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OPTIMALITY CONDITIONS FOR EFFICIENT SOLUTION TO THE VECTOR EQUILIBRIUM PROBLEMS WITH CONSTRAINTS

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Abstract. In this paper, by using the generalization of Ljusternik theorem, the open mapping theorem of convex process, and the convex sets separation theorem, we give the necessary conditions for the efficient solution to the constrained vector equilibrium problems without requiring that the ordering cone in the objective space has a nonempty interior and without requiring that the the convexity conditions. By the assumption of the convexity, we give the sufficient conditions for the efficient solution. As applications, we give the necessary conditions and the sufficient conditions for efficient solution to the constrained vector variational inequalities and constrained vector optimization problems.

1. INTRODUCTION

Vector equilibrium problems is an important part of non-linear analysis. Vector variational inequalities, vector optimization, vector Nash equilibrium, and vector complementarity problem are all special cases of the vector equilibrium problem (see [1]). An important subject of the vector equilibrium problems is to study its optimality condition. Giannessi, Mastroeni, Pellegrini [2] turned the vector variational inequalities with the constraints into another vector variational inequalities without the constraints. They gave sufficient conditions for efficient solution and weakly efficient solution to the vector variational inequalities in finite dimensional spaces. By using the concept of subdifferential of the function, Morgan and Romaniello [3] investigated the scalarization and Kuhn-Tucker-like conditions for weak vector generalized quasivariational inequalities in Hilbert space. Yang and Zheng [4] provided the optimality condition for the approximate solutions of vector variational inequalities in Banach space. Gong [5] investigated the optimality conditions for weakly efficient solution, Henig solution,

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superefficient solution, and globally efficient solution to the vector equilibrium problems with convexity conditions, and showed that the weakly efficient solution, Henig efficient solution, globally efficient solution, and superefficient solution to the vector equilibrium problems with constraints are equivalent to solution of corresponding scalar optimization problems without constraints, respectively. Qiu [6] presented the necessary and sufficient conditions for globally efficient solution of the vector equilibrium problems with constraints under generalized cone-subconvexlikeness and proved that the Kuhn- Tucker condition for the vector equilibrium problems with constraints is both necessary and sufficient under the condition of cone-preinvexity. Using nonsmooth analysis and the scalarization results, Gong [7] provided the necessary conditions for weakly efficient solutions, Henig efficient solutions, globally efficient solutions, and superefficient solutions to the vector equilibrium problems.

Since the above investigation relies on the assumption that the ordering cones have a nonempty interior, they therefore depend on the advantage of the openness of interior of the ordering cone. In many cases, however, the ordering cone has an empty interior. For example, for each $1 < p < +\infty,$ the positive cone of the normed linear spaces l^p and $L^p(\Omega)$ has an empty interior.

The efficient solution is an important solution to the vector equilibrium problems, the concept of efficient solution does not require the condition that the ordering cone has an nonempty interior. Gong and Yao [8, 9] studied the connectedness of the set of efficient solutions and the lower semicontinuity of the efficient solution mapping for vector equilibrium problems. So far, there has been no study on the optimality conditions for efficient solutions to the vector equilibrium problems with the condition that the ordering cone in the objective space has an empty interior. The difficulty lies in the fact that we can not use the separation theorem of convex sets directly.

In this paper, by using the concept of Fréchet differentiability of mapping, we study the optimality conditions for efficient solution of the vector equilibrium problems with constraints. We use the generalization of Ljusternik theorem, the open mapping theorem of convex process, and the convex sets separation theorem to give the necessary conditions for the efficient solution of the constrained vector equilibrium problems requiring neither that the ordering cone in the objective space has a nonempty interior and nor that the the convexity conditions. By the assumption of the convex, we also give the sufficient conditions for the efficient solution. As applications, we give the necessary conditions and the sufficient conditions for efficient solution to the constrained vector variational inequalities and constrained vector optimization problems.

2. Preliminaries and Definitions

Throughout this paper, let $X, Y,$ and Z be real Banach spaces. Let $C \subset Y$ and $K \subset Z$ be closed convex pointed cones with $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the interior of the set K.

Let Y^* and Z^* be the topological dual spaces of Y and Z, respectively. Let

$$C^* = \{y^* \in Y^* : y^*(y) \geq 0 \text{ for all } y \in C\}$$

and

$$K^* = \{z^* \in Z^* : z^*(z) \geq 0 \text{ for all } z \in K\}$$

be the dual cones of C and K , respectively. Denote the quasi-interior of C^* by $C^{\#}$, i.e.

$$C^{\#} = \{y^* \in Y^* : y^*(y) > 0 \quad \text{for all } y \in C \backslash \{0\}\}.$$

Let $S \subset X$ be a nonempty open convex subset, and $F : S \times S \to Y$, $g : S \to Z$ be mappings.

We define the constraint set

$$A = \{x \in S : g(x) \in -K\},$$

and consider the vector equilibrium problems with constraints (for short, VEPC): find $x \in A$ such that

$$F(x, y) \notin -P \backslash \{0\} \quad \text{for all } y \in A,$$

where P is a convex cone in Y.

Definition 2.1. A vector $x \in A$ satisfying

$$F(x, y) \notin -C \backslash \{0\} \quad \text{for all } y \in A,$$

is called an efficient solution to the VEPC.

Let $L(X, Y)$ be the space of all bounded linear mapping from X to Y.

VEPC includes as a special case a vector variational inequality with constraints (for short, VVIC) involving

$$F(x, y) = (Tx)(y - x), \quad x, y \in S,$$

where T is a mapping from S to $L(X, Y)$.

Definition 2.2. If $F(x, y) = (Tx)(y - x), x, y \in S$, and if $x \in A$ is an efficient solution to the VEPC, then $x \in A$ is called an efficient solution to the VVIC.

Another special case of VEPC is a vector optimization problem with constraints (for short, VOPC) involving

$$F(x, y) = f(y) - f(x), \quad x, y \in S,$$

where $f : S \to Y$ be a mapping.

Definition 2.3. If $F(x, y) = f(y) - f(x),\ x, y \in S$, and if $x \in A$ is an efficient solution to the VEPC, then $x \in A$ is called an efficient solution to the VOPC.

Definition 2.4. Let X be a real linear space, and Y be a real topological linear space. Let S_2 be a nonempty subset of X, and let a mapping $f : S_2 \to Y$ and some $\bar{x} \in S_2$ be given. If for some $h \in X$ the limit

$$f'(\bar{x})(h) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left(f\left(\bar{x} + \lambda h\right) - f(\bar{x})\right)$$

exists, then $f'(\bar{x})(h)$ is called the Gâteaux derivative of f at $\bar{x}$ in the direction h. If this limit exists for each direction h, the mapping f is called Gâteaux differentiable at $\bar{x}$.

Definition 2.5. Let X and Y be real normed spaces, and let D be a nonempty open subset of X . Moreover, let a mapping $f : D \to Y$ and some $\bar{x} \in D$ be given. If there exists a continuous linear mapping $f'(\bar{x}) : X \to Y$ with the property

$$\lim_{\|h\| \to 0} \frac{\|f(\bar{x} + h) - f(\bar{x}) - f'(\bar{x})(h)\|}{\|h\|} = 0,$$

then $f'(\bar{x})$ is called the Fréchet derivative of f at $\bar{x}$ and f is called Fréchet differentiable at $\bar{x}$.

Remark 2.1. By the Lemma 2.18 of ([10]), we can see that, if f is Fréchet differentiable at $\bar{x}$, then f is Gâteaux differentiable at $\bar{x}$ and the Fréchet derivative of f at $\bar{x}$ is equal to the Gâteaux derivative of f at $\bar{x}$ in each direction h.

Definition 2.6. Let X and Y be real linear spaces, C be a pointed convex cone in Y, and let A be a nonempty convex subset of X. A mapping $f : A \to Y$ is called $C-$ convex, if for all $x, y \in A$ and all $\lambda \in [0, 1]$

$$\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y) \in C.$$

Definition 2.7. Let T be a set-valued mapping from X to Y with $T(x) \neq \emptyset$ for all $x \in X$ is called a convex process from X to Y if

(a) $Tx + Ty \subset T(x + y)$ for all $x, y \in X$,

(b) $T(\lambda x) = \lambda T(x)$ for all $x \in X, \lambda > 0$,

(c) $0 \in T(0)$.

A convex process T from X to Y is said to be closed if $\{(x, y) : y \in Tx\}$ is closed in $X \times Y$.

Chen [12] gave the following the generalization of Ljusternik theorem.

Theorem 2.1. (see [12], p.94) Let X and Y be real Banach spaces, let $f : X \to Y$ be a C^1 mapping. If $f'(\bar{x})(X) = Y$, then for any $x \in X$ with $\|x\|$ small enough, there exists $u \in X$ with $\|u\| = \circ(\|x\|)$ such that

$$f(\bar{x} + x + u) - f(\bar{x}) - f'(\bar{x})(x) = 0.$$

In a fashion similar to Theorem 2.1, we can get the following Lemma 2.1. To facilitate readers, we will give a proof of Lemma 2.1 in the appendix.

Lemma 2.1. Let X and Y be real Banach spaces, let S be a nonempty open convex subset of X, and let $f : S \to Y$ be continuous Fr$\acute{e}$chet differentiable on a neighborhood of $\bar{x} \in S$. If $f'(\bar{x})(X) = Y$, then for any $x \in X$ with $\|x\|$ small enough, there exists $u \in X$ with $\|u\| = \circ(\|x\|)$ such that

$$f(\bar{x} + x + u) - f(\bar{x}) - f'(\bar{x})(x) = 0.$$

Lemma 2.2. (see [11], Theorem 2.2.1) Let X and Y be Banach spaces. Assume that a closed convex process $T : X \to 2^Y$ is surjective (in the sense that Im$(T) = Y$). Then T^{-1} is Lipschitz: there exists a constant $l > 0$ such that, for all $x_1 \in T^{-1}(y_1)$ and for any $y_2 \in Y$, we can find a solution $x_2 \in T^{-1}(y_2)$ satisfying:

$$\|x_1 - x_2\| \leq l \parallel y_1 - y_2 \parallel .$$

Remark 2.2. If T satisfying the condition of Lemma 2.2, then T is an open mapping. In fact, for any open subset D of X, and for any $\bar{y} \in T(D)$, there exists $\bar{x} \in D$ such that $\bar{y} \in T(\bar{x})$, that is $\bar{x} \in T^{-1}(\bar{y})$. As D is an open set, there a positive real number r such that $U(\bar{x}, r) = \{x \in X : \|x - \bar{x}\| < r\} \subset D$. For any $y \in U(\bar{y}, r/l) = \{y \in Y : \|y - \bar{y}\| < r/l\}$, by Lemma 2.2, there exists $x \in T^{-1}(y)$ satisfying:

$$\parallel x - \bar{x} \parallel \leq l \parallel y - \bar{y} \parallel < l(r/l) = r.$$

Thus, $x \in U(\bar{x}, r)$, and hence $y \in T(U(\bar{x}, r)) \subset T(D)$. This means that $U(\bar{y}, r/l) \subset T(D)$. Thus, $\bar{y}$ is a interior point of $T(D)$. By the arbitrarily of $\bar{y} \in T(D)$, $T(D)$ is an open set.

3. Optimality Condition

In this section, we give the necessary conditions and the sufficient conditions for the efficient solution to the vector equilibrium problems with constraints.

By the proof of the theorem 2.20 in [10] or from the definition, we have the following lemma.

Lemma 3.1. Let X, Y be real normed spaces, let S be a nonempty open convex subset of X, and let C be closed convex pointed cones in Y. Assume that $f : S \to Y$ is $C-$ convex and f is G$\hat{a}$teaux differentiable at $\bar{x} \in S$. Then

$$f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x}) \in C \text{ for all } x \in S.$$

Let $\bar{x} \in S$ be given. Denote the mapping $F_{\bar{x}} : S \to Y$ by

$$F_{\bar{x}}(y) = F(\bar{x}, y),\ y \in S.$$

Theorem 3.1. Let X, Y, and Z be real Banach spaces, and let C and K be closed convex pointed cones in Y and Z, respectively. Let $\text{int}K \neq \emptyset$. Let S be a nonempty open convex subset of $X, \bar{x} \in A$, and let $F(\bar{x}, \bar{x}) = 0$. Let $F_{\bar{x}}(\cdot) : S \to Y$ be continuous Fréchet differentiable on a neighborhood of $\bar{x}$, and $g(\cdot) : S \to Z$ be Fréchet differentiable at $\bar{x}$. Suppose that $F'_{\bar{x}}(\bar{x})(X) = Y$. If $\bar{x}$ is an efficient solution to the VEPC, then there exist $y^* \in C^*, z^* \in K^* \backslash \{0\}$, such that

$$(y^* \circ F'_{\bar{x}}(\bar{x}) + z^* \circ g'(\bar{x}))\,(x - \bar{x}) \geq 0 \quad \text{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0.$$

If in addition, $g'(\bar{x})(X) = Z$, then $y^* \neq 0$.

Proof. Assume that $\bar{x} \in A$ is an efficient solution to the VEPC. Pick $e \in \text{int}K$, then $V = (e - \text{int}K) \cap (-e + \text{int}K)$ is a neighborhood of zero in Z. Since $g'(\bar{x})(0) = 0$ and $g'(\bar{x})(\cdot)$ is continuous on X and $\bar{x} \in A \subset S = \text{int}S$, there exists a symmetric open neighborhood U of zero in X with $\bar{x} + U \subset S$ such that

$$g'(\bar{x})(x) \in V \quad \text{for all } x \in U. \tag{1}$$

Define the set

$$M = \{(y, z) \in Y \times Z : \text{ there exists } x \in S \text{ such that } y - F'_{\bar{x}}(\bar{x})(x - \bar{x}) \in C,$$
$$z - (g(\bar{x}) + g'(\bar{x})(x - \bar{x})) \in \text{int}K\}.$$

Since S is a convex set, and $F'_{\bar{x}}(\bar{x})$ and $g'(\bar{x})$ are linear operators, we can see that M is a nonempty convex set. We first show that $\text{int}M \neq \emptyset$. For any $(y, z) \in (F'_{\bar{x}}(\bar{x})(U) + C, e + \text{int}K)$, we have $y \in F'_{\bar{x}}(\bar{x})(U) + C$, and $z \in e + \text{int}K$. By $U \subset S - \bar{x}$, there exists $x \in S$ with $x - \bar{x} \in U$, and $c \in C$ such that

$$y = F'_{\bar{x}}(\bar{x})(x - \bar{x}) + c.$$

As U is symmetric, we have $-(x - \bar{x}) \in U$. By (1), we have

$$g'(\bar{x})(-(x - \bar{x})) \in V \subset -e + \text{int}K.$$

Therefore, we have

$$\begin{aligned} &z - (g(\bar{x}) + g'(\bar{x})(x - \bar{x})) \in e + \text{int}K - g(\bar{x}) + g'(\bar{x})(-(x - \bar{x})) \\ &= e + \text{int}K + K - e + \text{int}K \subset \text{int}K, \end{aligned}$$

and

$$y - F'_{\bar{x}}(\bar{x})(x - \bar{x}) = F'_{\bar{x}}(\bar{x})(x - \bar{x}) + c - F'_{\bar{x}}(\bar{x})(x - \bar{x}) = c \in C.$$

Noting that $x \in S$, we have $(y, z) \in M$. By the arbitrary of $(y, z) \in (F'_{\bar{x}}(\bar{x})(U) + C, e + \text{int}K)$, we have

$$(F'_{\bar{x}}(\bar{x})(U) + C, e + \text{int}K) \subset M.$$

Define $T : X \to 2^Y$ by

$$Tx = F'_{\bar{x}}(\bar{x})(x) + C \quad \text{for all } x \in X.$$

It is easy to see that T is a closed convex process. By assumption, $T(X) = Y$. In view of Lemma 2.2 and Remark 2.2, we can see that T is an open mapping. Since U is an open set, $T(U) = F'_{\bar{x}}(\bar{x})(U) + C$ is an open set, and

(2) $$(F'_{\bar{x}}(\bar{x})(U) + C, e + \text{int}K) \subset \text{int}M.$$

Next, we show that $(0, 0) \notin \text{int}M$. If not, $(0, 0) \in \text{int}M$, then there exist y and z such that $-y \in C\backslash\{0\}, -z \in \text{int}K$ and $(y, z) \in M$. Hence, there exists $x \in S$ such that

(3) $$y - F'_{\bar{x}}(\bar{x})(x - \bar{x}) \in C, \qquad z - (g(\bar{x}) + g'(\bar{x})(x - \bar{x})) \in \text{int}K.$$

Noting that $-y \in C\backslash\{0\}$, C is a pointed cone, and (3), we have

(4) $$F'_{\bar{x}}(\bar{x})(x - \bar{x}) \neq 0.$$

Since $F_{\bar{x}}(\cdot)$ is continuous Fréchet differentiable on a neighborhood of $\bar{x}$, and $F'_{\bar{x}}(\bar{x})(X) = Y$, by Lemma 2.1, for each positive integer n large enough, and for $(x - \bar{x})/t_n)$, where $t_n \to +\infty$, there exists $u_n \in X$ with $\|u_n\| = \circ(\|x - \bar{x}\|/t_n)$ such that

$$F_{\bar{x}}(\bar{x} + ((x - \bar{x})/t_n) + u_n) - F_{\bar{x}}(\bar{x}) - F'_{\bar{x}}(\bar{x})((x - \bar{x})/t_n) = 0.$$

Noting that $F_{\bar{x}}(\bar{x}) = 0$, we have

(5) $$F_{\bar{x}}(\bar{x} + ((x - \bar{x})/t_n) + u_n) - F'_{\bar{x}}(\bar{x})((x - \bar{x})/t_n) = 0.$$

Set $x_n = \bar{x} + ((x - \bar{x})/t_n) + u_n$. We have $x_n \to \bar{x}$. By (4), $x - \bar{x} \neq 0$. Since $\|u_n\| = \circ(\|x - \bar{x}\|/t_n)$, we have

$$t_n \parallel u_n \parallel / \parallel x - \bar{x}\| \to 0,$$

and hence

$$t_n\|u_n\| \to 0.$$

Thus,

$$t_n(x_n - \bar{x}) = t_n(\bar{x} + ((x - \bar{x})/t_n) + u_n - \bar{x}) = x - \bar{x} + t_n u_n \to x - \bar{x}.$$

By (4) and (5), we have

$$t_n F(\bar{x}, x_n) = t_n F_{\bar{x}}(x_n) = F'_{\bar{x}}(\bar{x})(x - \bar{x}) \neq 0. \tag{6}$$

This together with (3) yields

$$y - t_n F_{\bar{x}}(x_n) \in C.$$

Hence, there exist $c_n \in C$ such that

$$F_{\bar{x}}(x_n) = (y - c_n)/t_n, \quad \text{for } n \text{ large enough}.$$

Since $-y \in C \backslash \{0\}$, we obtain

$$F_{\bar{x}}(x_n) \in -C \backslash \{0\}.$$

That is

$$F(\bar{x}, x_n) \in -C \backslash \{0\} \quad \text{for } n \text{ large enough}. \tag{7}$$

By (3),

$$-z + (g(\bar{x}) + g'(\bar{x})(x - \bar{x})) \in -\text{int}K.$$

Noting that $-z \in \text{int}K$, we get

$$g(\bar{x}) + g'(\bar{x})(x - \bar{x}) \in z - \text{int}k \subset -\text{int}K. \tag{8}$$

Since g is Fr$\acute{e}$chet differentiable at $\bar{x}$,

$$g(x_n) - g(\bar{x}) - g'(\bar{x})(x_n - \bar{x}) = \omega(\bar{x}; x_n - \bar{x}), \tag{9}$$

where $\|\omega(\bar{x}; x_n - \bar{x})\| = \circ(\|x_n - \bar{x}\|)$. By (6) and $F(\bar{x}, \bar{x}) = 0$, we have $x_n \neq \bar{x}$. From (9) and $t_n(x_n - \bar{x}) \to x - \bar{x}$, we get

$$g'(\bar{x})(t_n(x_n - \bar{x})) + t_n\|x_n - \bar{x}\|(\omega(\bar{x}; x_n - \bar{x})/\|x_n - \bar{x}\|) = t_n(g(x_n) - g(\bar{x})). \tag{10}$$

As $t_n(x_n - \bar{x}) \to x - \bar{x}, \{t_n\|x_n - \bar{x}\|\}$ is a bounded sequence, by $\|\omega(\bar{x}; x_n - \bar{x})\| = \circ(\|x_n - \bar{x}\|)$, we get

$$\|\omega(\bar{x}; x_n - \bar{x})\|/\|x_n - \bar{x}\| \to 0.$$

Taking the limit on the both side of (10), we get

$$g'(\bar{x})(x - \bar{x})) = \lim_{n\to\infty} t_n(g(x_n) - g(\bar{x})),$$

hence,

$$g(\bar{x}) + g'(\bar{x})(x - \bar{x})) = g(\bar{x}) + \lim_{n \to \infty} t_n(g(x_n) - g(\bar{x})).$$

By (8), there exists N_1, when $n \geq N_1$, we have

$$g(\bar{x}) + t_n(g(x_n) - g(\bar{x})) \in -\text{int}K,$$

and $t_n > 1$ because $t_n \to +\infty$. Hence

$$t_n g(x_n) \in -(1 - t_n)g(\bar{x}) - \text{int}K \subset -\text{int}K.$$

Therefore,

$$g(x_n) \in -\text{int}K \subset -K. \tag{11}$$

Since $x_n \to \bar{x} \in A \subset S = \text{int}S$, there exists $n_0 > N_1$ large enough such that

$$x_{n_0} \in S. \tag{12}$$

By (11) and (12), we have $x_{n_0} \in A$. By (7),

$$F(\bar{x}, x_{n_0}) \in -C \backslash \{0\}.$$

This is a contradicts that $\bar{x}$ is an efficient solution to the VEPC. Hence, $(0, 0) \notin \text{int}M$. By the separation theorem of convex sets (see [10]), there exists $(0, 0) \neq (y^*, z^*) \in (Y \times Z)^* = Y^* \times Z^*$ such that

$$y^*(y) + z^*(z) > 0 \quad \text{for all } (y, z) \in \text{int}M. \tag{13}$$

Noting that M is convex set and $\text{int}M \neq \emptyset$, $\overline{M} = \overline{\text{int}M}$, we have

$$y^*(y) + z^*(z) \geq 0 \quad \text{for all } (y, z) \in M. \tag{14}$$

Let $(y, z) \in M$. Then there exists $x \in S$ such that

$$y - F'_{\bar{x}}(\bar{x})(x - \bar{x}) \in C, \quad z - (g(\bar{x}) + g'(\bar{x})(x - \bar{x})) \in \text{int}K.$$

Hence, for every $c \in C, k \in \text{int}K, t > 0, t' > 0$, we have $(y+tc, z) \in M, (y, z+t'k) \in M$. By (14), we have

$$y^*(y + tc) + z^*(z) \geq 0 \quad \text{for all } c \in C, t > 0.$$

It implies that $y^* \in C^*$. Similarly, we can show that $z^* \in K^*$. We claim that $z^* \neq 0$. In fact, if $z^* = 0$, from (13), we get

$$y^*(y) > 0 \quad \text{for all } (y, z) \in \text{int}M. \tag{15}$$

Taking U as in (2), by (2), we have

$$(F'_{\bar{x}}(\bar{x})(x) + c, e + k) \in \text{int}M, \text{ for all } x \in U, \ c \in C, k \in \text{int}K.$$

By (15),

$$y^*(F'_{\bar{x}}(\bar{x})(0)) > 0.$$

But, $F'_{\bar{x}}(\bar{x})(0) = 0$, and $y^*(0) = 0$, we get a contradiction. Thus, $z^* \neq 0$. It follows from

$$(F'_{\bar{x}}(\bar{x})(x - \bar{x}) + c, g(\bar{x}) + g'(\bar{x})(x - \bar{x}) + k) \in M, \quad \text{for all } x \in S, c \in C, k \in \text{int}K,$$

and (14) that

$$y^*(F'_{\bar{x}}(\bar{x})(x - \bar{x}) + c) + z^*(g(\bar{x}) + g'(\bar{x})(x - \bar{x}) + k) \geq 0 \text{ for all } x \in S, c \in C, k \in \text{int}K.$$

It implies that

$$\begin{aligned} &y^*(F'_{\bar{x}}(\bar{x})(x - \bar{x}) + c) + z^*(g(\bar{x}) + g'(\bar{x})(x - \bar{x}) + k) \geq 0, \\ &\quad \text{for all } x \in S, c \in C, k \in K. \end{aligned} \tag{16}$$

Taking $c = 0$, we obtain

$$y^*(F'_{\bar{x}}(\bar{x})(x - \bar{x})) + z^*(g(\bar{x}) + g'(\bar{x})(x - \bar{x}) + k) \geq 0, \text{ for all } x \in S, k \in K. \tag{17}$$

Since $\bar{x} \in S$, by (17), we get

$$z^*(g(\bar{x})) \geq 0.$$

By $g(\bar{x}) \in -K$, and $z^* \in K^*$, we have $z^*(g(\bar{x})) \leq 0$. Thus,

$$z^*(g(\bar{x})) = 0. \tag{18}$$

By (17) and (18), we get

$$(y^* \circ F'_{\bar{x}}(\bar{x}) + z^* \circ g'(\bar{x}))(x - \bar{x}) \geq 0 \qquad \text{for all } x \in S.$$

If in addition, $g'(\bar{x})(X) = Z$, then $y^* \neq 0$. In fact, if $y^* = 0$, then by (16),

$$z^*(g(\bar{x}) + g'(\bar{x})(x - \bar{x}) + k) \geq 0, \text{ for all } x \in S, k \in K.$$

Noting that $g(\bar{x}) \in -K$, we have

$$z^*(g'(\bar{x})(x - \bar{x})) \geq 0 \text{ for all } x \in S. \tag{19}$$

Since $\bar{x} \in S = \text{int}S$, there exists some neighborhood W of zero such that $W + \bar{x} \subset S$. By (19), we have

$$z^*(g'(\bar{x})(x')) \geq 0 \text{ for all } x' \in W. \tag{20}$$

This implies that

$$z^*(g'(\bar{x})(x)) \geq 0 \text{ for all } x \in X,$$

therefore,

$$z^*(g'(\bar{x})(x)) = 0 \text{ for all } x \in X.$$

This together with $g'(\bar{x})(X) = Z$ implies that $z^*(z) = 0$ for all $z \in Z$. This means that $z^* = 0$. This contradicts that $z^* \neq 0$. ∎

Theorem 3.2. Let X, Y, and Z be real Banach spaces, and let $C \subset Y$ be a closed convex pointed cone, and $K \subset Z$ be a closed convex pointed cone with $\text{int}K \neq \emptyset$. Let S be a nonempty open convex subset of X. Let $\bar{x} \in A$, and $F(\bar{x}, \bar{x}) = 0$. Assume that $F_{\bar{x}}(\cdot)$ and $g(\cdot)$ are Gâteaux differentiable at $\bar{x}$, and $F_{\bar{x}}(\cdot)$ is $C-$ convex on S, and $g(\cdot)$ is $K-$ convex on S. If there exist $y^* \in C^\sharp$ and $z^* \in K^*$ such that

(21) $$\left(y^* \circ F'_{\bar{x}}(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \text{for all } x \in S,$$

and

(22) $$z^*(g(\bar{x})) = 0,$$

then $\bar{x}$ is an efficient solution to the VEPC.

Proof. Since the mappings $F_{\bar{x}}(\cdot)$ and $g(\cdot)$ are Gâteaux differentiable at $\bar{x} \in A$, and $F_{\bar{x}}(\cdot)$ is $C-$ convex on S, and $g(\cdot)$ is $K-$ convex on S. From Lemma 3.1, we have

(23) $$F'_{\bar{x}}(\bar{x})(x - \bar{x}) \in F_{\bar{x}}(x) - F_{\bar{x}}(\bar{x}) - C = F_{\bar{x}}(x) - C \quad \text{for all } x \in S,$$

(24) $$g'(\bar{x})(x - \bar{x}) \in g(x) - g(\bar{x}) - K \quad \text{for all } x \in S.$$

From $y^* \in C^\sharp, z^* \in K^*$ and (21), (23), and (24), we get

$$y^*\left(F_{\bar{x}}(x)\right) + z^*\left(g(x) - g(\bar{x})\right) \geq \left(y^* \circ F'_{\bar{x}}(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \text{for all } x \in S.$$

This together with (22), we get

(25) $$y^*\left(F_{\bar{x}}(x)\right) + z^*\left(g(x)\right) \geq 0 \quad \text{for all } x \in S.$$

We will show that $\bar{x} \in A$ is an efficient solution to the VEPC. If not, then there exists $y_0 \in A$ such that

$$F(\bar{x}, y_0) \in -C \setminus \{0\}.$$

From $y^* \in C^\sharp$ we have

$$y^*\left(F(\bar{x}, y_0)\right) < 0.$$

Notice $y_0 \in A$, we have $g(y_0) \in -K$, therefore, $z^*(g(y_0)) \leq 0$ because of $z^* \in K^*$. Hence,

$$y^*\left(F_{\bar{x}}(y_0)\right) + z^*\left(g(y_0)\right) < 0.$$

This contradicts (25). Hence, $\bar{x}$ is an efficient solution to the VEPC. ∎

4. Application

In this section, we use the results of Section 3 to get the optimality conditions for efficient solution to the vector variational inequalities with constraints (for short, VVIC) and vector optimization problems with constraints (for short, VOPC), respectively.

Theorem 4.1. Let X, Y, and Z be real Banach spaces, and let $C \subset Y$ and $K \subset Z$ be closed convex pointed cones, and let int$K \neq \emptyset$. Let S be a nonempty open convex subset of X, and let that $T : S \to L(X, Y)$ be a mapping, $g(\cdot) : S \to Z$ be Fréchet differentiable at $\bar{x} \in A$. Suppose that $T(\bar{x})(X) = Y$. If $\bar{x}$ is an efficient solution to the VVIC, then there exist $y^* \in C^*, z^* \in K^* \backslash \{0\}$, such that

$$(y^* \circ T(\bar{x}) + z^* \circ g'(\bar{x}))\,(x - \bar{x}) \geq 0 \quad \text{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0.$$

If in addition, $g'(\bar{x})(X) = Z$, then $y^* \neq 0$.

Proof. Let $\bar{x} \in A$ be an efficient solution to the VVIC. Let

$$F(x, y) = (Tx)(y - x),\ x, y \in S.$$

Then, $\bar{x}$ is an efficient solution to the VEPC. It is clear that $F(\bar{x}, \bar{x}) = 0$.

For any $u \in S$,

$$\begin{aligned}
&\lim_{\|h\|\to 0} \frac{\|F_{\bar{x}}(u+h) - F_{\bar{x}}(u) - T(\bar{x})(h)\|}{\|h\|} \\
&= \lim_{\|h\|\to 0} \frac{\|F(\bar{x}, u+h) - F(\bar{x}, u) - T(\bar{x})(h)\|}{\|h\|} \\
&= \lim_{\|h\|\to 0} \frac{\|(T\bar{x})(u+h-\bar{x}) - (T\bar{x})(u-\bar{x}) - T(\bar{x})(h)\|}{\|h\|} \\
&= \lim_{\|h\|\to 0} \frac{\|(T\bar{x})(u-\bar{x}) + (T\bar{x})(h) - (T\bar{x})(u-\bar{x}) - T(\bar{x})(h)\|}{\|h\|} = 0.
\end{aligned}$$

Since $T\bar{x} \in L(X, Y)$ and Fréchet derivative is uniquely determined,

$$F'_{\bar{x}}(u) = T\bar{x} \ \text{ for all } u \in S.$$

So $F'_{\bar{x}}(\cdot)$ is continuous on a neighborhood of $\bar{x} \in S$ (let $L(X, Y)$ be equipped with norm topology), that is $F_{\bar{x}}(\cdot) : S \to Y$ be continuous Fréchet differentiable on a neighborhood of $\bar{x} \in S$. Noting that $F'_{\bar{x}}(\bar{x}) = T(\bar{x})$, by assumption, we have that $F'_{\bar{x}}(\bar{x})(X) = Y$ and $g(\cdot)$ is Fréchet differentiable at $\bar{x} \in S$. In view of Theorem 3.1, there exist $y^* \in C^*, z^* \in K^* \backslash \{0\}$, such that

$$(y^* \circ T(\bar{x}) + z^* \circ g'(\bar{x}))\,(x - \bar{x}) \geq 0 \quad \text{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0.$$

If in addition, $g'(\bar{x})(X) = Z$, then $y^* \neq 0$. ■

Theorem 4.2. *Let X, Y, and Z be real Banach spaces, and let $C \subset Y$ be a closed convex pointed cone, and $K \subset Z$ be a closed convex pointed cone with int$K \neq \emptyset$. Let S be a nonempty open convex subset of X. Let $\bar{x} \in A$. Assume that that $T : S \to L(X, Y)$ is a mapping, $g(\cdot)$ is Gâteaux differentiable at $\bar{x}$, and $g(\cdot)$ is $K-$ convex on S. If there exist $y^* \in C^\sharp$ and $z^* \in K^*$ such that*

$$\left(y^* \circ T(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \textit{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0,$$

then $\bar{x}$ is an efficient solution to the VVIC.

Proof. Let

$$F(x, y) = (Tx)(y - x),\ x, y \in S.$$

It is clear that $F(\bar{x}, \bar{x}) = 0$. $F_{\bar{x}}(\cdot) = (T\bar{x})(\cdot - \bar{x})$ and $g(\cdot)$ are Gâteaux differentiable at $\bar{x} \in A$. For any $h \in X$,

$$\begin{aligned} F'_{\bar{x}}(\bar{x})(h) &= \lim_{\lambda \to 0} \frac{F_{\bar{x}}(\bar{x} + \lambda h) - F_{\bar{x}}(\bar{x})}{\lambda} \\ &= \lim_{\lambda \to 0} \frac{(T\bar{x})(\bar{x} + \lambda h - \bar{x})}{\lambda} \\ &= \lim_{\lambda \to 0} \frac{\lambda (T\bar{x})(h)}{\lambda} = (T\bar{x})(h). \end{aligned}$$

It is clear that $F_{\bar{x}}(\cdot)$ is $C-$ convex on S, and $g(\cdot)$ is $K-$ convex on S. If there exist $y^* \in C^\sharp$ and $z^* \in K^*$ such that

$$\left(y^* \circ T(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \text{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0,$$

then

$$\left(y^* \circ F'_{\bar{x}}(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \text{for all } x \in S.$$

By Theorem 3.2, $\bar{x}$ is an efficient solution to the VEPC, hence $\bar{x}$ is an efficient solution to the VVIC. ■

Theorem 4.3. *Let X, Y, and Z be real Banach spaces, let $C \subset Y$ and $K \subset Z$ be closed convex pointed cones, and let int$K \neq \emptyset$. Let S be a nonempty open convex subset of X. Let $f : S \to Y$ be continuous Fréchet differentiable on a neighborhood of $\bar{x} \in A$, and $g(\cdot) : S \to Z$ be Fréchet differentiable at $\bar{x}$. Suppose that $f'(\bar{x})(X) = Y$. If $\bar{x}$ is an efficient solution to the VOPC, then there exist $y^* \in C^*, z^* \in K^* \backslash \{0\}$, such that*

$$(y^* \circ f'(\bar{x}) + z^* \circ g'(\bar{x}))(x - \bar{x}) \geq 0 \quad \textit{for all } x \in S$$

and

$$z^*(g(\bar{x})) = 0.$$

If in addition, $g'(\bar{x})(X) = Z$, then $y^* \neq 0$.

Proof. Let $\bar{x} \in A$ be an efficient solution to the VOPC. Let

$$F(x, y) = f(y) - f(x), \; x, y \in S.$$

Then, $\bar{x}$ is an efficient solution to the VEPC. It is clear that $F(\bar{x}, \bar{x}) = 0$. Since f is continuous Fréchet differentiable on a neighborhood U of $\bar{x} \in S$, for any $u \in U$,

$$\begin{aligned}
&\lim_{\|h\| \to 0} \frac{\|F_{\bar{x}}(u+h) - F_{\bar{x}}(u) - f'(u)(h)\|}{\|h\|} \\
&= \lim_{\|h\| \to 0} \frac{\|F(\bar{x}, u+h) - F(\bar{x}, u) - f'(u)(h)\|}{\|h\|} \\
&= \lim_{\|h\| \to 0} \frac{\|f(u+h) - f(\bar{x}) - (f(u) - f(\bar{x})) - f'(u)(h)\|}{\|h\|} \\
&= \lim_{\|h\| \to 0} \frac{\|f(u+h) - f(u) - f'(u)(h)\|}{\|h\|} = 0.
\end{aligned}$$

Thus $F'_{\bar{x}}(u) = f'(u)$ for all $u \in U$. In particular, $F'_{\bar{x}}(\bar{x}) = f'(\bar{x})$. $F_{\bar{x}}(\cdot)$ is continuous Fréchet differentiable on a neighborhood of $\bar{x} \in S$. By assumption and using Theorem 3.1, we can see that there exist $y^* \in C^*, z^* \in K^* \backslash \{0\}$, such that

$$(y^* \circ F'_{\bar{x}}(\bar{x}) + z^* \circ g'(\bar{x}))(x - \bar{x}) \geq 0 \quad \text{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0.$$

That is

$$(y^* \circ f'(\bar{x}) + z^* \circ g'(\bar{x}))(x - \bar{x}) \geq 0 \quad \text{for all } x \in S$$

and

$$z^*(g(\bar{x})) = 0.$$

If in addition, $g'(\bar{x})(X) = Z$, then $y^* \neq 0$. ■

Theorem 4.4. *Let X, Y, and Z be real Banach spaces, and let $C \subset Y$ be a closed convex pointed cone, and $K \subset Z$ be a closed convex pointed cone with int$K \neq \emptyset$. Let S be a nonempty open convex subset of X. Let $\bar{x} \in A$, Assume that $f(\cdot)$ and $g(\cdot)$ are Gâteaux differentiable at $\bar{x}$, and $f(\cdot)$ is $C-$ convex on S, and $g(\cdot)$ is $K-$ convex on S. If there exist $y^* \in C^\sharp$ and $z^* \in K^*$ such that*

$$\left(y^* \circ f'(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \textit{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0,$$

then $\bar{x}$ is an efficient solution to the VOPC.

Proof. Let

$$F(x, y) = f(y) - f(x),\ x, y \in S.$$

It is clear that $F(\bar{x}, \bar{x}) = 0$. It is easy to see that the Gâteaux derivative of $F_{\bar{x}}(\cdot)$ at $\bar{x} \in A$ in the direction h is $f'(\bar{x})(h)$. We can see that $F_{\bar{x}}(\cdot) = f(\cdot) - f(\bar{x})$ and $g(\cdot)$ are Gâteaux differentiable at $\bar{x}$, and $F_{\bar{x}}(\cdot)$ is $C-$ convex on S, and $g(\cdot)$ is $K-$ convex on S. If there exist $y^* \in C^\sharp$ and $z^* \in K^*$ such that

$$\left(y^* \circ f'(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \text{for all } x \in S,$$

and

$$z^*(g(\bar{x})) = 0,$$

then

$$\left(y^* \circ F'_{\bar{x}}(\bar{x}) + z^* \circ g'(\bar{x})\right)(x - \bar{x}) \geq 0 \quad \text{for all } x \in S.$$

Then by Theorem 3.2, then $\bar{x}$ is an efficient solution to the VEPC, hence $\bar{x}$ is an efficient solution to the VOPC. ■

5. Appendix

Lemma A. *Let X and Y be Banach spaces, S be a nonempty open convex subset of X, and let $f : S \to Y$ be a mapping, $\bar{x} \in S, h \in X$ with $L = \{\bar{x}+th : 0 \leq t \leq 1\} \subset S$. Suppose that f is continuous Fréchet differentiable on L, then*

$$f(\bar{x} + h) - f(\bar{x}) = \int_0^1 f'(\bar{x} + th)(h)dt.$$

Proof. Denote the topological dual space of Y by Y^*. For $\bar{x} \in S, h \in X$ with $L = \{\bar{x} + th : 0 \leq t \leq 1\} \subset S$, and $y^* \in Y^*$, we define the following real-valued function

$$g(t) = y^* \circ f(\bar{x} + th), t \in [0, 1].$$

It is easy to know that for each $t \in [0,1]$, we have

$$\begin{aligned} g'(t) &= \lim_{\triangle t \to 0} \frac{g(t+\triangle t) - g(t)}{\triangle t} \\ &= \lim_{\triangle t \to 0} \frac{y^* f(\bar{x} + (t+\triangle t)h) - y^* f(\bar{x} + th)}{\triangle t} \\ &= y^* \circ f'(\bar{x} + th)(h). \end{aligned}$$

According to Newton-Leibniz formula, we have

$$\begin{aligned} y^*(f(\bar{x}+h) - f(\bar{x})) &= g(1) - g(0) = \int_0^1 g'(t)dt \\ &= \int_0^1 y^*(f'(\bar{x}+th)(h))dt = y^*(\int_0^1 f'(\bar{x}+th)(h)dt). \end{aligned}$$

By the arbitrariness of $y^* \in Y^*$, it follows that

$$f(\bar{x}+h) - f(\bar{x}) = \int_0^1 f'(\bar{x}+th)(h)dt.$$

The proof is complited. ■

The proof of Lemma 2.1. Set

$$X_0 = \{x \in X : f'(\bar{x})(x) = 0\}.$$

Since $f'(\bar{x})$ is continuous, X_0 is a closed linear subspace of X. Taking into account the quotient space X/X_0, it is easy to known that X/X_0 is a linear space. When $\dot{x} \in X/X_0$, that is $\dot{x} = x + X_0$, we define the norm $\|\dot{x}\| = \inf_{x \in \dot{x}} \|x\|$ in the quotient space X/X_0. It is clear that X/X_0 is a Banach space since is X_0 closed. If $x, x' \in \dot{x}$, then $f'(\bar{x})(x) = f'(\bar{x})(x')$. Now we define $A : X/X_0 \to Y$ by

$$A\dot{x} = f'(\bar{x})(x), \text{ for all } \dot{x} \in X/X_0 \text{ and } x \in \dot{x}.$$

It is clear that A is well defined and A is a linear operator from the linearity of $f'(\bar{x})$. For arbitrary $\dot{x} \in X/X_0$,

$$\|A\dot{x}\| = \|f'(\bar{x})(x)\| \leq \|f'(\bar{x})\|\|x\| \text{ for all } x \in \dot{x}.$$

Therefore,

$$\|A\dot{x}\| \leq \|f'(\bar{x})\| \cdot \inf_{x \in \dot{x}} \|x\| = \|f'(\bar{x})\| \cdot \|\dot{x}\| \text{ for all } \dot{x} \in X/X_0.$$

Hence, A is a bounded linear operator. The zero element of X/X_0 is $\dot{0} = X_0$. When $\dot{x} \neq \dot{0}$, it is obvious that $A\dot{x} \neq 0$. Hence A is injective. On the other hand, the assumption that $f'(\bar{x})(X) = Y$ implies that $A(X/X_0) = Y$. From the Banach's inverse operator theorem, A has a bounded inverse operator denoted by A^{-1} since X/X_0 and Y are both Banach spaces.

By assumption, f is continuous Fr$\acute{e}$chet differentiable on a neighborhood of $\bar{x} \in S$, then there exists a positive number r such that $U(\bar{x}, r) = \bar{x} + U(0, r) \subset S$, where $U(0, r) = \{x \in X : \|x - 0\| < r\}$, and $f : S \to Y$ is continuous Fr$\acute{e}$chet differentiable on $U(\bar{x}, r)$. Thus, there exists $0 < \sigma < r/2$ such that

(26) $$\sup_{\|z-\bar{x}\|<\sigma} \|f'(z) - f'(\bar{x})\| < 1/(4\|A^{-1}\|).$$

Take x with $\|x\| < \sigma/2$. Since f is Fr$\acute{e}$chet differentiable at $\bar{x}$,

(27) $$f(\bar{x} + x) - f(\bar{x}) - f'(\bar{x})(x) = \omega(\bar{x}; x),$$

where, $\lim\limits_{\|x\|\to 0} \frac{\|\omega(\bar{x};x)\|}{\|x\|} = 0$. We can pick $\|x\| < \sigma/2$ small enough such that

(28) $$\|\omega(\bar{x}; x)\| < \sigma/8\|A^{-1}\|.$$

For this x, we use iteration method to solve the following equation

(29) $$f(\bar{x} + x + z) - f(\bar{x}) - f'(\bar{x})(x) = 0.$$

Let $\dot{z_0} = \dot{0}$, and

(30) $$\dot{z_1} = \dot{z_0} - A^{-1}[f(\bar{x} + x + z_0) - f(\bar{x}) - f'(\bar{x})(x)],$$

where $z_0 = 0$. If $f(\bar{x} + x + z_0) - f(\bar{x}) - f'(\bar{x})(x) = 0$, then $z = z_0 = 0$ is a solution of (29). Obviously, $z_0 = \circ(\|x\|)$. Otherwise, $\dot{z_1} \neq \dot{0}$. Since $\|\dot{z_1}\| = \inf\limits_{z\in\dot{z_1}} \|z\|$ and $\|\dot{z_1}\| < 2\|\dot{z_1}\|$, by the definition of infimum, there exists some $z_1 \in \dot{z_1}$ such that $\inf\limits_{z\in\dot{z_1}} \|z\| \leq \|z_1\| < 2\|\dot{z_1}\|$. This together with (27), (28), and (30), we get

(31) $$\|z_1\| < 2\|A^{-1}\|\|f(\bar{x} + x) - f(\bar{x}) - f'(\bar{x})(x)\| = 2\|A^{-1}\|\|\omega(\bar{x}; x)\| < \sigma/4.$$

Thus, $\bar{x} + x + z_1 \in U(\bar{x}, \sigma)$. Let

(32) $$\dot{z_2} = \dot{z_1} - A^{-1}[f(\bar{x} + x + z_1) - f(\bar{x}) - f'(\bar{x})(x)].$$

If $f(\bar{x} + x + z_1) - f(\bar{x}) - f'(\bar{x})(x) = 0$, then z_1 is a solution of the equation (29). Since $\lim\limits_{\|x\|\to 0} \frac{\|\omega(\bar{x};x)\|}{\|x\|} = 0$, By (31), we have $\lim\limits_{\|x\|\to 0} \frac{\|z_1\|}{\|x\|} = 0$, thus, $\|z_1\| = \circ(\|x\|)$. Otherwise, $\dot{z_2} - \dot{z_1} \neq \dot{0}$. Set $\dot{y_1} = A^{-1}[f(\bar{x} + x + z_1) - f(\bar{x}) - f'(\bar{x})(x)]$. By (32), $\dot{z_2} = \dot{z_1} - \dot{y_1} = z_1 - y_1 + X_0$, where $y_1 \in \dot{y_1}$. Noting that

(33) $$\dot{z_2} - \dot{z_1} = -y_1 + X_0,$$

$\|\dot{z_2}-\dot{z_1}\| = \inf_{x'\in X_0} \|-y_1+x'\|$, and $\inf_{x'\in X_0} \|-y_1+x'\| = \|\dot{z_2}-\dot{z_1}\| < 2\|\dot{z_2}-\dot{z_1}\|$. By the definition of infimum, there exists some $x_0 \in X_0$ such that $\|-y_1+x_0\| < 2\|\dot{z_2}-\dot{z_1}\|$. We have $\|z_1 - y_1 + x_0 - z_1\| < 2\|\dot{z_2} - \dot{z_1}\|$. By (33), $z_1 - y_1 + x_0 \in \dot{z_2}$. Let $z_2 = z_1 - y_1 + x_0$, then $z_2 \in \dot{z_2}$. Thus,

$$\|z_2 - z_1\| < 2\|\dot{z_2} - \dot{z_1}\|. \tag{34}$$

By Lemma A, we have

$$f(\bar{x} + x + z_1) - f(\bar{x} + x) = \int_0^1 f'(\bar{x} + x + tz_1)(z_1)dt.$$

This together with (34), (30), (32), (26), and $\|x\| + \|z_1\| < \sigma$, we get

$$\begin{aligned}
\|z_2 - z_1\| < 2\|\dot{z_2} - \dot{z_1}\| &= 2\|\dot{z_1} - \dot{z_0} - A^{-1}[f(\bar{x} + x + z_1) - f(\bar{x} + x + z_0)]\| \\
&\le 2\|A^{-1}\| \cdot \|f(\bar{x} + x + z_1) - f(\bar{x} + x + z_0) - A(\dot{z_1})\| \\
&= 2\|A^{-1}\| \cdot \| \int_0^1 f'(\bar{x} + x + tz_1)(z_1)dt - f'(\bar{x})(z_1)\| \\
&= 2\|A^{-1}\| \cdot \| \int_0^1 f'(\bar{x} + x + tz_1)(z_1)dt - \int_0^1 f'(\bar{x})(z_1)dt\| \\
&= 2\|A^{-1}\| \cdot \int_0^1 \|f'(\bar{x} + x + tz_1) - f'(\bar{x})\|\|(z_1)\|dt \\
&= 2\|A^{-1}\| \cdot \int_0^1 \sup_{\|z-\bar{x}\|<\sigma} \|f'(z) - f'(\bar{x})\|\|(z_1)\|dt \\
&= 2\|A^{-1}\| \cdot \frac{1}{4\|A^{-1}\|}\|z_1\| = \|z_1\|/2.
\end{aligned}$$

By (31), we have $\|z_2\| < (1 + 1/2)\|z_1\| < (3/2) \cdot \sigma/4 < \sigma/2$. This together with $\|x\| < \sigma/2$, we get

$$\|x + tz_2\| \le \|x\| + \|z_2\| < \sigma/2 + \sigma/2 = \sigma \text{ for all } t \in [0, 1].$$

Thus, $\bar{x} + x + z_2 \in U(\bar{x}, \sigma)$. Since $\|z_1\| = \circ(\|x\|)$ and $\|z_2\| \le 3\|z_1\|/2$, we also have $\|z_2\| = \circ(\|x\|)$. Let

$$\dot{z_3} = \dot{z_2} - A^{-1}[f(\bar{x} + x + z_2) - f(\bar{x}) - f'(\bar{x})(x)].$$

So we can obtain the conclusion that or we can get the solution of (29) through finite iterations or else we can get a sequence $\{\dot{z_n}\}$ by induction with

$$\dot{z_n} = \dot{z_{n-1}} - A^{-1}[f(\bar{x} + x + z_{n-1}) - f(\bar{x}) - f'(\bar{x})(x)], \tag{35}$$

where $z_{n-1} \in \dot{z_{n-1}}$ with $\bar{x} + x + z_{n-1} \in U(\bar{x}, \sigma)$ and

$$\|z_n - z_{n-1}\| < 2\|\dot{z_n} - \dot{z_{n-1}}\| \text{ and } \|z_n\| < \sigma/2. \tag{36}$$

By (35), and $\|x + tz_{k-1} + (1-t)z_{k-2}\| < \sigma$, we have

$$\begin{aligned}
&\|z_k - z_{k-1}\| < 2\|\dot{z_k} - \dot{z_{k-1}}\| \\
=\ & 2\|\dot{z_{k-1}} - \dot{z_{k-2}} - A^{-1}[f(\bar{x} + x + z_{k-1}) - f(\bar{x} + x + z_{k-2})]\| \\
\leq\ & 2\|A^{-1}\| \cdot \|f(\bar{x} + x + z_{k-1}) - f(\bar{x} + x + z_{k-2}) - A(\dot{z_{k-1}} - \dot{z_{k-2}})\| \\
\leq\ & 2\|A^{-1}\| \cdot \|f(\bar{x} + x + z_{k-1}) - f(\bar{x} + x + z_{k-2}) - f'(\bar{x})(z_{k-1} - z_{k-2})\| \\
=\ & 2\|A^{-1}\| \cdot \| \int_0^1 f'(\bar{x} + x + z_{k-1} + t(z_{k-1} - z_{k-2}))(z_{k-1} - z_{k-2})dt \\
&- \int_0^1 f'(\bar{x})(z_{k-1} - z_{k-2})dt\| \\
=\ & 2\|A^{-1}\| \cdot \| \int_0^1 [f'(\bar{x} + x + z_{k-2} + t(z_{k-1} - z_{k-2})) - f'(\bar{x})](z_{k-1} - z_{k-2})dt\| \\
\leq\ & 2\|A^{-1}\| \cdot \int_0^1 \sup_{\|z - \bar{x}\| < \sigma} \|f'(z) - f'(\bar{x})\| \|(z_{k-1} - z_{k-2})\| dt \\
\leq\ & 2\|A^{-1}\| \cdot \frac{1}{4\|A^{-1}\|} \|z_{k-1} - z_{k-2}\| = \|z_{k-1} - z_{k-2}\|/2, \quad \text{for all } k \geq 3.
\end{aligned} \tag{37}$$

and

$$\|z_{k-1}\| < (1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-2}})\|z_1\|, \quad \text{for all } k \geq 3. \tag{38}$$

For any $n > 1$, noting that $z_0 = 0$, we have

$$\|z_n - z_{n-1}\| < \frac{1}{2}\|z_{n-1} - z_{n-2}\| < \cdots < \frac{1}{2^{n-1}}\|z_1\|.$$

Thus,

$$\lim_{n \to \infty} \|z_n - z_{n-1}\| = 0.$$

For any natural number p, we have

$$\begin{aligned}
\|z_{n+p} - z_n\| &\leq \|z_{n+p} - z_{n+p-1}\| + \|z_{n+p-1} - z_{n+p-2}\| + \cdots + \|z_{n+1} - z_n\| \\
&\leq \left(\frac{1}{2^{p-1}} + \frac{1}{2^{p-2}} + \cdots + 1\right)\|z_{n+1} - z_n\| < 2\|z_{n+1} - z_n\|.
\end{aligned}$$

We can see that $\{z_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some z such that $z_n \to z$ $(n \to \infty)$. By the definition of norm in the quotient space

X/X_0, it is clear that $\dot{z_n} \to \dot{z}$, where $z \in \dot{z}$. Taking limit on both side of (35), by the continuity of A^{-1} and f, we have

$$\dot{z} = \dot{z} - A^{-1}[f(\bar{x} + x + z) - f(\bar{x}) - f'(\bar{x})(x)].$$

So

$$A^{-1}[f(\bar{x} + x + z) - f(\bar{x}) - f'(\bar{x})(x)] = \dot{0}.$$

Since A is injective, so is A^{-1}. Hence

$$f(\bar{x} + x + z) - f(\bar{x}) - f'(\bar{x})(x) = 0.$$

Furthermore, the conclusion $\|z\| = \circ(\|x\|)$ follows immediately from the fact that $\|z_n\| \leq 2\|z_1\| = \circ(\|x\|)$.

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