phi_Q\Big\rangle \psi_Q \\ &= \sum_Q \langle g, \varphi_Q\rangle |Q|^{-1/2}\Big(\sum_P \langle \psi_P, \Phi_Q\rangle \langle f, \varphi_P\rangle\Big) \psi_Q. \end{aligned} \tag{4}$$

Let G be the matrix $\{\langle\psi_P, \Phi_Q\rangle\}_{Q,P}$. Also let $d_Q = \langle g, \varphi_Q\rangle$ and $d = \{d_Q\}_Q$. Define the diagonal operator $T_d^{(0)}$ as follows. For a sequence $s = \{s_Q\}_Q$, $T_d^{(0)}$ sends s to $T_d^{(0)}s$, where

$$(T_d^{(0)}s)_Q = |Q|^{-1/2}s_Q d_Q$$

denotes the Q^{th} entry of the sequence $T_d^{(0)}s$. In fact, $T_d^{(0)} = \mathrm{diag}\{|Q|^{-1/2}d_Q\}$ is a diagonal matrix determined by the given sequence d. By Proposition 1.1 and equality (4), we have

$$\begin{aligned}\|\Pi_g f\|_{\dot F_p^{\alpha,q}} &\approx \Big\|\Big\{\langle g, \varphi_Q\rangle|Q|^{-1/2}\Big(\sum_P\langle\psi_P, \Phi_Q\rangle\langle f, \varphi_P\rangle\Big)\Big\}_Q\Big\|_{\dot f_p^{\alpha,q}}\\ &= \|T_d^{(0)}Gs\|_{\dot f_p^{\alpha,q}},\end{aligned} \tag{5}$$

where $s = \{\langle f, \varphi_P\rangle\}_P$. So, to show the boundedness of Π_g from $\dot F_p^{0,q}$ into $\dot F_p^{\alpha,q}$ is equivalent to show the boundedness of $T_d^{(0)}G$ from $\dot f_p^{0,q}$ into $\dot f_p^{\alpha,q}$. We will give a characterization of boundedness of paraproduct operators on weighted Besov spaces in Section 5.

Let us recall Haar multipliers introduced in [12, 19, 20]. Precisely, given a sequence $t = \{t_I\}_{I \text{ dyadic}}$, a *Haar multiplier* is an operator of the form

$$T_t f(x) := \sum_{I \text{ dyadic}} t_I\langle f, h_I\rangle h_I(x), \quad \text{for} \quad f \in L^2(\mathbb{R}),$$

where the sum runs over all dyadic intervals in $\mathbb{R}$, h_I is the Haar function associated to I and $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product.

Motivated by [12, 19, 20], let us consider the generalized Haar multipliers in $\mathbb{R}^n$. For a sequence $t = \{t_Q\}$, define the *Fourier-Haar multiplier* T_t by

$$T_t(f) := \sum_{Q\in\mathcal{Q}} |Q|^{-1/2}t_Q\langle f, \varphi_Q\rangle\psi_Q. \tag{6}$$

By Proposition 1.1, $\|T_t(f)\|_{\dot F_p^{\alpha,q}} \approx \|\{|Q|^{-1/2}t_Q\langle f, \varphi_Q\rangle\}_Q\|_{\dot f_p^{\alpha,q}}$. Thus, to study the boundedness of T_t on $\dot F_p^{\alpha,q}$ is equivalent to study the corresponding diagonal matrix on $\dot f_p^{\alpha,q}$. We will study the boundedness of Fourier-Haar multipliers on weighted Besov spaces in Section 5.

In this article, we focus on that a matrix operator is mapped from one weighted sequence space to another one. In the following, we introduce the weighted Besov space $\dot B_{p,w}^{\alpha,q}$ and weighted sequence Besov space $\dot b_{p,w}^{\alpha,q}$. We say w is a *weight* means that w is a non-negative, locally integrable function.

Definition 1.3. (Weighed Besov space $\dot B_{p,w}^{\alpha,q}$). Select a function $\varphi \in \mathcal{S}$ satisfing condition (1). For $\alpha \in \mathbb{R}$, $0 < p,\ q \le \infty$, w a weight and $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, define the *homogeneous weighted Besov space* $\dot B_{p,w}^{\alpha,q}$ via the norm

$$\|f\|_{\dot{B}^{\alpha,q}_{p,w}} := \left\| \left\{ 2^{\nu\alpha} \|\varphi_\nu * f\|_{L^p(w)} \right\}_\nu \right\|_{l^q} < \infty.$$

Note that the definition of homogeneous weighted Besov spaces is independent of the choice of φ if the weight w satisfies doubling condition, see [8, 21] for more details on matrix-weighted Besov spaces. For a weight w, let $\mathcal{Q}(w)$ denote the collection of all cubes $Q \in \mathcal{Q}$ such that $w(Q) := \int_Q w(x)dx \neq 0$ and $\mathcal{Q}_\nu(w)$ denote the collection of all cubes $Q \in \mathcal{Q}_\nu$ such that $w(Q) \neq 0$ for $\nu \in \mathbb{Z}$. It is clear that $\cup_{\nu\in\mathbb{Z}}\mathcal{Q}_\nu(w) = \mathcal{Q}(w)$ and $\mathcal{Q}(w) = \mathcal{Q}$ if $w > 0$ almost everywhere.

Definition 1.4. (Weighed sequence Besov space $\dot{b}^{\alpha,q}_{p,w}$). For $\alpha \in \mathbb{R}$, $0 < p,\ q \leq \infty$, and w a weight, the space $\dot{b}^{\alpha,q}_{p,w}$ consists of all sequence $s = \{s_Q\}_Q$, enumerated by the dyadic cubes Q contained in $\mathbb{R}^n$, such that

$$\|s\|_{\dot{b}^{\alpha,q}_{p,w}} := \left\| \left\{ 2^{\nu\alpha} \Big\| \sum_{Q\in\mathcal{Q}_\nu(w)} |Q|^{-\frac{1}{2}} s_Q \chi_Q \Big\|_{L^p(w)} \right\}_{\nu\in\mathbb{Z}} \right\|_{l^q} < \infty.$$

The main conclusion is the norm equivalence between the homogeneous weighted Besov space $\dot{B}^{\alpha,q}_{p,w}$ and the weighted sequence Besov space $\dot{b}^{\alpha,q}_{p,w}$ under the A_p condition. For the detailed description of A_p condition, refer to [9, 11]. Under the A_p condition on w, $\mathcal{Q}(w)$ is the same as $\mathcal{Q}$.

Proposition 1.5. ([8, Theorem 1.1], [21, Theorem 1.4]). *Let* $\alpha \in \mathbb{R}$, $0 < p,\ q \leq \infty$, $w \in A_p$. *Then*

$$\|f\|_{\dot{B}^{\alpha,q}_{p,w}} = \Big\| \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle \psi_Q \Big\|_{\dot{B}^{\alpha,q}_{p,w}} \approx \Big\| \big\{ s_Q(f) \big\}_Q \Big\|_{\dot{b}^{\alpha,q}_{p,w}},$$

where $\big\{s_Q(f)\big\}_Q = \big\{\langle f, \varphi_Q\rangle\big\}_Q$ *is the sequence of* φ*-transform coefficients of* f.

Remark 1.6.

(a) When $w \equiv 1$, the sequence space $\dot{b}^{\alpha,q}_{p,1}$ is the usual unweighted sequence space $\dot{b}^{\alpha,q}_p$ given by Frazier, Jawerth and Weiss in [7].

(b) When $0 < p < \infty$, we have

$$\|s\|_{\dot{b}^{\alpha,q}_{p,w}} = \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{\substack{Q \\ \ell(Q)=2^{-\nu}}} \left(|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \right)^p w(Q) \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

and

$$\|s\|_{\dot{b}^{\alpha,\infty}_{\infty,w}} = \sup_{Q\in\mathcal{Q}(w)} |Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q|.$$

This article is organized as follows. In Section 2, we characterize completely for diagonal matrix operators acting from one weighted sequence Besov space to another one. Also, in this section, we characterize the dual space of $\dot{b}_{p,w}^{\alpha,q}$. In section 3, we define a class of almost diagonal matrices $\mathbf{ad}_p^{\alpha}(\beta)$ for the weighted sequence Besov spaces and show the boundedness of these matrices on $\dot{b}_{p,w}^{\alpha,q}$ if w is a weight with a doubling exponent β. In section 4, we treat more general matrix operators. In some special cases, we obtain necessary and sufficient conditions for boundedness of operators acting on weighted sequence Besov spaces. Consequently, we characterize an algebra of bounded matrix operators on $\dot{b}_{p,w}^{\alpha,q}$ for fixed $\alpha \in \mathbb{R}$ and for all $1 \le p,\ q \le \infty$. We apply our previous results to study the boundedness of Fourier-Haar multipliers and paraproduct operators acting from one weighted Besov space to another in the last section.

Through the article, a cube means a dyadic cube in $\mathbb{R}^n$, and C denotes a positive constant independent of the main variables, which may vary from line to line. We also denote by q' the index conjugate to q; that is, $q' = q/(q-1)$ for $1 \le q \le \infty$. When $0 < q \le 1$, q' is defined as ∞.

2. Diagonal Matrices and Duality

As in [13, 14, 24, 25], to study singular integral operators acting on homogeneous Triebel-Lizorkin spaces or Besov spaces, it suffices to study the boundedness for paraproduct operators acting on the same spaces, equivalently, it does study the boundedness of matrix operators deduced from paraproduct operators acting on corresponding sequence sapces, as described in (5).

Here we start with the diagonal matrices acting on weighted sequence Besov spaces. For $\gamma \in \mathbb{R}$ and a fixed sequence $d = \{d_Q\}_Q$, define a linear operator $T_d^{(\gamma)}$ acting on sequence spaces by

$$T_d^{(\gamma)} s := \left\{ |Q|^{-1/2-\gamma/n} d_Q s_Q \right\}_Q \qquad \text{for every sequence} \quad s. \tag{7}$$

Let $D_d^{(\gamma)}$ be the diagonal matrix operator with diagonal entries $\{|Q|^{-1/2-\gamma/n} d_Q\}_Q$. Then $T_d^{(\gamma)} = D_d^{(\gamma)}$. In this section, let us first study the boundedness of $T_d^{(\gamma)}$.

Proposition 2.1. [10, Theorem 3.1]). *Let* $\alpha_1, \alpha_2, \gamma \in \mathbb{R}$, $0 < p,\ q_1,\ q_2 \le \infty$ *and let* w *be a weight.*

(a) *For* $q_1 > q_2$, $T_d^{(\gamma)}$ *is bounded from* $\dot{b}_{p,w}^{\alpha_1,q_1}$ *into* $\dot{b}_{p,w}^{\alpha_1+\alpha_2,q_2}$ *if and only if* $d \in \dot{b}_{\infty,w}^{\alpha_2+\gamma,\frac{q_1 q_2}{q_1-q_2}}$.

(b) *For* $q_1 \le q_2$, $T_d^{(\gamma)}$ *is bounded from* $\dot{b}_{p,w}^{\alpha_1,q_1}$ *into* $\dot{b}_{p,w}^{\alpha_1+\alpha_2,q_2}$ *if and only if* $d \in \dot{b}_{\infty,w}^{\alpha_2+\gamma,\infty}$.

More generally, we have the following result for different indices.

Theorem 2.2. *Let* $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < p_1,\ p_2,\ q_1,\ q_2 \le \infty$ *and* $\gamma \in \mathbb{R}$. *Also let* w *be a weight. Then* $T_d^{(\gamma)}$ *is bounded from* $\dot{b}^{\alpha_1,q_1}_{p_1,w}$ *into* $\dot{b}^{\alpha_2,q_2}_{p_2,w}$ *if one of following cases holds:*

(a) $p_1 > p_2,\ q_1 > q_2$ *and* $d \in \dot{b}^{\alpha_2-\alpha_1+\gamma,\frac{q_1q_2}{q_1-q_2}}_{\frac{p_1p_2}{p_1-p_2},w}$;

(b) $p_1 > p_2,\ q_1 \le q_2$ *and* $d \in \dot{b}^{\alpha_2-\alpha_1+\gamma,\infty}_{\frac{p_1p_2}{p_1-p_2},w}$;

(c) $p_1 \le p_2,\ q_1 > q_2$ *and* $d \in \dot{b}^{\alpha_2-\alpha_1+\gamma,\frac{q_1q_2}{q_1-q_2}}_{\infty,w}$;

(d) $p_1 \le p_2,\ q_1 \le q_2$ *and* $d \in \dot{b}^{\alpha_2-\alpha_1+\gamma,\infty}_{\infty,w}$.

Proof. Without loss of generality, we may assume $\alpha_1 = \alpha_2 = 0$. Let $s \in \dot{b}^{0,q_1}_{p_1,w}$ and suppose $0 < p_1,\ p_2,\ q_1,\ q_2 \le \infty$.

For part (a), let $\delta = p_1/p_2$ and $\rho = q_1/q_2$. Applying Hölder's inequality twice, then we have

$$\begin{aligned}
\|T_d^{(\gamma)} s\|_{\dot{b}^{0,q_2}_{p_2,w}} &= \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q| \right)^{p_2} \left(|Q|^{-\frac{1}{2}} |s_Q| \right)^{p_2} w(Q) \right]^{\frac{q_2}{p_2}} \right\}^{\frac{1}{q_2}} \\
&\le \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q| \right)^{p_2\delta'} w(Q) \right]^{\frac{q_2}{p_2\delta'}} \right. \\
&\quad \left. \times \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}} |s_Q| \right)^{p_1} w(Q) \right]^{\frac{q_2}{p_1}} \right\}^{\frac{q_1}{q_2}\cdot\frac{1}{q_1}} \\
&\le \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q| \right)^{p_2\delta'} w(Q) \right]^{\frac{q_2\rho'}{p_2\delta'}} \right\}^{\frac{1}{q_2\rho'}} \\
&\quad \times \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}} |s_Q| \right)^{p_1} w(Q) \right]^{\frac{q_1}{p_1}} \right\}^{\frac{1}{q_1}} \\
&= \|d\|_{\dot{b}^{\gamma,q_2\rho'}_{p_2\delta',w}} \cdot \|s\|_{\dot{b}^{0,q_1}_{p_1,w}},
\end{aligned}$$

where $p_2\delta' = p_2(p_1/p_2)' = \frac{p_1p_2}{p_1-p_2}$ and $q_2\rho' = q_2(q_1/q_2)' = \frac{q_1q_2}{q_1-q_2}$.

For part (b), let $\delta = p_1/p_2$. Since $q_1 \le q_2$, $q_1/q_2 \le 1$. Applying Hölder's inequality and triangle inequality, we obtain

$$\begin{aligned}
\|T_d^{(\gamma)} s\|_{\dot{b}^{0,q_2}_{p_2,w}} &= \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q| \right)^{p_2} \left(|Q|^{-\frac{1}{2}} |s_Q| \right)^{p_2} w(Q) \right]^{\frac{q_2}{p_2}} \right\}^{\frac{1}{q_2}} \\
&\le \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q| \right)^{p_2\delta'} w(Q) \right]^{\frac{q_2}{p_2\delta'}} \right.
\end{aligned}$$

$$
\begin{aligned}
&\times\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12}|s_Q|\Big)^{p_1}w(Q)\Bigg]^{\frac{q_2}{p_1}}\Bigg\}^{\frac{1}{q_2}}\\
\le\ &\sup_{\nu\in\mathbb{Z}}\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12-\frac{\gamma}{n}}|d_Q|\Big)^{p_2\delta'}w(Q)\Bigg]^{\frac{1}{p_2\delta'}}\\
&\times\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12}|s_Q|\Big)^{p_1}w(Q)\Bigg]^{\frac{q_2}{p_1}}\Bigg\}^{\frac{1}{q_2}}\\
\le\ &\|d\|_{\dot b^{\gamma,\infty}_{p_2\delta',w}}\cdot\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12}|s_Q|\Big)^{p_1}w(Q)\Bigg]^{\frac{q_1}{p_1}}\Bigg\}^{\frac{1}{q_1}}\\
=\ &\|d\|_{\dot b^{\gamma,\infty}_{p_2\delta',w}}\cdot\|s\|_{\dot b^{0,q_1}_{p_1,w}}.
\end{aligned}
$$

For part (c), let $\rho=q_1/q_2$. Since $p_1\le p_2$, $p_1/p_2\le 1$. Applying triangle inequality and then Hölder's inequality, we obtain

$$
\begin{aligned}
\|T_d^{(\gamma)}s\|_{\dot b^{0,q_2}_{p_2,w}}&=\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12-\frac{\gamma}{n}}|d_Q|\Big)^{p_2}\Big(|Q|^{-\frac12}|s_Q|\Big)^{p_2}w(Q)\Bigg]^{\frac{q_2}{p_2}}\Bigg\}^{\frac{1}{q_2}}\\
&\le\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12}|Q|^{-\frac12-\frac{\gamma}{n}}|d_Q||s_Q|\Big)^{p_1}w(Q)\Bigg]^{\frac{q_2}{p_1}}\Bigg\}^{\frac{1}{q_2}}\\
&\le\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sup_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12-\frac{\gamma}{n}}|d_Q|\Big)^{p_1}\Bigg]^{\frac{q_2}{p_1}}\\
&\quad\times\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12}|s_Q|\Big)^{p_1}w(Q)\Bigg]^{\frac{q_2}{p_1}}\Bigg\}^{\frac{q_1}{q_2}\cdot\frac{1}{q_1}}\\
&\le\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sup_{Q\in\mathcal{Q}_\nu(w)}|Q|^{-\frac12-\frac{\gamma}{n}}|d_Q|\Bigg]^{q_2\rho'}\Bigg\}^{\frac{1}{q_2\rho'}}\\
&\quad\times\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12}|s_Q|\Big)^{p_1}w(Q)\Bigg]^{\frac{q_1}{p_1}}\Bigg\}^{\frac{1}{q_1}}\\
&=\|d\|_{\dot b^{\gamma,q_2\rho'}_{\infty}}\cdot\|s\|_{\dot b^{0,q_1}_{p_1,w}}.
\end{aligned}
$$

For part (d), since $p_1\le p_2$ and $q_1\le q_2$,we have $p_1/p_2\le 1$ and $q_1/q_2\le 1$. Applying triangle inequality twice, we obtain the result.

$$
\|T_d^{(\gamma)}s\|_{\dot b^{0,q_2}_{p_2,w}}=\Bigg\{\sum_{\nu\in\mathbb{Z}}\Bigg[\sum_{Q\in\mathcal{Q}_\nu(w)}\Big(|Q|^{-\frac12-\frac{\gamma}{n}}|d_Q|\Big)^{p_2}\Big(|Q|^{-\frac12}|s_Q|\Big)^{p_2}w(Q)\Bigg]^{\frac{q_2}{p_2}}\Bigg\}^{\frac{1}{q_2}}
$$

$$
\begin{aligned}
&\le \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}} |Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q||s_Q| \right)^{p_1} w(Q) \right]^{\frac{q_2}{p_1}} \right\}^{\frac{1}{q_2}} \\
&\le \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sup_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q| \right)^{p_1} \right]^{\frac{q_2}{p_1}} \right. \\
&\quad \times \left. \left[\sum_{\substack{Q \\ \ell(Q)=2^{-\nu}}} \left(|Q|^{-\frac{1}{2}} |s_Q| \right)^{p_1} w(Q) \right]^{\frac{q_2}{p_1}} \right\}^{\frac{1}{q_2}} \\
&\le \sup_{Q\in\mathcal{Q}(w)} |Q|^{-\frac{1}{2}-\frac{\gamma}{n}} |d_Q| \cdot \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{-\frac{1}{2}} |s_Q| \right)^{p_1} w(Q) \right]^{\frac{q_1}{p_1}} \right\}^{\frac{1}{q_1}} \\
&= \|d\|_{\dot{b}^{\gamma,\infty}_{\infty,w}} \cdot \|s\|_{\dot{b}^{0,q_1}_{p_1,w}}.
\end{aligned}
$$

For the case $p_i = \infty$ or $q_i = \infty$, with modification, we could obtain the results. This completes the proof of Theorem 2.2. ∎

At the end of this section, let us consider the duals of weighted sequence Besov spaces.

Proposition 2.3. [10, Theorem 1.3]). *Let $\alpha \in \mathbb{R}$, $1 < p \le \infty$, $0 < q \le \infty$, and w be a weight. Then the dual of $\dot{b}^{\alpha,q}_{p,w}$ is $\dot{b}^{-\alpha,q'}_{p',w}$ in the following sense.*

(*i*) *For $\mathbf{t} = \{t_Q\}_{Q\in\mathcal{Q}(w)} \in \dot{b}^{-\alpha,q'}_{p',w}$, the linear functional $L_{\mathbf{t}}$ on $\dot{b}^{\alpha,q}_{p,w}$, given by $L_{\mathbf{t}}(\mathbf{s}) = \langle \mathbf{s}, \mathbf{t}\rangle_w = \sum_{Q\in\mathcal{Q}(w)} s_Q \overline{t_Q} \frac{w(Q)}{|Q|}$ for $\mathbf{s} = \{s_Q\}_{Q\in\mathcal{Q}(w)} \in \dot{b}^{\alpha,q}_{p,w}$, is continuous with $\|L_{\mathbf{t}}\| \le C\|\mathbf{t}\|_{\dot{b}^{-\alpha,q'}_{p',w}}$.*

(*ii*) *Conversely, every continuous linear functional L on $\dot{b}^{\alpha,q}_{p,w}$ satisfies $L = L_{\mathbf{t}}$ for some $\mathbf{t} = \{t_Q\}_{Q\in\mathcal{Q}(w)} \in \dot{b}^{-\alpha,q'}_{p',w}$ with $\|\mathbf{t}\|_{\dot{b}^{-\alpha,q'}_{p',w}} \le C\|L\|$ provided that w is a "double measure", i.e. $w(2B) \le Cw(B)$ for every ball B in $\mathbb{R}^n$.*

In order to find the dual space of $\dot{b}^{\alpha,q}_{p,w}$ for $0 < p \le 1$ and $0 < q \le \infty$, we need to define a sequence space $\dot{c}^{\alpha,q}_{p,w}$ given in [10].

Definition 2.4. For $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$, and a weight w, we say that $t = \{t_Q\}_Q \in \dot{c}^{\alpha,q}_{p,w}$ if $\|t\|_{\dot{c}^{\alpha,q}_{p,w}}$ is finite, where $\|t\|_{\dot{c}^{\alpha,q}_{p,w}}$ is defined by

$$
\|t\|_{\dot{c}^{\alpha,q}_{p,w}} = \left[\sum_{\nu\in\mathbb{Z}} \left(\sup_{Q\in\mathcal{Q}_\nu(w)} |Q|^{-\frac{\alpha}{n}-\frac{1}{2}} |t_Q| w(Q)^{1-\frac{1}{p}} \right)^q \right]^{\frac{1}{q}}.
$$

Remark 2.5.

(i) If $p = 1$, then $\dot{c}^{\alpha,q}_{1,w} = \dot{b}^{\alpha,q}_{\infty,w}$.

(ii) If $w \equiv 1$, then $\dot{c}_{p,1}^{\alpha,q} = \dot{b}_{\infty}^{\alpha-n+n/p,q}$ by a direct calculation.

Here is a characterization of the dual of $\dot{b}_{p,w}^{\alpha,q}$ for $\alpha \in \mathbb{R}$, $0 < p \leq 1$ and $0 < q \leq \infty$.

Proposition 2.6. [10, Theorem 1.6]). *Let $\alpha \in \mathbb{R}$, $0 < p \leq 1$, $0 < q \leq \infty$, and w be a weight. Then the dual of $\dot{b}_{p,w}^{\alpha,q}$ is $\dot{c}_{p,w}^{-\alpha,q'}$ in the following sense.*

(i) *For $\mathbf{t} = \{t_Q\}_{Q\in\mathfrak{Q}(w)} \in \dot{c}_{p,w}^{-\alpha,q'}$, the linear functional $L_{\mathbf{t}}$ on $\dot{b}_{p,w}^{\alpha,q}$, given by $L_{\mathbf{t}}(\mathbf{s}) = \langle \mathbf{s}, \mathbf{t}\rangle_w$ for $\mathbf{s} = \{s_Q\}_{Q\in\mathfrak{Q}(w)} \in \dot{b}_{p,w}^{\alpha,q}$, is continuous with $\|L_{\mathbf{t}}\| \leq C\|\mathbf{t}\|_{\dot{c}_{p,w}^{-\alpha,q'}}$.*

(ii) *Conversely, every continuous linear functional L on $\dot{b}_{p,w}^{\alpha,q}$ satisfies $L = L_{\mathbf{t}}$ for some $\mathbf{t} = \{t_Q\}_{Q\in\mathfrak{Q}(w)} \in \dot{c}_{p,w}^{-\alpha,q'}$ with $\|\mathbf{t}\|_{\dot{c}_{p,w}^{-\alpha,q'}} \leq C\|L\|$.*

Remark 2.7. Observe that

$$\langle \mathbf{s}, \mathbf{t}\rangle_w = \sum_{Q\in\mathfrak{Q}} s_Q \overline{t_Q} \frac{w(Q)}{|Q|} = \sum_{Q\in\mathfrak{Q}} s_Q \overline{h_Q} = \langle \mathbf{s}, \mathbf{h}\rangle,$$

where $h_Q = t_Q \frac{w(Q)}{|Q|}$ and $\mathbf{h} = \{h_Q\}_Q$.

The characterization presented in Proposition 2.3 says that the dual of $\dot{b}_{p,w}^{\alpha,q}$ with respect to a weighted pairing can be identified with $\dot{b}_{p',w}^{-\alpha,q'}$ for any doubling weight w. Let us denote the dual with respect to the weighted pairing by $\left(\dot{b}_{p,w}^{\alpha,q}\right)'$. The difference arises because the pairing used as above observation. In Roudenko's case she has two sequences $\mathbf{s} = \{s_Q\}_{Q\in\mathfrak{Q}}$, and $\mathbf{h} = \{h_Q\}_{Q\in\mathfrak{Q}}$, indexed on the dyadic cubes and the pairing is: $\langle \mathbf{s}, \mathbf{h}\rangle = \sum_Q s_Q \overline{h_Q}$, whereas in this article the pairing is $\langle \mathbf{s}, \mathbf{t}\rangle_w = \sum_{Q\in\mathfrak{Q}(w)} s_Q \overline{t_Q} \frac{w(Q)}{|Q|}$. When dealing with any doubling weight, it may occur that $w(Q) = 0$ for some Q, which would imply $w = 0$ a.e. on Q; so two sequences will be equal in such space if and only if they coincide off those cubes (we are working with equivalence classes). In the case of the weighted pairing since there is no need to invoque the reciprocal of the weight (or a power of the weight), the above idetification work well, unlike when using the usual pairing.

In the case when both $w > 0$ a.e, and $w^{-1} > 0$ a.e, it would be interesting to explicitly state that the map that takes a sequence h_Q into the sequence $t_Q = h_Q \frac{|Q|}{w(Q)}$ is a one-to-one and continuous mapping from $\dot{b}_{p',w^{1-p'}}^{-\alpha,q'}$ into $\dot{b}_{p',w}^{-\alpha,q'}$ for all weight. To see this,

$$\|\mathbf{t}\|_{\dot{b}_{p',w}^{-\alpha,q'}} = \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathfrak{Q}_\nu(w)} \left(|Q|^{\alpha/n-1/2} |h_Q| \frac{|Q|}{w(Q)} \right)^{p'} w(Q) \right]^{q'/p'} \right\}^{1/q'}$$

$$
\begin{aligned}
&\le \left\{ \sum_{\nu\in\mathbb{Z}} \left[\sum_{Q\in\mathcal{Q}_\nu(w)} \left(|Q|^{\alpha/n-1/2} |h_Q| \right)^{p'} w^{1-p'}(Q) \right]^{q'/p'} \right\}^{1/q'} \\
&= \|\mathbf{h}\|_{\dot{b}^{-\alpha,q'}_{p',w^{1-p'}}},
\end{aligned}
$$

since

$$|Q| = \int_Q w^{1/p}(x) w^{-1/p}(x)dx \le \big[w(Q)\big]^{1/p} \big[w^{1-p'}(Q)\big]^{1/p'},$$

by Hölder's inequality. However the reverse embedding holds only if $w \in A_p$ because $|Q| \le \big[w(Q)\big]^{1/p}\big[w^{1-p'}(Q)\big]^{1/p'} \le C|Q|$, where C is dependent only on the A_p constant. Effectively the duals with respective to the different pairings are different spaces when the weight w is not in A_p, that explains the discrepancy.

By Remark 2.5 and Proposition 2.6, we have a characterization of the dual space of unweighted sequence Besov space $\dot{b}^{\alpha,q}_p$ for $\alpha \in \mathbb{R}$, $0 < p \le 1$ and $0 < q \le +\infty$.

Corollary 2.8. *Let* $\alpha \in \mathbb{R}$, $0 < p \le 1$ *and* $0 < q \le \infty$. *Then*

$$\big(\dot{b}^{\alpha,q}_p\big)' = \big(\dot{b}^{\alpha,q}_{p,1}\big)' \approx \dot{c}^{-\alpha,q'}_{p,1} = \dot{b}^{-\alpha-n+\frac{n}{p},q'}_{\infty}.$$

3. Almost Diagonal Matrices

At the beginning of this section, let us recall a definition about "doubling condition".

Definition 3.1. A weight w is called a *doubling measure*, if there exists a constant $C = C_n$ such that for any $\delta > 0$ and any $z \in \mathbb{R}^n$,

$$\int_{B_{2\delta}(z)} w(t)dt \le C \int_{B_\delta(z)} w(t)dt, \tag{8}$$

where $B_\delta(z)$ is an open ball in $\mathbb{R}^n$ centered at z with radius δ. If $C = 2^\beta$ is the smallest constant for the inequality (8) holds, then β is called the *doubling exponent* of w.

In this section, we always assume that w is a weight which is a doubling measure with doubling exponent β. For such a weight, we study the almost diagonality given by Roudenko [21] with matrix-weight for $p \ge 1$ and by Bownik [1] for scalar case. Here we adopt Bownik's definition, but we emphasize that, for $p \ge 1$, both definitions are equivalent.

Definition 3.2. Let w be a doubling measure with doubling exponent β. For $\alpha \in \mathbb{R}$, $0 < p,\ q \le \infty$, let $J = \frac{\beta}{p} + \max\left\{0, n - \frac{n}{p}\right\}$, we say that a matrix $A = \{a_{QP}\}_{Q,P}$ is (α, p, w) *almost diagonal*, denoted by $A \in \mathbf{ad}^\alpha_p(\beta)$, if there exist an $\varepsilon > 0$ and $C > 0$ such that for all dyadic cubes $Q,\ P$,

$$|a_{QP}| \leq C\left[\frac{\ell(Q)}{\ell(P)}\right]^{\alpha} \min\left(\left[\frac{\ell(Q)}{\ell(P)}\right]^{\frac{n+\varepsilon}{2}}, \left[\frac{\ell(P)}{\ell(Q)}\right]^{\frac{n+\varepsilon}{2}+J-n}\right)\left(1+\frac{|x_Q - x_P|}{\max(\ell(Q), \ell(P))}\right)^{-J-\varepsilon}.$$

Remark 3.3. Note that if $p \geq 1$ then $J = n + (\beta - n)/p$ and if $0 < p < 1$ then $J = \beta/p$. Also note that if the weight $w \equiv 1$, then $\beta = n$. Thus the definition of almost diagonality in Definition 3.2 is the same as the one given by M. Frazier and B. Jawerth in [5] under $q \geq 1$ and $w \equiv 1$. Also note that in general the exponent J is independent of q for Besov case, while in case of the Triebel-Lizorkin spaces $J = n/\min(1, p, q)$ in unweighted cases.

Basically, the proof was showed by Roudenko for $p \geq 1$ in [21] and showed by Bownik in more general setting in [2].

Proposition 3.4. ([2, 10, 21]). *Let $\alpha \in \mathbb{R}$, $0 < p,\ q \leq \infty$, and w a doubling measure with exponent β. If $A \in \mathbf{ad}_p^{\alpha}(\beta)$, then A is bounded on $\dot{b}_{p,w}^{\alpha,q}$.*

Now we may state that the class of almost diagonal matrices is closed under composition. The class of all operators on the distribution space level, which corresponds to almost diagonal matrices, is then also an algebra under composition. For $\gamma > 0$, $\delta > 0$, $J = \frac{\beta}{p} + \max\left\{0, n - \frac{n}{p}\right\}$ and $P,\ Q$ dyadic, denote

$$w_{QP}(\delta, \gamma) : = \left[\frac{\ell(Q)}{\ell(P)}\right]^{\alpha} \min\left(\left[\frac{\ell(Q)}{\ell(P)}\right]^{\frac{n+\gamma}{2}}, \left[\frac{\ell(P)}{\ell(Q)}\right]^{\frac{n+\gamma}{2}+J-n}\right) \left(1 + \frac{|x_Q - x_P|}{\max(\ell(Q), \ell(P))}\right)^{-J-\delta}$$

and

$$W_{QP}(\delta, \gamma_1, \gamma_2) := \sum_R w_{QR}(\delta, \gamma_1) w_{RP}(\delta, \gamma_2).$$

Theorem 3.5. *Suppose $\alpha \in \mathbb{R}$ and $0 < p,\ q \leq \infty$. If $A,\ B \in \mathbf{ad}_p^{\alpha}(\beta)$, then $A \circ B \in \mathbf{ad}_p^{\alpha}(\beta)$. Consequently, $\mathbf{ad}_p^{\alpha}(\beta)$ is an algebra.*

Before proving the Theorem 3.5, we need the following lemma, which is a modification of Theorem D.2 in [5].

Lemma 3.6. [5, Theorem D.2]). *Suppose $\delta, \gamma_1, \gamma_2 > 0$, $\gamma_1 \neq \gamma_2$, and $2\delta < \gamma_1 + \gamma_2$. Then there exists a constant C such that*

$$W_{QP}(\delta, \gamma_1, \gamma_2) \leq C w_{QP}(\delta, \min(\gamma_1, \gamma_2)).$$

Proof. [Proof of Theorem 3.5.] By the proof for [5, Theorem 9.1], we have the desired result immediately by Lemma 3.6. ∎

4. An Algebra of Matrix Operators on Weighted Sequence Besov Spaces

In this section, we will treat more general matrices on special weighted sequence Besov space. Let $\mathbf{b}$ denote the class of matrices A such that $|A|$ is bounded on $\dot{f}_p^{0,q}$ for all $1 \le p,\ q \le \infty$, where $|A| = \{|a_{QP}|\}_{Q,P}$ if $A = \{a_{QP}\}_{Q,P}$. Frazier and Jawerth [5] characterized the following result.

Proposition 4.1. [5, Corollary 10.2]). *A matrix $\{a_{QP}\}_{Q,P}$ belongs to $\mathbf{b}$ if and only if $\{|a_{QP}|\}_{Q,P}$ satisfies all conditions in the following:*

$$\sup_{P\in\mathcal{Q}} \sum_{Q\in\mathcal{Q}} |a_{QP}|(|Q|/|P|)^{1/2} < \infty;$$
$$\sup_{Q\in\mathcal{Q}} \sum_{P\in\mathcal{Q}} |a_{QP}|(|Q|/|P|)^{-1/2} < \infty;$$
$$\sup_{P_0\in\mathcal{Q}} \frac{1}{|P_0|}\left\| \left\{ \sum_{P\in\mathcal{Q}_{P_0}} |a_{QP}||P|^{1/2}\right\}\right\|_{\dot{f}_1^{0,\infty}} < \infty;$$
$$\sup_{Q_0\in\mathcal{Q}} \frac{1}{|Q_0|}\left\| \left\{ \sum_{Q\in\mathcal{Q}_{Q_0}} |a_{QP}||Q|^{1/2}\right\}\right\|_{\dot{f}_1^{0,\infty}} < \infty.$$

The main purpose of this section is to characterize an algebra of bounded matrix operators acting on weighted sequence Besov spaces $\dot{b}_{p,w}^{\alpha,q}$ for all $1 \le p, q \le \infty$, where α is fixed in $\mathbb{R}$. Let us observe some special cases.

Theorem 4.2. *Suppose $\alpha \in \mathbb{R}$, $0 < p = q \le 1$, w is a weight, and $A = \{a_{QP}\}$ is a matrix. Then A is bounded on $\dot{b}_{q,w}^{\alpha,q}$ if and only if*

$$\sup_{P\in\mathcal{Q}(w)} \left\{ \sum_{Q\in\mathcal{Q}(w)} \left[\left(\frac{|Q|}{|P|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}} |a_{QP}|\right]^q \frac{w(Q)}{w(P)}\right\}^{\frac{1}{q}} < \infty. \tag{9}$$

Proof. First let us suppose that A is bounded on $\dot{b}_{q,w}^{\alpha,q}$. Fix a dyadic cube $P \in \mathcal{Q}(w)$ and define a sequence s^P by

$$(s^P)_Q := \begin{cases} |P|^{\frac{\alpha}{n}+\frac{1}{2}} w(P)^{-\frac{1}{q}} & \text{if} \quad Q = P \\ 0 & \text{if} \quad Q \ne P \end{cases}.$$

Then $\|s^P\|_{\dot{b}_{q,w}^{\alpha,q}} = 1$. Since $(As^P)_Q = a_{QP}|P|^{\frac{\alpha}{n}+\frac{1}{2}} w(P)^{-\frac{1}{q}}$ for $Q \in \mathcal{Q}(w)$, we have

$$\begin{aligned}
&\left\{ \sum_{Q\in\mathcal{Q}(w)} \left[\left(\frac{|Q|}{|P|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}} |a_{QP}|\right]^q \frac{w(Q)}{w(P)}\right\}^{\frac{1}{q}} \\
&\quad = \left[\sum_{Q\in\mathcal{Q}(w)} \left(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}|a_{QP}||P|^{\frac{\alpha}{n}+\frac{1}{2}} w(P)^{-\frac{1}{q}}\right)^q w(Q)\right]^{\frac{1}{q}} \\
&\quad = \|As^P\|_{\dot{b}_{q,w}^{\alpha,q}} \le \|A\| \|s^P\|_{\dot{b}_{q,w}^{\alpha,q}} = \|A\|.
\end{aligned}$$

Thus, after taking supremum over all dyadic cubes P in $\mathcal{Q}(w)$, we have condition (9).

Conversely, suppose that condition (9) holds. Since $s \in \dot{b}^{\alpha,q}_{q,w}$, we have

$$\|s\|_{\dot{b}^{\alpha,q}_{q,w}} = \left[\sum_{P \in \mathcal{Q}(w)} \left(|P|^{-\frac{\alpha}{n}-\frac{1}{2}} |s_P| \right)^q w(P) \right]^{\frac{1}{q}}$$

and so

$$\begin{aligned}
\|As\|^q_{\dot{b}^{\alpha,q}_{q,w}} &= \sum_{Q \in \mathcal{Q}(w)} \left(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}} \left| \sum_{P \in \mathcal{Q}(w)} a_{QP} s_P \right| \right)^q w(Q) \\
&\leq \sum_{P \in \mathcal{Q}(w)} \sum_{Q \in \mathcal{Q}(w)} \left[\left(\frac{|Q|}{|P|} \right)^{-\frac{\alpha}{n}-\frac{1}{2}} |a_{QP}| \right]^q \frac{w(Q)}{w(P)} \cdot \left(|P|^{-\frac{\alpha}{n}-\frac{1}{2}} |s_P| \right)^q w(P) \\
&\leq \sup_{P \in \mathcal{Q}(w)} \left\{ \sum_{Q \in \mathcal{Q}(w)} \left[\left(\frac{|Q|}{|P|} \right)^{-\frac{\alpha}{n}-\frac{1}{2}} |a_{QP}| \right]^q \frac{w(Q)}{w(P)} \right\} \cdot \|s\|^q_{\dot{b}^{\alpha,q}_{q,w}},
\end{aligned}$$

where we apply the triangle inequlaity in the first inequality for index q. Hence A is bounded on $\dot{b}^{\alpha,q}_{q,w}$. ∎

By a duality argument, we have the following result.

Corollary 4.3. *Suppose $\alpha \in \mathbb{R}$, w is a weight, and $A = \{a_{QP}\}$ is a matrix. Then A is bounded on $\dot{b}^{\alpha,\infty}_{\infty,w}$ if and only if*

$$\sup_Q \sum_P \left(\frac{|P|}{|Q|} \right)^{\frac{\alpha}{n}-\frac{1}{2}} |a_{QP}| \frac{w(P)}{w(Q)} < \infty. \tag{10}$$

Proof. Note that a matrix A is bounded on $\dot{b}^{\alpha,\infty}_{\infty,w}$ if and only if its adjoint A^* is bounded on $\dot{b}^{-\alpha,1}_{1,w}$ where $a^*_{QP} = \overline{a_{PQ}}$. Thus, by Theorem 4.2, A is bounded on $\dot{b}^{\alpha,\infty}_{\infty,w}$ if and only if condition (10) holds. ∎

Theorem 4.4. *Suppose $\alpha \in \mathbb{R}$, w is a weight and $A = \{a_{QP}\}$ is a matrix operator with $a_{QP} \geq 0$ for all dyadic cubes P and Q. Then A is bounded on $\dot{b}^{\alpha,\infty}_{1,w}$ if and only if*

$$\sup_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} \sup_{P \in \mathcal{Q}_\mu(w)} \sum_{Q \in \mathcal{Q}_\nu(w)} \left(\frac{|Q|}{|P|} \right)^{-\frac{\alpha}{n}-\frac{1}{2}} a_{QP} \frac{w(Q)}{w(P)} < \infty. \tag{11}$$

Proof. Suppose A is bounded on $\dot{b}^{\alpha,\infty}_{1,w}$. For each pair of μ and ν in $\mathbb{Z}$, let

$$K_{\mu,\nu} := \sup_{P \in \mathcal{Q}_\mu(w)} \sum_{Q \in \mathcal{Q}_\nu(w)} \left(\frac{|Q|}{|P|} \right)^{-\frac{\alpha}{n}-\frac{1}{2}} a_{QP} \frac{w(Q)}{w(P)}.$$

Claim that $K_{\mu,\nu} < \infty$ for every pair of μ and ν. Suppose that there exist $\mu_0, \nu_0 \in \mathbb{Z}$ such that $K_{\mu_0,\nu_0} = \infty$. For $j \in \mathbb{N}$, there exists a dyadic cube P_j such that $\ell(P_j) = 2^{-\mu_0}$ and

$$\sum_{Q \in \mathcal{Q}_{\nu_0}(w)} \left(\frac{|Q|}{|P_j|} \right)^{-\frac{\alpha}{n}-\frac{1}{2}} a_{QP_j} \frac{w(Q)}{w(P_j)} \geq j.$$

For $j \in \mathbb{N}$, let s^j be a sequence defiend by, for $P \in \mathcal{Q}(w)$,

$$(s^j)_P := \begin{cases} |P_j|^{\frac{\alpha}{n}+\frac{1}{2}} w(P_j)^{-1} & \text{if} \quad P = P_j \\ 0 & \text{if} \quad P \neq P_j \end{cases}.$$

Then $\|s^j\|_{\dot{b}^{\alpha,\infty}_{1,w}} = 1$ and for $Q \in \mathcal{Q}(w)$

$$(As^j)_Q = \sum_{P \in \mathcal{Q}(w)} a_{QP} (s^j)_P = a_{QP_j} |P_j|^{\frac{\alpha}{n}+\frac{1}{2}} w(P_j)^{-1}.$$

Thus,

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_{\nu_0}(w)} \left(\frac{|Q|}{|P_j|} \right)^{-\frac{\alpha}{n}-\frac{1}{2}} a_{QP_j} \frac{w(Q)}{w(P_j)} &= \sum_{Q \in \mathcal{Q}_{\nu_0}(w)} |Q|^{-\frac{\alpha}{n}-\frac{1}{2}} |(As^j)| w(Q) \\ &\leq \|As^j\|_{\dot{b}^{\alpha,\infty}_{1,w}} \leq C \|s^j\|_{\dot{b}^{\alpha,\infty}_{1,w}} = C, \end{aligned}$$

where we apply the boundedness of the matrix A on $\dot{b}^{\alpha,\infty}_{1,w}$. This contradiction yields $K_{\mu,\nu} < \infty$ for each $\mu,\ \nu \in \mathbb{Z}$.

Fix $\nu \in \mathbb{Z}$ and, for each $\mu \in \mathbb{Z}$, choose a dyadic cube P_μ satisfying $\ell(P_\mu) = 2^{-\mu}$ and

$$\sum_{Q \in \mathcal{Q}_{\nu}(w)} \left(\frac{|Q|}{|P_\mu|} \right)^{-\frac{\alpha}{n}-\frac{1}{2}} a_{QP_\mu} \frac{w(Q)}{w(P_\mu)} \geq \frac{1}{2} K_{\mu,\nu}.$$

Let s^ν be a sequence defiend by

$$(s^\nu)_P := \begin{cases} |P_\mu|^{\frac{\alpha}{n}+\frac{1}{2}} w(P_\mu)^{-1} & \text{if} \quad P = P_\mu \\ 0 & \text{if} \quad P \neq P_\mu \end{cases}.$$

Then $\|s^\nu\|_{\dot{b}^{\alpha,\infty}_{1,w}} = 1$ and

$$(As^\nu)_Q = \sum_{\mu \in \mathbb{Z}} \sum_{P \in \mathcal{Q}_\mu(w)} a_{QP} (s^\nu)_P = \sum_{\mu \in \mathbb{Z}} a_{QP_\mu} |P_\mu|^{\frac{\alpha}{n}+\frac{1}{2}} w(P_\mu)^{-1}.$$

Since A is bounded on $\dot{b}^{\alpha,\infty}_{1,w}$, we have

$$
\begin{aligned}
&\sum_{\mu\in\mathbb{Z}}\sup_{P\in\mathcal{Q}_\mu(w)}\sum_{Q\in\mathcal{Q}_\nu(w)}\left(\frac{|Q|}{|P|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}}a_{QP}\frac{w(Q)}{w(P)}\\
&\leq 2\sum_{\mu\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_\nu(w)}\left(\frac{|Q|}{|P_\mu|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}}a_{QP_\mu}\frac{w(Q)}{w(P_\mu)}\\
&=2\sum_{Q\in\mathcal{Q}_\nu(w)}|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}\left(\sum_{\mu\in\mathbb{Z}}a_{QP_\mu}|P_\mu|^{\frac{\alpha}{n}+\frac{1}{2}}w(P_\mu)^{-1}\right)w(Q)\\
&\leq 2\|As^\nu\|_{\dot b^{\alpha,\infty}_{1,w}}\leq C\|s^\nu\|_{\dot b^{\alpha,\infty}_{1,w}}=C.
\end{aligned}
$$

Thus, after taking the supremum over $\nu\in\mathbb{Z}$, we have condition (11).

Conversely, suppose that condition (11) holds. Since $s\in\dot b^{\alpha,\infty}_{1,w}$, we have

$$
\|s\|_{\dot b^{\alpha,\infty}_{1,w}}=\sup_{\mu\in\mathbb{Z}}\sum_{P\in\mathcal{Q}_\mu(w)}|P|^{-\frac{\alpha}{n}-\frac{1}{2}}|s_P|w(P)
$$

and so

$$
\begin{aligned}
\|As\|_{\dot b^{\alpha,\infty}_{1,w}}&\leq\sup_{\nu\in\mathbb{Z}}\sum_{\mu\in\mathbb{Z}}\sum_{P\in\mathcal{Q}_\mu(w)}|P|^{-\frac{\alpha}{n}-\frac{1}{2}}|s_P|w(P)\sum_{Q\in\mathcal{Q}_\nu(w)}\left(\frac{|Q|}{|P|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}}a_{QP}\frac{w(Q)}{w(P)}\\
&\leq\sup_{\nu\in\mathbb{Z}}\sum_{\mu\in\mathbb{Z}}\left[\sup_{P\in\mathcal{Q}_\mu(w)}\sum_{Q\in\mathcal{Q}_\nu(w)}\left(\frac{|Q|}{|P|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}}a_{QP}\frac{w(Q)}{w(P)}\right]\\
&\quad\times\left[\sum_{P\in\mathcal{Q}_\mu(w)}|P|^{-\frac{\alpha}{n}-\frac{1}{2}}|s_P|w(P)\right]\\
&\leq\sup_{\nu\in\mathbb{Z}}\left[\sum_{\mu\in\mathbb{Z}}\sup_{P\in\mathcal{Q}_\mu(w)}\sum_{Q\in\mathcal{Q}_\nu(w)}\left(\frac{|Q|}{|P|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}}a_{QP}\frac{w(Q)}{w(P)}\right]\cdot\|s\|_{\dot b^{\alpha,\infty}_{1,w}}.
\end{aligned}
$$

Hence A is bounded on $\dot b^{\alpha,\infty}_{1,w}$. ■

Corollary 4.5. *Suppose $\alpha\in\mathbb{R}$, w is a weight and $A=\{a_{QP}\}$ is a matrix operator with $a_{QP}\geq 0$ for all dyadic cubes P and Q. Then A is bounded on $\dot b^{\alpha,1}_{\infty,w}$ if and only if*

$$
\sup_{\mu\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}}\sup_{Q\in\mathcal{Q}_\nu(w)}\sum_{P\in\mathcal{Q}_\mu(w)}\left(\frac{|P|}{|Q|}\right)^{\frac{\alpha}{n}-\frac{1}{2}}a_{QP}\frac{w(P)}{w(Q)}<\infty. \tag{12}
$$

Proof. By a duality argument, the result follows immediately. ■

Definition 4.6. Let w be a weight and $\alpha\in\mathbb{R}$. We say that a matrix operator $A=\{a_{QP}\}$ is an element of an algebra of bounded matrix operator, denoted by $A\in\mathbf{amo}^\alpha(w)$, if $|A|$ is bounded on $\dot b^{\alpha,q}_{p,w}$ for all $1\leq p,\ q\leq\infty$.

Here is a characterization of $\mathbf{amo}^{\alpha}(w)$.

Theorem 4.7. *Let* $\alpha \in \mathbb{R}$, w *be a weight, and* $A = \{a_{QP}\}$ *be a matrix operator. Then* $A \in \mathbf{amo}^{\alpha}(w)$ *if and only if* A *satisfies* (9–10),

$$\sup_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} \sup_{P \in \mathcal{Q}_{\mu}(w)} \sum_{Q \in \mathcal{Q}_{\nu}(w)} \left(\frac{|Q|}{|P|}\right)^{-\frac{\alpha}{n}-\frac{1}{2}} |a_{QP}| \frac{w(Q)}{w(P)} < \infty, \tag{13}$$

and

$$\sup_{\mu \in \mathbb{Z}} \sum_{\nu \in \mathbb{Z}} \sup_{Q \in \mathcal{Q}_{\nu}(w)} \sum_{P \in \mathcal{Q}_{\mu}(w)} \left(\frac{|P|}{|Q|}\right)^{\frac{\alpha}{n}-\frac{1}{2}} |a_{QP}| \frac{w(P)}{w(Q)} < \infty. \tag{14}$$

Proof. By Definition 4.6, the "if" part follows immediately by Theorem 4.2 with $p = q = 1$, Corollary 4.3, Theorem 4.4 and Corollary 4.5. Conversely,

(a) by Theorem 4.2 with $p = q = 1$ and condition (9), A is bounded on $\dot{b}^{\alpha,1}_{1,w}$;

(b) by Corollary 4.3 and condition (10), A is bounded on $\dot{b}^{\alpha,\infty}_{\infty,w}$;

(c) by Theorem 4.4 and condition (13), A is bounded on $\dot{b}^{\alpha,\infty}_{1,w}$;

(d) by Corollary 4.5 and condition (14), A is bounded on $\dot{b}^{\alpha,1}_{\infty,w}$.

Hence it follows from interpolation theorem that A is bounded on $\dot{b}^{\alpha,q}_{p,w}$ for all $1 \leq p,\ q \leq \infty$, i.e., $A \in \mathbf{amo}^{\alpha}(w)$. ∎

Remark 4.8.

(a) It is routine to check that $\mathbf{amo}^{\alpha}(w)$ is an algebra with composition.

(b) Because $\beta/p + n/p' \leq \beta$ for $p \geq 1$, it follows from Theorem 3.4 that we have $\mathbf{ad}^{\alpha}_{1}(\beta) \subseteq \mathbf{amo}^{\alpha}(w)$.

(c) If the weight $w \equiv 1$, then $\beta = n$. So we have the following result: if a matrix A is almost diagonal then (i) the estimate w_{QP} is independent of p for $p \geq 1$, (ii) the matrix A is bounded on $\dot{b}^{\alpha,q}_{p}$ for $1 \leq p,\ q \leq \infty$ and (iii) $A \in \mathbf{amo}^{\alpha}(1)$.

5. Applications

Consider that an operator T is linear from the Schwartz space $\mathcal{S}$ to its dual $\mathcal{S}'$ and has a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ which gives the action of T away from the diagonal. The kernel K is a function which is locally integrable on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}$ and there exist a constant $C > 0$ and a regularity exponent $\varepsilon \in (0,1]$ such that

$$|K(x,y)| \leq C|x-y|^{-n} \qquad \text{for} \quad x \neq y; \tag{15}$$

$$|K(x,y) - K(x',y)| \leq C\frac{|x-x'|^{\varepsilon}}{|x-y|^{n+\varepsilon}} \qquad \text{for} \quad |x-x'| \leq \frac{|x-y|}{2}; \tag{16}$$

$$|K(x,y) - K(x,y')| \leq C\frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}} \qquad \text{for} \quad |y-y'| \leq \frac{|x-y|}{2}. \tag{17}$$

That K gives the action of T away from the diagonal means that is for any two functions f and g in $\mathcal{S}$ and which have disjoint support, we have that

$$Tf(g) = \langle Tf, g\rangle = \int_{\mathbb{R}^{2n}} K(x,y)f(y)g(x)dxdy,$$

then T is called a *singular integral operator*, denoted by $T \in SIO(\varepsilon)$.

Next, we recall the definition of A_p weight in questions.

Definition 5.1. (A_p Weights). For every cube Q in $\mathbb{R}^n$, and a non-negative and locally integrable function w on $\mathbb{R}^n$. We say that $w \in A_p$ if $\|w\|_{A_p}$ is finte, where $\|w\|_{A_p}$ is defined by

$$\|w\|_{A_p} := \begin{cases} \sup\limits_Q \operatorname*{ess\,sup}\limits_{y\in Q} w^{-1}(y)\dfrac{1}{|Q|}\displaystyle\int_Q w(t)dt & \text{if} \quad 0 < p \le 1 \\ \sup\limits_Q \left(\dfrac{1}{|Q|}\displaystyle\int_Q w(x)dx\right)\left(\dfrac{1}{|Q|}\displaystyle\int_Q w(x)^{1-p'}dx\right)^{p-1} & \text{if} \quad 1 < p < \infty \end{cases},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Also let $A_\infty = \bigcup_{0<p<\infty} A_p$.

Remark 5.2.

(a) Let us recall matrix-A_p weights given in [8, 18, 21]. Let $\mathcal{M}$ be the cone of non-negative definite $m \times m$ complex-valued matrices. By definition, a ***matrix weight*** W is an almost everywhere invertible map $W : \mathbb{R}^n \to \mathcal{M}$, W and W^{-1} are locally integrable. We say that W is a matrix-A_p weight if it is a matrix weight satisfying

$$\|W\|_{A_p} := \begin{cases} \sup\limits_Q \operatorname*{ess\,sup}\limits_{y\in Q} \dfrac{1}{|Q|}\displaystyle\int_Q \left\|W^{\frac{1}{p}}(t)W^{-\frac{1}{p}}(y)\right\|^p dt < \infty & \text{if} \quad 0 < p \le 1 \\ \sup\limits_Q \displaystyle\int_Q \left(\int_Q \left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(t)\right\|^{p'} \frac{dt}{|Q|}\right)^{\frac{p}{p'}} \frac{dx}{|Q|} < \infty & \text{if} \quad 1 < p < \infty \end{cases},$$

where the first supremum is taken over all cubes Q in $\mathbb{R}^n$.

(b) In the scalar case, an A_p weight is an A_1 weight in the sense of Muckenhoupt [11] for $0 < p \le 1$. Since there exists a constant $C > 0$ such that

$$\frac{1}{|Q|}\int_Q w(t)dt \le C \cdot w(y) \qquad \text{for a.e. } y \in Q, \text{ for all } Q \subseteq \mathbb{R}^n.$$

In terms of the maximal function, this condition is

$$Mw(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q w(t)dt \le C \cdot w(x) \qquad \text{for a.e. } x,$$

i.e. $w \in A_1$, where M is the Hardy-Littlewood maximal operator.

(c) If a scalar weight $w \in A_p$ for $1 \le p \le \infty$, then $w(x)dx$ is a doubling measure.

Definition 5.3. Let w be a weight and $T \in SIO(\varepsilon)$. Then $T \in \mathbf{AMO}^0(w)$ if T is bounded on $\dot{B}^{0,q}_{p,w}$ for all $1 \le p,\ q \le \infty$.

Theorem 5.4. *Suppose w is an A_1 weight and $T \in SIO(\varepsilon)$. Let $A(T) = \{\langle T\psi_P, \varphi_Q\rangle\}$. Then the following statements are equivalent.*

(a) $T \in \mathbf{AMO}^0(w)$*;*

(b) $A(T) \in \mathbf{amo}^0(w)$*;*

(c) $A(T)$ *satisfies* (9–10) and (13-14) with $\alpha = 0$ and $a_{QP} = \langle T\psi_P, \varphi_Q\rangle$, simultaneously.

Proof. By Proposition 1.5, (a) implies (b) and by Theorem 4.7, (b) implies (c). Finally, by Definition 5.3, Proposition 1.5 and Theorem 4.7, (c) implies (a). Hence we establish the equivalence of all three statements. ∎

Here is an application for the boundedness of Fourier-Haar multipliers.

Theorem 5.5. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < p,\ q_1,\ q_2 \le \infty$ and $w \in A_p$.*

(a) *For $q_1 > q_2$, the Fourier-Haar multiplier T_t is bounded from $\dot{B}^{\alpha_1,q_1}_{p,w}$ into $\dot{B}^{\alpha_1+\alpha_2,q_2}_{p,w}$ if and only if $t \in \dot{b}^{\alpha_2,q_2\rho'}_{\infty,w}$, where ρ' is the index conjugate of $\rho = q_1/q_2$.*

(b) *For $q_1 \le q_2$, the Fourier-Haar multiplier T_t is bounded from $\dot{B}^{\alpha_1,q_1}_{p,w}$ into $\dot{B}^{\alpha_1+\alpha_2,q_2}_{p,w}$ if and only if $t \in \dot{b}^{\alpha_2,\infty}_{\infty,w}$.*

Proof. For part (a), suppose $t = \{t_Q\}_Q \in \dot{b}^{\alpha_2,q_2\rho'}_{\infty,w}$ and $f \in \dot{B}^{\alpha_1,q_1}_{p,w}$ where $\rho = \frac{q_1}{q_2}$ and $\rho' = \frac{q_1}{q_1-q_2}$. Then $\left\{\langle f, \varphi_Q\rangle\right\}_Q \in \dot{b}^{\alpha_1,q_1}_{p,w}$, by Proposition 1.5. Thus, by (6), Propositions 1.5 and 2.1(a), we have

$$\begin{aligned}\|T_t f\|_{\dot{B}^{\alpha_1+\alpha_2,q_2}_{p,w}} &\approx \left\|\left\{|Q|^{-\frac{1}{2}} t_Q \langle f, \varphi_Q\rangle\right\}_Q\right\|_{\dot{b}^{\alpha_1+\alpha_2,q_2}_{p,w}}\\ &\le C\|t\|_{\dot{b}^{\alpha_2,q_2\rho'}_{\infty,w}} \cdot \left\|\left\{\langle f, \varphi_Q\rangle\right\}_Q\right\|_{\dot{b}^{\alpha_1,q_1}_{p,w}}\\ &\le C\|t\|_{\dot{b}^{\alpha_2,q_2\rho'}_{\infty,w}} \cdot \|f\|_{\dot{B}^{\alpha_1,q_1}_{p,w}}.\end{aligned}$$

Conversely, suppose that T_t is bounded from $\dot{B}^{\alpha_1,q_1}_{p,w}$ into $\dot{B}^{\alpha_1+\alpha_2,q_2}_{p,w}$. Then, by Proposition 1.5, we obtain

$$\begin{aligned}\left\|\left\{|Q|^{-\frac{1}{2}} t_Q \langle f, \varphi_Q\rangle\right\}_Q\right\|_{\dot{b}^{\alpha_1+\alpha_2,q_2}_{p,w}} &\approx \|T_t f\|_{\dot{B}^{\alpha_1+\alpha_2,q_2}_{p,w}}\\ &\le C\|f\|_{\dot{B}^{\alpha_1,q_1}_{p,w}}\\ &\le C\left\|\left\{\langle f, \varphi_Q\rangle\right\}_Q\right\|_{\dot{b}^{\alpha_1,q_1}_{p,w}}.\end{aligned}$$

Thus, by Proposition2.1 (a), we get $t \in \dot{b}_{\infty,w}^{\alpha_2,q_2\rho'}$.

Similarly, for part (b), suppose $t = \{t_Q\}_Q \in \dot{b}_{\infty,w}^{\alpha_2,\infty}$ and $f \in \dot{B}_{p,w}^{\alpha_1,q_1}$. Then $\{\langle f,\varphi_Q\rangle\}_Q \in \dot{b}_{p,w}^{\alpha_1,q_1}$, and so T_t is bounded from $\dot{B}_{p,w}^{\alpha_1,q_1}$ into $\dot{B}_{p,w}^{\alpha_1+\alpha_2,q_2}$, by Propositions 1.5 and 2.1 (b).

Conversely, suppose that T_t is bounded from $\dot{B}_{p,w}^{\alpha_1,q_1}$ into $\dot{B}_{p,w}^{\alpha_1+\alpha_2,q_2}$. Then

$$\Big\|\Big\{|Q|^{-\frac{1}{2}}t_Q\langle f,\varphi_Q\rangle\Big\}_Q\Big\|_{\dot{b}_{p,w}^{\alpha_1+\alpha_2,q_2}} \le C\Big\|\Big\{\langle f,\varphi_Q\rangle\Big\}_Q\Big\|_{\dot{b}_{p,w}^{\alpha_1,q_1}}.$$

Thus, by Proposition 2.1(b), we obtain $t \in \dot{b}_{\infty,w}^{\alpha_2,\infty}$. ■

To prove the boundedness of paraproduct operators, we need the following lemma.

Lemma 5.6. *Let Φ be the function given in Definition 1.2. Define $\Phi_Q(x) = |Q|^{-\frac{1}{2}}\Phi\big(\frac{x-x_Q}{\ell(Q)}\big)$. Suppose $G=\{g_{QP}\}_{Q,P}$ where $g_{QP}=\langle\psi_P,\Phi_Q\rangle$ for all dyadic cubes P and Q. For $\alpha<0$, $0<p,\ q\le\infty$, $G\in\mathbf{ad}_p^{\alpha}(\beta)$, hence is bounded on $\dot{b}_{p,w}^{\alpha,q}$.*

Proof. For $\ell(P) \le \ell(Q)$, since $\int x^\gamma\psi_P(x)dx = 0$ for all γ, by [5, p. 150, Lemma B.1], we have

$$|\langle\psi_P,\Phi_Q\rangle| \le C\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\alpha}\Big(1+\frac{|x_Q-x_P|}{\ell(Q)}\Big)^{-J-\varepsilon}\Big(\frac{\ell(P)}{\ell(Q)}\Big)^{\frac{n+\varepsilon}{2}+J-n}, \quad \alpha\in\mathbb{R} \text{ and } \varepsilon>0,$$

where C depends on J only.

For $\ell(Q) < \ell(P)$, by [5, p. 152, Lemma B.2], we obtain

$$\begin{aligned}|\langle\psi_P,\Phi_Q\rangle| &\le C\Big(1+\frac{|x_Q-x_P|}{\ell(P)}\Big)^{-J-\varepsilon}\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\frac{n}{2}}\\ &= C\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\alpha}\Big(1+\frac{|x_Q-x_P|}{\ell(P)}\Big)^{-J-\varepsilon}\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\frac{n-2\alpha}{2}}.\end{aligned}$$

So choosing $\varepsilon = -2\alpha$, we obtain the result. ■

Here is an application to paraproduct operators.

Theorem 5.7. *For $\alpha < 0$, $\beta \in \mathbb{R}$ and $0 < p,\ q \le \infty$, let w be an A_p-weight and Π_g be the paraproduct operator defined in Definition 1.2.*

(*i*) *If $0 < r < p$ and $g \in \dot{B}_{pr/(p-r),w}^{\beta,qr/(q-r)}$, then Π_g is bounded from $\dot{B}_{p,w}^{\alpha,q}$ into $\dot{B}_{r,w}^{\alpha+\beta,r}$.*

(*ii*) *If $0 < p \le r$ and $S_\varphi(g) = \{\langle g,\varphi_Q\rangle\}_{Q\in\mathcal{Q}} \in \dot{c}_{p/r,w}^{\beta,qr/(q-r)}$, then Π_g is bounded from $\dot{B}_{p,w}^{\alpha,q}$ into $\dot{B}_{r,w}^{\alpha+\beta,r}$.*

Proof. Let $f \in \dot{B}^{\alpha,q}_{p,w}$. By equation (4) and Proposition 1.5, we have

$$\begin{aligned} \|\Pi_g f\|^r_{\dot{B}^{\alpha+\beta,r}_{r,w}} &\approx \Big\| \Big\{ \langle g, \varphi_Q \rangle |Q|^{-1/2} \Big(\sum_P \langle \psi_P, \Phi_Q \rangle \langle f, \varphi_P \rangle \Big) \Big\}_Q \Big\|^r_{\dot{b}^{\alpha+\beta,r}_{r,w}} \\ &= \sum_{Q \in \mathcal{Q}} \Big(|Q|^{-\alpha/n-1/2+1/(2r)} |(G\mathbf{s})_Q| \Big)^r \\ &\quad \Big(|Q|^{-\beta/n-1/2+1/(2r)} |\langle g, \varphi_Q \rangle| \Big)^r \frac{w(Q)}{|Q|}, \end{aligned} \tag{18}$$

where $\mathbf{s} = \{\langle f, \varphi_P \rangle\}_{P \in \mathcal{Q}} = S_\varphi(f)$. For case (i), by Proposition 2.3, the last inequality is dominated by a multiple of

$$\Big\| \Big\{ \Big(|Q|^{-\alpha/n-1/2+1/(2r)} (G\mathbf{s})_Q \Big)^r \Big\}_{Q \in \mathcal{Q}} \Big\|_{\dot{b}^{0,q/r}_{p/r,w}} \times \Big\| \Big\{ \Big(|Q|^{-\beta/n-1/2+1/(2r)} \langle g, \varphi_Q \rangle \Big)^r \Big\}_{Q \in \mathcal{Q}} \Big\|_{\dot{b}^{0,(q/r)'}_{(p/r)',w}},$$

provided $\Big\{ \Big(|Q|^{-\beta/n-1/2+1/(2r)} \langle g, \varphi_Q \rangle \Big)^r \Big\}_{Q \in \mathcal{Q}} \in \dot{b}^{0,(q/r)'}_{(p/r)',w}$. A calculation shows that

$$\begin{aligned} &\Big\| \Big\{ \Big(|Q|^{-\alpha/n-1/2+1/(2r)} (G\mathbf{s})_Q \Big)^r \Big\}_{Q \in \mathcal{Q}} \Big\|_{\dot{b}^{0,q/r}_{p/r,w}} \\ &= \Big\{ \sum_{\nu \in \mathbb{Z}} \Big[\sum_{Q \in \mathcal{Q}_\nu} \Big(|Q|^{-\alpha/n-1/2} |(G\mathbf{s})_Q| \Big)^p w(Q) \Big]^{q/p} \Big\}^{r/p} \\ &= \|G\mathbf{s}\|^r_{\dot{b}^{\alpha,q}_{p,w}} \le C \|\mathbf{s}\|^r_{\dot{b}^{\alpha,q}_{p,w}} \le C \|f\|^r_{\dot{B}^{\alpha,q}_{p,w}}, \end{aligned}$$

by Proposition 1.5 and Lemma 5.6. Also

$$\begin{aligned} &\Big\| \Big\{ \Big(|Q|^{-\beta/n-1/2+1/(2r)} \langle g, \varphi_Q \rangle \Big)^r \Big\}_{Q \in \mathcal{Q}} \Big\|_{\dot{b}^{0,(q/r)'}_{(p/r)',w}} \\ &= \Big\{ \sum_{\nu \in \mathbb{Z}} \Big[\sum_{Q \in \mathcal{Q}_\nu} \Big(|Q|^{-\beta/n-1/2} |\langle g, \varphi_Q \rangle| \Big)^{pr/(p-r)} w(Q) \Big]^{\frac{qr/(q-r)}{pr/(p-r)}} \Big\}^{(q-r)/r} \\ &= \big\| \{ \langle g, \varphi_Q \rangle \} \big\|^r_{\dot{b}^{\beta,qr/(q-r)}_{pr/(p-r),w}}, \end{aligned}$$

which is equivalent to $\|g\|^r_{\dot{B}^{\beta,qr/(q-r)}_{pr/(p-r),w}}$ by Proposition 1.5.

For case (ii), apply Proposition 2.6 to (18) to yield that

$$\begin{aligned} \|\Pi_g f\|^r_{\dot{B}^{\alpha+\beta,r}_{r,w}} \le C &\Big\| \Big\{ \Big(|Q|^{-\alpha/n-1/2+1/(2r)} (G\mathbf{s})_Q \Big)^r \Big\}_{Q \in \mathcal{Q}} \Big\|_{\dot{b}^{0,q/r}_{p/r,w}} \\ &\times \Big\| \Big\{ \Big(|Q|^{-\beta/n-1/2+1/(2r)} \langle g, \varphi_Q \rangle \Big)^r \Big\}_{Q \in \mathcal{Q}} \Big\|_{\dot{c}^{0,(q/r)'}_{p/r,w}}. \end{aligned}$$

It is clear that

$$
\begin{aligned}
&\left\|\left\{\left(|Q|^{-\beta/n-1/2+1/(2r)}\langle g,\varphi_Q\rangle\right)^r\right\}_{Q\in\mathcal{Q}}\right\|_{\dot{c}^{0,(q/r)'}_{p/r,w}}\\
&\quad=\left\{\sum_{\nu\in\mathbb{Z}}\left(|Q|^{-\beta/n-1/2}|\langle g,\varphi_Q\rangle|w(Q)^{1/r-1/p}\right)^{qr/(q-r)}\right\}^{(q-r)/(qr)}\\
&\quad=\left\|\left\{\langle g,\varphi_Q\rangle\right\}_{Q\in\mathcal{Q}}\right\|_{\dot{c}^{\beta,qr/(q-r)}_{p_0,w}},
\end{aligned}
$$

where p_0 satisfies $1-1/p_0=1/r-1/p$; that is $p_0=pr/(pr+r-p)$. Therefore Π_g is bounded from $\dot{B}^{\alpha,q}_{p,w}$ into $\dot{B}^{\alpha+\beta,r}_{r,w}$ and the proof is finished. ■

Remark 5.8. In 1989, M. Meyer [16] proved that a singular integral operator T is bounded on $\dot{B}^{0,1}_1$ if and only if $T^*1=0$, $T1\in\dot{B}^{0\infty}_\infty$, Π_{T1} is bounded on $\dot{B}^{0,1}_1$, and T satisfies the weak boundedness property. In 1995, Youssfi [27] showed that for $\beta\in\mathbb{R}$, $1<p<\infty$, $1\le q\le 2$, and $g\in\dot{B}^{\beta,\infty}_\infty$, Π_g is bounded from $\dot{F}^{0,q}_p$ into $\dot{B}^{\beta,p}_p$ if and only if $g\in\dot{F}^{\beta,q}_\infty$. In Theorem 5.7, we give a sufficient condition for the boundedness of paraproduct operators acting on homogeneous weighted Besov spaces.

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