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AN INTERVAL-TYPE ALGORITHM FOR CONTINUOUS-TIME LINEAR FRACTIONAL PROGRAMMING PROBLEMS

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Abstract. An interval-type computational procedure by combining the parametric method and discretization approach is proposed in this paper to solve a class of continuous-time linear fractional programming problems (CLFP). Using the different step sizes of discretization problems, we construct a sequence of convex, piecewise linear and strictly decreasing upper and lower bound functions. The zeros of upper and lower bound functions then determine a sequence of intervals shrinking to the optimal value of (CLFP) as the size of discretization getting larger. By using the intervals we can find corresponding approximate solutions to (CLFP). We also establish upper bounds of lengths of these intervals, and thereby we can determine the size of discretization in advance such that the accuracy of the corresponding approximate solution can be controlled within the predefined error tolerance. Moreover, we prove that the searched sequence of approximate solution functions weakly-star converges to an optimal solution of (CLFP). Finally, we provide some numerical examples to implement our proposed method.

1. INTRODUCTION

The theory of continuous-time linear programming problem has received considerable attention for a long time. Tyndall [36, 37] treated rigorously a continuous-time linear programming problem with the constant matrices, which was originated from the "bottleneck problem" proposed by Bellman [4]. Levison [14] generalized the results of Tyndall by considering the time-dependent matrices in which the functions shown in the objective and constraints were assumed as continuous on the time interval $[0, T]$.

Meidan and Perold [15], Papageorgiou [18] and Schechter [28] have also obtained some interesting results of continuous-time linear programming problem. Anderson et al. [1, 2, 3], Fleischer and Sethuraman [8], Pullan [19, 20, 21, 22, 23] and Wang et al.

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[38] investigated a subclass of continuous-time linear programming problem, which is called the separated continuous-time linear programming problem and can be used to model the job-shop scheduling problems. Recently, Weiss [39] proposed a simplex-like algorithm to solve the separated continuous-time linear programming problem. Wen et al. [41] developed a numerical method to solve the non-separated continuous-time linear programming problem.

On the other hand, the nonlinear type of continuous-time optimization problems was also studied by Farr and Hanson [6, 7], Grinold [11, 12], Hanson and Mond [13], Reiland [24, 25], Reiland and Hanson [26] and Singh [32]. The nonsmooth continuous-time optimization problems was studied by Rojas-Medar et al. [27] and Singh and Farr [33]. The nonsmooth continuous-time multiobjective programming problems was also studied by Nobakhtian and Pouryayevali [16, 17].

The optimization problem in which the objective function appears as a ratio of two real-valued function is known as a fractional programming problem. Due to its significance appearing in the information theory, stochastic programming and decomposition algorithms for large linear systems, the various theoretical and computational issues have received particular attention in the last decades. For more details on this topic, we may refer to Stancu-Minasian [34] and Schaible et al. [10, 29, 30, 31]. On the other hand, Zalmai [42, 43, 44, 45] investigated the continuous-time fractional programming problems. Moreover, Stancu-Minasian and Tigan [35] studied the stochastic continuous-time linear fractional programming problem. Under some positivity conditions, by using the minimum-risk approach, the stochastic continuous-time linear fractional programming problem can be shown to be equivalent to the deterministic continuous-time linear fractional programming problem.

Let $L^\infty([0,T],\mathbb{R}^p)$ be the space of all measurable and essentially bounded functions from a time space $[0,T]$ into the p-dimensional Euclidean space $\mathbb{R}^p$ and let $C([0,T],\mathbb{R}^p)$ be the space of all continuous functions from $[0,T]$ into the $\mathbb{R}^p$. In this paper, we consider the continuous-time linear fractional programming problem (CLFP) that is formulated as follows:

$$\text{(CLFP)} \qquad \max \quad \frac{\mu + \int_0^T (\mathbf{f}(t))^\top \mathbf{x}(t)dt}{\xi + \int_0^T (\mathbf{h}(t))^\top \mathbf{x}(t)dt}$$

$$\text{subject to} \quad B\mathbf{x}(t) \le \mathbf{g}(t) + \int_0^t K\mathbf{x}(s)ds \text{ for all } t \in [0,T]$$

$$\mathbf{x} \in L^\infty([0,T],\mathbb{R}^q_+),$$

where $\mathbf{f} \in C([0,T],\mathbb{R}^q)$, $\mathbf{h} \in C([0,T],\mathbb{R}^q_+)$, $\mathbf{g} \in C([0,T],\mathbb{R}^p_+)$ and $\xi > 0$. It is obvious that the problem (CLFP) is feasible with the trivial feasible solution $\mathbf{x}(t) = \mathbf{0}$ for all $t \in [0,T]$. We also assume that $B = [B_{ij}]_{p\times q}$ and $K = [K_{ij}]_{p\times q}$ are $p \times q$ constant matrices satisfying

- $K_{ij} \geq 0$ for all $i = 1, \cdots, p$ and $j = 1, \cdots, q$;
- $B_{ij} \geq 0$ and $\sum_{i=1}^{p} B_{ij} > 0$ for all $i = 1, \cdots, p$ and $j = 1, \cdots, q$,

and the superscript "$^\top$" denotes the transpose operation of matrices. Recently, Wen and Wu [40] have developed a computational procedure by combining the parametric method and discrete approximation method to solve a special class of the present problem (CLFP). In this paper, by extending the methodology of [40], we shall develop a more practical and efficient computational procedure which can generate an approximate solution with predefined error bound. On the other hand, the convergent properties of approximate solutions will be studied in this paper, which was not taken into account in Wen and Wu [40].

This paper is organized as follows. In Section 2, we propose the auxiliary parametric optimization problems, and establish many useful relations between the parametric problem and (CLFP), which will be used to design a practical computational procedure. In Section 3, we introduce the discrete approximation method for the auxiliary parametric optimization problems. In Section 4, by using the different step sizes of discretization problems, we construct a sequence of convex, piecewise linear and strictly decreasing upper and lower bound functions with the unique zeros, respectively. In Section 5, we show that the zeros of upper and lower bound functions determine a sequence of intervals which will shrink to the optimal value of (CLFP) as the size of discretization getting larger. Besides, we establish upper bounds of lengths of these intervals, thereby we can determine the size of discretization in advance such that the accuracy of the corresponding approximate solution can be controlled within the predefined error tolerance. Moreover, we prove that the searched sequence of approximate solution functions weakly-star converges to an optimal solution of (CLFP). In the final Section 6, the computational procedure is proposed and two numerical examples are provided to demonstrate the usefulness of this practical algorithm.

2. Parametric Continuous-time Linear Programming Problems

Let us write

$$\lambda = \frac{\mu + \int_0^T (\mathbf{f}(t))^\top \mathbf{x}(t)dt}{\xi + \int_0^T (\mathbf{h}(t))^\top \mathbf{x}(t)dt}. \tag{1}$$

Then the problem (CLFP) is equivalent to the following continuous-time optimization problem

$$
\begin{aligned}
\text{(CP)} \quad & \max \quad \lambda \\
& \text{subject to} \quad \mu + \int_0^T (\mathbf{f}(t))^\top \mathbf{x}(t)dt = \lambda \cdot \left(\xi + \int_0^T (\mathbf{h}(t))^\top \mathbf{x}(t)dt \right) \\
& \qquad\qquad B\mathbf{x}(t) \leq \mathbf{g}(t) + \int_0^t K\mathbf{x}(s)ds \text{ for all } t \in [0,T] \\
& \qquad\qquad \mathbf{x} \in L^\infty([0,T], \mathbb{R}^q_+) \text{ and } \lambda \in \mathbb{R}.
\end{aligned}
$$

Remark 2.1. When we say that $(\mathbf{x}^*, \lambda^*)$ is an optimal solution of (CP), it means that the optimal objective value of (CP) is λ^*. However, when we say that the optimal objective value of (CP) is λ^*, it does not necessary say that the problem (CP) has an optimal solution $(\mathbf{x}^*, \lambda^*)$, and it just means that the optimal objective value λ^* is obtained by taking the supremum.

Given any optimization problem (P), we denote by $V(\text{P})$ the optimal objective value of (P); that is, $V(\text{P})$ will be obtained by taking the supremum or infimum.

Since problem (CP) is not a linear programming problem, an auxiliary problem associated with (CP) will be proposed and formulated as the parametric continuous-time linear programming problem. For any $\lambda \in \mathbb{R}$, we consider the following parametric continuous-time linear programming problem:

$$
\begin{aligned}
(\text{CLP}_\lambda) \quad & \max \quad \mu - \lambda\xi + \int_0^T (\mathbf{f}(t) - \lambda\mathbf{h}(t))^\top \mathbf{x}(t)dt \\
& \text{subject to} \quad B\mathbf{x}(t) \leq \mathbf{g}(t) + \int_0^t K\mathbf{x}(s)ds \text{ for all } t \in [0,T] \\
& \qquad\qquad \mathbf{x}(t) \in L^\infty([0,T], \mathbb{R}^q_+).
\end{aligned} \tag{2}
$$

The dual problem (DCLP_λ) can be defined as follows:

$$
\begin{aligned}
(\text{DCLP}_\lambda) \quad & \min \quad \mu - \lambda\xi + \int_0^T (\mathbf{g}(t))^\top \mathbf{y}(t)dt \\
& \text{subject to} \quad B^\top \mathbf{y}(t) - \int_t^T K^\top \mathbf{y}(s)ds \geq \mathbf{f}(t) - \lambda\mathbf{h}(t) \text{ for } t \in [0,T] \\
& \qquad\qquad \mathbf{y}(t) \in L^\infty([0,T], \mathbb{R}^p_+).
\end{aligned} \tag{3}
$$

According to the same arguments given in Tyndall [36], the weak and strong duality properties can be realized below.

Theorem 2.1. (Weak Duality between (CLP_λ) and (DCLP_λ)). *Considering the primal-dual pair problems* (CLP_λ) *and* (DCLP_λ), *for any feasible solutions* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *of problems* (CLP_λ) *and* (DCLP_λ), *respectively, we have*

$$
\mu - \lambda\xi + \int_0^T (\mathbf{f}(t) - \lambda\mathbf{h}(t))^\top \mathbf{x}(t)dt \leq \mu - \lambda\xi + \int_0^T (\mathbf{g}(t))^\top \mathbf{y}(t)dt;
$$

that is, $V(\text{CLP}_\lambda) \leq V(\text{DCLP}_\lambda)$.

Theorem 2.2. (Strong Duality between (CLP_λ) and (DCLP_λ)). *There exist optimal solutions* $\mathbf{x}^{(*,\lambda)}(t)$ *and* $\mathbf{y}^{(*,\lambda)}(t)$ *of the primal-dual pair problems* (CLP_λ) *and* (DCLP_λ)*, respectively, such that*

$$\mu - \lambda\xi + \int_0^T (\mathbf{f}(t) - \lambda\mathbf{h}(t))^\top \mathbf{x}^{(*,\lambda)}(t)dt = \mu - \lambda\xi + \int_0^T (\mathbf{g}(t))^\top \mathbf{y}^{(*,\lambda)}(t)dt;$$

that is, $V(\text{CLP}_\lambda) = V(\text{DCLP}_\lambda)$.

Using the solvability of the problem (CLP_λ) and by the same arguments given in Wen and Wu [40], the relations between (CP) and its associated auxiliary problem (CLP_λ) can also be realized. To see this, we denote by $Q(\lambda) = V(\text{CLP}_\lambda)$ the optimal objective value of (CLP_λ), which says that $Q(\cdot)$ is a real-valued function.

First, we can see that the function $Q(\lambda)$ is continuous and strictly decreasing.

Proposition 2.1. *The following statements hold true.*

(i) *The real-valued function* $Q(\lambda)$ *is convex, hence is continuous.*

(ii) *If* $\lambda_1 < \lambda_2$*, then* $Q(\lambda_1) > Q(\lambda_2)$*; that is, the real-valued function* $Q(\cdot)$ *is strictly decreasing.*

Many useful relations between (CLP_λ) and (CP) are given below.

Proposition 2.2. *The following statements hold true.*

(i) *Given any* $\lambda \in \mathbb{R}$, $Q(\lambda) > 0$ *if and only if* $\lambda < V(\text{CP})$. *Equivalently,* $Q(\lambda) \leq 0$ *if and only if* $\lambda \geq V(\text{CP})$.

(ii) *Suppose that* $(\mathbf{x}^*_{\lambda^*}(t), \lambda^*)$ *is an optimal solution of* (CP) *such that* $\mathbf{x}^*_{\lambda^*}(t)$ *is an optimal solution of* (CLP_{λ^*}). *Then* $Q(\lambda^*) = 0$.

(iii) *If problem* (CLP_λ) *has an optimal solution* $\bar{\mathbf{x}}_\lambda(t)$ *such that* $Q(\lambda) = 0$*, then* $(\bar{\mathbf{x}}_\lambda(t), \lambda)$ *is an optimal solution of problem* (CP) *with* $V(\text{CP}) = \lambda$. *In this case,* $\bar{\mathbf{x}}_\lambda(t)$ *is also an optimal solution of* (CLFP).

From Proposition 2.2, it follows that the optimal solution of (CLFP) is equivalent to determine the root of the nonlinear equation $Q(\lambda) = 0$. If the equation $Q(\lambda) = 0$ has a root, then Proposition 2.1 also says that the root is unique. However, it is notoriously difficult to find the exact solution of every (CLP_λ). In the next section, we shall use the discrete approximation procedure developed by Wen et al. [41] to find the approximate value of $Q(\lambda)$ and estimate its error bound.

3. A Discrete Approximation Method for (CLP_λ)

Now, we are going to propose the discrete approximation method to solve the parametric problem (CLP_λ). In this case, the discrete problem derived from problem (CLP_λ) will be a finite-dimensional linear programming problem.

For each $n \in \mathbb{N}$, we take

$$\mathcal{P}_n = \left\{0, \frac{T}{n}, \frac{2T}{n}, \cdots, \frac{(n-1)T}{n}, T\right\}$$

as a partition of $[0, T]$, which divides $[0, T]$ into n subintervals with equal length T/n. For $l = 1, \cdots, n$, let

$$\mathbf{a}_l^{(n,\lambda)} = \left(a_{1l}^{(n,\lambda)}, a_{2l}^{(n,\lambda)}, \cdots, a_{ql}^{(n,\lambda)}\right)^\top \in \mathbb{R}^q \tag{4}$$

and

$$\mathbf{b}_l^{(n)} = \left(b_{1l}^{(n)}, b_{2l}^{(n)}, \cdots, b_{pl}^{(n)}\right)^\top \in \mathbb{R}^p, \tag{5}$$

where

$$\begin{aligned} a_{jl}^{(n,\lambda)} &= \min\left\{f_j(t) - \lambda h_j(t) : t \in \left[\frac{(l-1)T}{n}, \frac{lT}{n}\right]\right\} \\ &\text{for } j = 1, \cdots, q \text{ and } l = 1, \cdots, n. \end{aligned} \tag{6}$$

and

$$b_{il}^{(n)} = \min\left\{g_i(t) : t \in \left[\frac{(l-1)T}{n}, \frac{lT}{n}\right]\right\} \text{ for } i = 1, \cdots, p \text{ and } l = 1, \cdots, n. \tag{7}$$

For convenience, the "empty sum" $\sum_{l=1}^{0} \mathbf{x}_l$ is defined to be the zero vector. According to the continuous-time linear programming problem (CLP_λ), its discrete version can be defined as the following finite-dimensional linear programming problem

$$\begin{aligned} (\mathrm{P}_n^{(\lambda)}) \quad \text{maximize} \quad & \mu - \lambda\xi + \frac{T}{n}\sum_{l=1}^{n}(\mathbf{a}_l^{(n,\lambda)})^\top \mathbf{x}_l \\ \text{subject to} \quad & B\mathbf{x}_l \leq \mathbf{b}_l^{(n)} + \frac{T}{n}K\sum_{\omega=1}^{l-1}\mathbf{x}_\omega \text{ for } l = 1, \cdots, n \\ & \mathbf{x}_l \in \mathbb{R}_+^q \text{ for } l = 1, \cdots, n. \end{aligned}$$

The dual problem $(\mathrm{D}_n^{(\lambda)})$ of $(\mathrm{P}_n^{(\lambda)})$ is defined by

$$\begin{aligned} (\mathrm{D}_n^{(\lambda)}) \quad \text{minimize} \quad & \mu - \lambda\xi + \frac{T}{n}\sum_{l=1}^{n}(\mathbf{b}_l^{(n)})^\top \mathbf{y}_l \\ \text{subject to} \quad & B^\top \mathbf{y}_l \geq \mathbf{a}_l^{(n,\lambda)} + \frac{T}{n}K^\top \sum_{\omega=l+1}^{n}\mathbf{y}_\omega \\ & \mathbf{y}_l \in \mathbb{R}_+^p \text{ for } l = 1, \cdots, n. \end{aligned} \tag{8}$$

where the "empty sum" $\sum_{l=n+1}^{n} \mathbf{y}_l$ is defined to be the zero vector.

Remark 3.1. We have the following observations.

- For each $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, the problem $(\mathrm{P}_n^{(\lambda)})$ is feasible, since $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$ with $\mathbf{x}_l = \mathbf{0}$ for all $l = 1, \cdots, n$ is a feasible solution of $(\mathrm{P}_n^{(\lambda)})$.
- The feasibility of $(\mathrm{D}_n^{(\lambda)})$ can be realized by Lemma 3.3 below.
- The above two observations say that the strong duality theorem holds true for the primal-dual pair problems $(\mathrm{P}_n^{(\lambda)})$ and $(\mathrm{D}_n^{(\lambda)})$, i.e., $-\infty < V(\mathrm{P}_n^{(\lambda)}) = V(\mathrm{D}_n^{(\lambda)}) < \infty$.

It can be shown that the feasible sets of the problems $(\mathrm{P}_n^{(\lambda)})$ are uniformly bounded for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. To see this, let

$$\sigma = \min\left\{B_{ij} : B_{ij} > 0\right\}, \tag{9}$$

$$\nu = \max_{j=1,\cdots,q}\left\{\sum_{i=1}^{p} K_{ij}\right\}, \tag{10}$$

$$\zeta = \max\left\{g_i(t) : i = 1, \cdots, p \text{ and } t \in [0, T]\right\} \tag{11}$$

and

$$\tau_\lambda = \max_{j=1,\cdots,q}\max_{t\in[0,T]}\max\left\{f_j(t) - \lambda h_j(t), 0\right\}. \tag{12}$$

From (6) and (7), it follows that $a_{jl}^{(n,\lambda)} \le \tau_\lambda$ and $b_{il}^{(n)} \le \zeta$ for $i = 1, \cdots, p$, $j = 1, \cdots, q$ and $l = 1, \cdots, n$.

By slightly modifying the arguments given in [41], we have the following useful results.

Lemma 3.1. *Given any $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, if $(\mathbf{x}_1^{(n,\lambda)}, \mathbf{x}_2^{(n,\lambda)}, \cdots, \mathbf{x}_n^{(n,\lambda)})$ is a feasible solution of the primal problem $(\mathrm{P}_n^{(\lambda)})$, where $\mathbf{x}_l^{(n,\lambda)} = (x_{1l}^{(n,\lambda)}, x_{2l}^{(n,\lambda)}, \cdots, x_{ql}^{(n,\lambda)})^\top \in \mathbb{R}_+^q$, then*

$$0 \le x_{jl}^{(n,\lambda)} \le \frac{\zeta}{\sigma}\cdot\exp\left(\frac{q\nu T}{\sigma}\right) \text{ for all } j = 1, \cdots, q \text{ and } l = 1, \cdots, n \tag{13}$$

and

$$\mu - \lambda\xi \le V(\mathrm{P}_n^{(\lambda)}) \le \mu - \lambda\xi + q\cdot\tau_\lambda\cdot T\cdot\frac{\zeta}{\sigma}\cdot\exp\left(\frac{q\nu T}{\sigma}\right). \tag{14}$$

This says that the feasible sets of the problems $(\mathrm{P}_n^{(\lambda)})$ are uniformly bounded in the sense that the bounds of $x_{jl}^{(n,\lambda)}$ are independent of n and λ.

Lemma 3.2. *Given any $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, let $(\hat{\mathbf{y}}_1^{(n,\lambda)}, \hat{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \hat{\mathbf{y}}_n^{(n,\lambda)})$ with $\hat{\mathbf{y}}_l^{(n,\lambda)} = (\hat{y}_{1l}^{(n,\lambda)}, \hat{y}_{2l}^{(n,\lambda)}, \cdots, \hat{y}_{pl}^{(n,\lambda)})^\top$ be defined by*

$$(15) \qquad \hat{y}_{il}^{(n,\lambda)} = \frac{\tau_\lambda}{\sigma} \cdot \left(1 + \frac{\nu T}{n\sigma}\right)^{n-l} \geq 0 \text{ for all } i = 1, \cdots, p \text{ and } l = 1, \cdots, n.$$

Then $(\hat{\mathbf{y}}_1^{(n,\lambda)}, \hat{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \hat{\mathbf{y}}_n^{(n,\lambda)})$ is a feasible solution of the problem $(\mathrm{D}_n^{(\lambda)})$ satisfying

$$(16) \qquad \hat{y}_{il}^{(n,\lambda)} \leq \frac{\tau_\lambda}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right) \text{ for all } i = 1, \cdots, p \text{ and } l = 1, \cdots, n.$$

Lemma 3.3. *Given any $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, let $\hat{y}_{il}^{(n,\lambda)}$ be defined in (15). Given a feasible solution $(\mathbf{y}_1^{(n,\lambda)}, \mathbf{y}_2^{(n,\lambda)}, \cdots, \mathbf{y}_n^{(n,\lambda)})$ of problem $(\mathrm{D}_n^{(\lambda)})$, let $(\bar{\mathbf{y}}_1^{(n,\lambda)}, \bar{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{y}}_n^{(n,\lambda)})$ with $\bar{\mathbf{y}}_l^{(n,\lambda)} = (\bar{y}_{1l}^{(n,\lambda)}, \bar{y}_{2l}^{(n,\lambda)}, \cdots, \bar{y}_{pl}^{(n,\lambda)})^\top$ be defined by*

$$(17) \qquad \bar{y}_{il}^{(n,\lambda)} = \min\left\{y_{il}^{(n,\lambda)}, \hat{y}_{il}^{(n,\lambda)}\right\} \text{ for all } i = 1, \cdots, p \text{ and } l = 1, \cdots, n.$$

Then $(\bar{\mathbf{y}}_1^{(n,\lambda)}, \bar{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{y}}_n^{(n,\lambda)})$ is a feasible solution of $(\mathrm{D}_n^{(\lambda)})$ satisfying

$$(18) \qquad 0 \leq \bar{y}_{il}^{(n,\lambda)} \leq \frac{\tau_\lambda}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right) \text{ for all } l = 1, \cdots, n \text{ and } i = 1, \cdots, p.$$

Moreover, if $(\mathbf{y}_1^{(n,\lambda)}, \mathbf{y}_2^{(n,\lambda)}, \cdots, \mathbf{y}_n^{(n)})$ is an optimal solution of $(\mathrm{D}_n^{(\lambda)})$, then the feasible solution $(\bar{\mathbf{y}}_1^{(n,\lambda)}, \bar{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{y}}_n^{(n,\lambda)})$ is also an optimal solution of $(\mathrm{D}_n^{(\lambda)})$.

Remark 3.2. We are going to claim that there exists an optimal solution $(\bar{\mathbf{y}}_1^{(n,\lambda)}, \bar{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{y}}_n^{(n,\lambda)})$ of $(\mathrm{D}_n^{(\lambda)})$ satisfying the inequalities (18). To see this, let $(\mathbf{y}_1^{(n,\lambda)}, \mathbf{y}_2^{(n,\lambda)}, \cdots, \mathbf{y}_n^{(n,\lambda)})$ be an optimal solution of $(\mathrm{D}_n^{(\lambda)})$. Using Lemma 3.2 and referring to (17), we can always construct an optimal solution $(\bar{\mathbf{y}}_1^{(n,\lambda)}, \bar{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{y}}_n^{(n,\lambda)})$ of $(\mathrm{D}_n^{(\lambda)})$ that is defined by

$$\bar{y}_{il}^{(n,\lambda)} = \begin{cases} y_{il}^{(n,\lambda)}, & \text{if } y_{il}^{(n,\lambda)} \leq \dfrac{\tau_\lambda}{\sigma}\left(1 + \dfrac{\nu T}{n\sigma}\right)^{n-l} \\ \hat{y}_{il}^{(n,\lambda)} = \dfrac{\tau_\lambda}{\sigma}\left(1 + \dfrac{\nu T}{n\sigma}\right)^{n-l}, & \text{otherwise} \end{cases}$$

such that the inequalities (18) are satisfied. Moreover, we have

$$(19) \qquad \mu - \lambda\xi \leq V(\mathrm{P}_n^{(\lambda)}) = V(\mathrm{D}_n^{(\lambda)}) \leq \mu - \lambda\xi + p \cdot \tau_\lambda \cdot T \cdot \frac{\zeta}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right).$$

Besides, we can construct the feasible solutions of the problems (CLP_λ) by virtue of the optimal solution of the problem $(\text{P}_n^{(\lambda)})$. Let $(\bar{\mathbf{x}}_1^{(n,\lambda)}, \bar{\mathbf{x}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{x}}_n^{(n,\lambda)})$ be an optimal solution of $(\text{P}_n^{(\lambda)})$. For $j = 1, \cdots, q$, we define the step functions $\bar{x}_j^{(n,\lambda)} : [0, T] \to \mathbb{R}$ as follows:

$$\bar{x}_j^{(n,\lambda)}(t) = \begin{cases} \bar{x}_{jl}^{(n,\lambda)}, & \text{if } \frac{(l-1)T}{n} \le t < \frac{lT}{n} \\ \bar{x}_{jn}^{(n,\lambda)}, & \text{if } t = T, \end{cases} \tag{20}$$

where $l = 1, \cdots, n$. Then we can form a vector-valued function $\bar{\mathbf{x}}^{(n,\lambda)} : [0, T] \to \mathbb{R}^q$ by

$$\bar{\mathbf{x}}^{(n,\lambda)}(t) = \left(\bar{x}_1^{(n,\lambda)}(t), \bar{x}_2^{(n,\lambda)}(t), \cdots, \bar{x}_q^{(n,\lambda)}(t)\right)^\top. \tag{21}$$

In this case, we say that $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ is a *natural solution* of (CLP_λ) constructed from $(\bar{\mathbf{x}}_1^{(n,\lambda)}, \bar{\mathbf{x}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{x}}_n^{(n,\lambda)})$. After some algebraic calculations, it is not hard to show the feasibility of natural solutions of (CLP_λ), which will be presented below.

Proposition 3.1. *Let* $(\bar{\mathbf{x}}_1^{(n,\lambda)}, \bar{\mathbf{x}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{x}}_n^{(n,\lambda)})$ *be an optimal solution of* $(\text{P}_n^{(\lambda)})$. *Then the natural solution* $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ *of problem* (CLP_λ) *constructed from* $(\bar{\mathbf{x}}_1^{(n,\lambda)}, \bar{\mathbf{x}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{x}}_n^{(n,\lambda)})$ *is a feasible solution of* (CLP_λ). *Moreover, we have*

$$Q(\lambda) = V(\text{CLP}_\lambda) \ge V(\text{P}_n^{(\lambda)}) \tag{22}$$

for all $n \in \mathbb{N}$.

Furthermore, by the forthcoming results, we can also see that

$$\lim_{n\to\infty} V(\text{P}_n^{(\lambda)}) = Q(\lambda).$$

For $i = 1, \cdots, p$ and $j = 1, \cdots, q$, we define the step functions as follows

$$\varphi_j^{(n,\lambda)}(t) = \begin{cases} a_{jl}^{(n,\lambda)}, & \text{if } \frac{(l-1)T}{n} \le t < \frac{lT}{n} \\ a_{jn}^{(n,\lambda)}, & \text{if } t = T \end{cases} \tag{23}$$

and

$$g_i^{(n)}(t) = \begin{cases} b_{il}^{(n)}, & \text{if } \frac{(l-1)T}{n} \le t < \frac{lT}{n} \\ b_{in}^{(n)}, & \text{if } t = T, \end{cases} \tag{24}$$

where $l = 1, \cdots, n$, and $a_{jl}^{(n,\lambda)}$ and $b_{il}^{(n)}$ are defined in (6) and (7), respectively.

Remark 3.3. Since each $f_j(t)-\lambda\, h_j(t)$ is continuous on the compact interval $[0,T]$ for $j=1,\cdots,q$, it follows that each $f_j(t)-\lambda\, h_j(t)$ is also uniformly continuous on the compact interval $[0,T]$ for all j. Therefore, the sequence of step functions $\{\varphi_j^{(n,\lambda)}(t)\}$ converges to $f_j(t)-\lambda\, h_j(t)$ on $[0,T]$ for j. Similarly, we can also conclude that the sequence of step functions $\{g_i^{(n)}(t)\}$ converges to $g_i(t)$ on $[0,T]$ for $i=1,\cdots,p$.

For further discussion, we define

$$\rho = \max_{j=1,\cdots,q}\left\{\frac{\sum_{i=1}^{p} K_{ij}}{\sum_{i=1}^{p} B_{ij}}, \frac{1}{\sum_{i=1}^{p} B_{ij}}\right\}. \tag{25}$$

Let $(\bar{\mathbf{y}}_1^{(n,\lambda)}, \bar{\mathbf{y}}_2^{(n,\lambda)},\cdots,\bar{\mathbf{y}}_n^{(n,\lambda)})$ be an optimal solution of $(\mathrm{D}_n^{(\lambda)})$. From Lemma 3.3, we can assume that this optimal solution satisfies the inequalities (18). We also adopt the following notations

$$\epsilon_n(\lambda) = \max_{j=1,\cdots,q}\sup_{t\in[0,T]}\left\{f_j(t)-\lambda h_j(t)-\varphi_j^{(n,\lambda)}(t)\right\} \tag{26}$$

$$\bar{\epsilon}_n = \max_{i=1,\cdots,p}\sup_{t\in[0,T]}\left\{g_i(t)-g_i^{(n)}(t)\right\} \tag{27}$$

$$\delta_n(\lambda) = \max_{i=1,\cdots,p}\max_{l=1,\cdots,n}\left\{\frac{T}{n}\bar{y}_{il}^{(n,\lambda)}\right\}. \tag{28}$$

By Remark 3.3 and Lemma 3.3, we see that for all λ,

$$\epsilon_n(\lambda)\to 0,\ \bar{\epsilon}_n\to 0,\ \text{and } \delta_n(\lambda)\to 0,\ \text{as } n\to\infty.$$

By slightly modifying the results given in [41], we see that the natural solution $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ of problem (CLP_λ) constructed from an optimal solution of $(\mathrm{P}_n^{(\lambda)})$ is an approximate solution of (CLP_λ), and its error bound can be estimated.

Proposition 3.2. *The following statements hold true.*

(i) *We have*

$$0\le Q(\lambda)-V(\mathrm{P}_n^{(\lambda)})\le \varepsilon_n(\lambda), \tag{29}$$

where

$$\begin{aligned}\varepsilon_n(\lambda) :&= \bar{\epsilon}_n\cdot p\cdot\delta_n(\lambda)\cdot(n+\exp(\rho T)-1)\\ &+(\epsilon_n(\lambda)+\delta_n(\lambda))\int_0^T \rho\cdot\exp(\rho(T-t))\,(\mathbf{g}(t))^\top\mathbf{1}dt.\end{aligned} \tag{30}$$

(ii) *We have*

$$\lim_{n\to\infty} V(\mathrm{D}_n^{(\lambda)}) = \lim_{n\to\infty} V(\mathrm{P}_n^{(\lambda)}) = Q(\lambda).$$

(*iii*) *Let* $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ *be the natural solutions of* (CLP_λ). *Then the error between the optimal objective value of* (CLP_λ) *and the objective value of* $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ *is less than or equal to* $\varepsilon_n(\lambda)$.

4. Lower and Upper Bound Functions

In the sequel, we are going to develop a computational procedure for solving problem (CLFP). From Proposition 2.2, we just need to obtain the root λ^* of $Q(\lambda) = 0$. Let $V_n(\lambda) := V(\mathrm{P}_n^{(\lambda)})$. Then for all n, $V_n(\lambda)$ is a function of λ. Wen and Wu [40] utilized the zero λ_n of the continuous function $V_n(\lambda)$ and the zero λ_n° of the continuous function $V_n(\lambda) + \varepsilon_n(\lambda)$ to construct an interval $I_n = [\lambda_n, \lambda_n^\circ]$ containing λ^*. By using the fact that the length of the interval I_n approaches to zero as n tends to infinity, the approximate solutions of (CLFP) can be obtained. Based on this methodology, we shall derive new lower and upper bound functions of $Q(\lambda)$ such that they are continuous, convex, piecewise linear and strictly decreasing. These functions will be used to derive the upper and lower bounds of λ^* in the sense that these bounds can form a closed interval containing λ^*. Besides, we can also establish upper bounds of length of these closed intervals containing λ^*. These new lower and upper bound functions of $Q(\lambda)$ make it possible to design a more practical and efficient algorithm for solving (CLFP).

Now we consider the following parametric linear programming problem:

$$\begin{aligned}
(\Psi\mathrm{P}_n^{(\lambda)}) \quad & \text{maximize} \quad \mu - \lambda\xi + \sum_{l=1}^{n} \Big(\int_{\frac{l-1}{n}T}^{\frac{l}{n}T} \mathbf{f}(t) - \lambda\, \mathbf{h}(t) dt \Big)^\top \mathbf{x}_l \\
& \text{subject to} \quad B\mathbf{x}_l \leq \mathbf{b}_l^{(n)} + \frac{T}{n} K \sum_{\omega=1}^{l-1} \mathbf{x}_\omega \text{ for } l = 1, \cdots, n \\
& \qquad \mathbf{x}_l \in \mathbb{R}_+^q \text{ for } l = 1, \cdots, n.
\end{aligned}$$

The dual problem $(\Psi\mathrm{D}_n^{(\lambda)})$ of $(\Psi\mathrm{P}_n^{(\lambda)})$ is defined by

$$\begin{aligned}
(\Psi\mathrm{D}_n^{(\lambda)}) \quad & \text{minimize} \quad \mu - \lambda\xi + \sum_{l=1}^{n} (\mathbf{b}_l^{(n)})^\top \mathbf{y}_l \\
& \text{subject to} \quad B^\top \mathbf{y}_l \geq \int_{\frac{l-1}{n}T}^{\frac{l}{n}T} \{\mathbf{f}(t) - \lambda\, \mathbf{h}(t)\} dt + \frac{T}{n} K^\top \sum_{\omega=l+1}^{n} \mathbf{y}_\omega \\
& \qquad \text{and } \mathbf{y}_l \in \mathbb{R}_+^p \text{ for } l = 1, \cdots, n.
\end{aligned}$$

In order to derive a lower bound function of $Q(\lambda)$, given any $n \in \mathbb{N}$, we define the function

$$L_n(\lambda) = V(\Psi\mathrm{P}_n^{(\lambda)}). \tag{31}$$

Since $(\Psi\mathrm{P}_n^{(\lambda)})$ and $(\mathrm{P}_n^{(\lambda)})$ have the same feasible regions, it says that $(\Psi\mathrm{P}_n^{(\lambda)})$ is also solvable. This also says that the strong duality theorem holds true for the primal-dual pair problems $(\Psi\mathrm{P}_n^{(\lambda)})$ and $(\Psi\mathrm{D}_n^{(\lambda)})$, i.e., $-\infty < V(\Psi\mathrm{P}_n^{(\lambda)}) = V(\Psi\mathrm{D}_n^{(\lambda)}) < \infty$. On the other hand, by the same arguments for proving Lemma 3.1, we can obtain

$$\mu - \lambda\xi \leq V(\Psi\mathrm{P}_n^{(\lambda)}) = L_n(\lambda) \leq \mu - \lambda\xi + q \cdot \tau_\lambda \cdot T \cdot \frac{\zeta}{\sigma} \cdot \exp\left(\frac{q\nu T}{\sigma}\right) \text{ for all } \lambda. \tag{32}$$

Besides, by the Mean Value Theorem for Definite Integrals, for all $l = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, q$,

$$\int_{\frac{l-1}{n}T}^{\frac{l}{n}T} \{f_j(t) - \lambda\, h_j(t)\}dt = \frac{T}{n}\{f_j(t_{jl}) - \lambda\, h(t_{jl})\} \text{ for some } t_{jl} \in [\frac{l-1}{n}T, \frac{l}{n}T].$$

Hence, by modifying the arguments for proving Lemmas 3.2 and 3.3, we have that there exists an optimal solution $(\bar{\mathbf{y}}_1^{(n,\lambda)}, \bar{\mathbf{y}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{y}}_n^{(n,\lambda)})$ of $(\Psi\mathrm{D}_n^{(\lambda)})$ satisfying the following inequalities:

$$0 \leq \bar{y}_{il}^{(n,\lambda)} \leq \frac{T}{n} \cdot \frac{\tau_\lambda}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right) \text{ for all } l = 1, \cdots, n \text{ and } i = 1, \cdots, p. \tag{33}$$

This implies

$$\begin{aligned} &\mu - \lambda\xi \leq V(\Psi\mathrm{P}_n^{(\lambda)}) \\ &= V(\Psi\mathrm{D}_n^{(\lambda)}) = L_n(\lambda) \leq \mu - \lambda\xi + p \cdot \tau_\lambda \cdot T \cdot \frac{\zeta}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right) \text{ fo all } \lambda. \end{aligned} \tag{34}$$

For further discussion, we define

$$c_1 = \max_{j=1,\cdots,q} \max_{t\in[0,T]} \max\left\{f_j(t) - \frac{\mu}{\xi} \cdot h_j(t), 0\right\}, \tag{35}$$

$$c_2 = p \cdot c_1 \cdot T \cdot \frac{\zeta}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right), \tag{36}$$

$$\widehat{c}_2 = q \cdot c_1 \cdot T \cdot \frac{\zeta}{\sigma} \cdot \exp\left(\frac{q\nu T}{\sigma}\right) \tag{37}$$

and

$$\Lambda = \min\{c_2, \widehat{c}_2\} \geq 0. \tag{38}$$

Since $h_j \geq 0$ for all $j = 1, \cdots, q$, from (12), we have

$$\tau_\lambda \leq c_1, \text{ for all } \lambda \geq \frac{\mu}{\xi}. \tag{39}$$

Hence, by (32), (34), (36), (37) and (38), we have

$$L_n(\lambda) \leq \mu - \lambda\xi + \Lambda \text{ for all } \lambda \geq \frac{\mu}{\xi}. \tag{40}$$

In the sequel, we shall provide some useful lemmas for further study.

Lemma 4.1. *The following statements hold true.*

(i) *For each* $n \in \mathbb{N}$, $L_n(\lambda)$ *is a continuous, convex, piecewise linear and strictly decreasing function of* λ. *Moreover, each linear piece of* $L_n(\lambda)$ *corresponds to an interval of* λ *over which the problem* $(\Psi\mathrm{P}_n^{(\lambda)})$ *has the same optimal solution.*

(ii) *For each* $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}$, *we have* $V(\mathrm{P}_n^{(\lambda)}) \leq L_n(\lambda) \leq Q(\lambda)$.

(iii) *Let*

$$\eta^L = \max\left\{\frac{\mu + \Lambda}{\xi}, 0\right\}. \tag{41}$$

Then $\eta^L \geq \mu/\xi$ *and there exists a unique* $\lambda_n^L \in [\mu/\xi, \eta^L]$ *such that* $L_n(\lambda_n^L) = 0$ *for each* $n \in \mathbb{N}$.

Proof. To prove part (i), using the same arguments for proving Proposition 2.1, we can prove that $L_n(\lambda)$ is convex, which also says that $L_n(\lambda)$ is continuous. Since $\xi > 0$ and $\mathbf{h}(t) \geq \mathbf{0}$, it follows that $L_n(\lambda)$ is strictly decreasing. Besides, by the knowledge of sensitivity analysis for linear programming, it is well known (refer to [5]) that there exist stable ranges of λ such that $(\Psi\mathrm{P}_n^{(\lambda)})$ has the same optimal solution when λ stays at each stable range. Hence, we see that $L_n(\lambda)$ is piecewise linear and each linear piece corresponds to an interval of λ over which the problem $(\Psi\mathrm{P}_n^{(\lambda)})$ has the same optimal solution.

To prove part (ii), let $(\bar{\mathbf{x}}_1^{(n,\lambda)}, \bar{\mathbf{x}}_2^{(n,\lambda)}, \cdots, \bar{\mathbf{x}}_n^{(n,\lambda)})$ be an optimal solution of $(\Psi\mathrm{P}_n^{(\lambda)})$. Then the corresponding natural solution $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ defined in (20) is also a feasible solution of problem (CLP_λ) by Proposition 3.1. Since the objective value of (CLP_λ) at $\bar{\mathbf{x}}^{(n,\lambda)}(t)$ is equal to $V(\Psi\mathrm{P}_n^{(\lambda)})$, it follows that

$$L_n(\lambda) = V(\Psi\mathrm{P}_n^{(\lambda)}) \leq V(\mathrm{CLP}_\lambda) = Q(\lambda).$$

On the other hand, since the objective function of $(\mathrm{P}_n^{(\lambda)})$ can be rewritten as

$$\mu - \lambda\xi + \frac{T}{n}\sum_{l=1}^{n}(\mathbf{a}_l^{(n,\lambda)})^\top \mathbf{x}_l = \mu - \lambda\xi + \sum_{l=1}^{n}\left(\int_{\frac{l-1}{n}T}^{\frac{l}{n}T} \mathbf{a}_l^{(n,\lambda)} dt\right)^\top \mathbf{x}_l,$$

by the definition of $\mathbf{a}_l^{(n,\lambda)}$, we see that $V(\mathrm{P}_n^{(\lambda)}) \leq L_n(\lambda)$ for all n.

To prove part (iii), it is obvious that $\eta^L \geq \mu/\xi$. Since $\eta^L \geq \frac{\mu+\Lambda}{\xi}$ and by (40), we have

$$L_n(\eta^L) \leq \mu - \eta^L \xi + \Lambda \leq 0.$$

On the other hand, by (32), we also have $L_n(\mu/\xi) \geq 0$. The continuity of $L_n(\lambda)$ says that there exists $\lambda_n^L \in [\mu/\xi, \eta^L]$ such that $L_n(\lambda_n^L) = 0$. Finally, by part (i), the strictly decreasing property of L_n shows the uniqueness of root λ_n^L. This completes the proof. ■

Remark 4.1. For further discussion, we define the function

$$\widehat{L}_n(\lambda) = V(\widehat{\Psi\mathrm{P}}_n^{(\lambda)}), \tag{42}$$

where the problem $(\widehat{\Psi\mathrm{P}}_n^{(\lambda)})$ is obtained from $(\Psi\mathrm{P}_n^{(\lambda)})$ by removing the term $\mu - \lambda\xi$ of the objective function. By the nonnegativity of $\mathbf{h}$, it is easy to see that $\widehat{L}_n(\lambda)$ is a decreasing function of λ . Besides, since $(\widehat{\Psi\mathrm{P}}_n^{(\lambda)})$ and $(\Psi\mathrm{P}_n^{(\lambda)})$ have the same feasible regions, they also have the same optimal solutions. Hence,

$$L_n(\lambda) = \mu - \lambda\,\xi + \widehat{L}_n(\lambda). \tag{43}$$

In order to derive the upper bound function of $Q(\lambda)$, let

$$\widehat{\delta}_n(\lambda) = \frac{T}{n} \cdot \frac{\tau_\lambda}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right) \tag{44}$$

and

$$\begin{aligned} \widehat{\varepsilon}_n(\lambda) &= \bar{\epsilon}_n \cdot p \cdot \widehat{\delta}_n(\lambda) \cdot (n + \exp(\rho T) - 1) \\ &+ \left(\epsilon_n(\lambda) + \widehat{\delta}_n(\lambda)\right) \int_0^T \rho \cdot \exp(\rho(T-t))\,(\mathbf{g}(t))^\top \mathbf{1} dt. \end{aligned} \tag{45}$$

Using (18), (28) and (30), we immediately have

$$\delta_n(\lambda) \leq \widehat{\delta}_n(\lambda) \text{ and } \varepsilon_n(\lambda) \leq \widehat{\varepsilon}_n(\lambda). \tag{46}$$

By (39), we also have

$$\widehat{\delta}_n(\lambda) \leq \frac{T}{n} \cdot \frac{c_1}{\sigma} \cdot \exp\left(\frac{\nu T}{\sigma}\right) \text{ for all } \lambda \geq \frac{\mu}{\xi}. \tag{47}$$

Lemma 4.2. *Suppose that the functions f_j and h_j satisfy the Lipschitz conditions for $1 \leq j \leq q$. Then, for all $n \in \mathbb{N}$ and $\lambda \geq \frac{\mu}{\xi}$, there exist $d \geq 0$ and $r_n \geq 0$ such that*

$$0 \leq \widehat{\varepsilon}_n(\lambda) \leq \frac{d}{n} \cdot (1+\lambda) + r_n \tag{48}$$

and $r_n \to 0$ as $n \to \infty$.

Proof. From (26), we have

$$\epsilon_n(\lambda) = \max_{j=1,\cdots,q} \max_{l=1,\cdots,n} \left\{ \max_{t\in I_l^{(n)}} \{f_j(t) - \lambda h_j(t)\} - a_{jl}^{(n,\lambda)} \right\}, \tag{49}$$

where

$$I_l^{(n)} = \left[\frac{l-1}{n}T, \frac{l}{n}T \right].$$

Therefore, there exist $j_0 \in \{1, \cdots, q\}$ and $t_1, t_2 \in I_l^{(n)}$ such that

$$\epsilon_n(\lambda) = f_{j_0}(t_1) - \lambda h_{j_0}(t_1) - [f_{j_0}(t_2) - \lambda h_{j_0}(t_2)]. \tag{50}$$

Since f_j and h_j satisfy the Lipschitz conditions for all $j = 1, \cdots, q$, there exists a constant c_3 such that

$$|f_j(t_1) - f_j(t_2)| \le c_3 |t_1 - t_2| \text{ and } |h_j(t_1) - h_j(t_2)| \le c_3 |t_1 - t_2| \tag{51}$$

for all $t_1, t_2 \in [0, T]$. Then. we have

$$\begin{aligned} \epsilon_n(\lambda) &= f_{j_0}(t_1) - f_{j_0}(t_2) - \lambda [h_{j_0}(t_1) - h_{j_0}(t_2)] \\ &\le c_3 |t_1 - t_2| + \lambda c_3 |t_1 - t_2| \\ &\le (1+\lambda) c_3 \cdot \frac{T}{n}. \end{aligned} \tag{52}$$

Now, we define

$$c_4 = \int_0^T \rho \cdot \exp(\rho(T-t)) (\mathbf{g}(t))^\top \mathbf{1} dt \tag{53}$$

$$d = c_3 c_4 T \tag{54}$$

$$r_n = \frac{c_2}{n\zeta} \left[\bar{\epsilon}_n (n + \exp(\rho T) - 1) + \frac{c_4}{p} \right]. \tag{55}$$

Then, for all $\lambda \ge \frac{\mu}{\xi}$, we have

$$\begin{aligned} \widehat{\varepsilon}_n(\lambda) &= \bar{\epsilon}_n \cdot p \cdot \widehat{\delta}_n(\lambda) \cdot (n + \exp(\rho T) - 1) \\ &\quad + \left(\epsilon_n(\lambda) + \widehat{\delta}_n(\lambda) \right) \int_0^T \rho \cdot \exp(\rho(T-t)) (\mathbf{g}(t))^\top \mathbf{1} dt. \\ &= n \cdot p \cdot \bar{\epsilon}_n \cdot \widehat{\delta}_n(\lambda) + \widehat{\delta}_n(\lambda) [p \cdot \bar{\epsilon}_n (\exp(\rho T) - 1) + c_4] + c_4 \cdot \epsilon_n(\lambda) \\ &\le r_n + c_4 \cdot \epsilon_n(\lambda) \text{ (by (36), (47) and (55))} \\ &\le r_n + c_3 c_4 (1+\lambda) \cdot \frac{T}{n} \text{ (by (52))} \\ &= \frac{d}{n} \cdot (1+\lambda) + r_n. \end{aligned}$$

It is easy to see that $\widehat{\varepsilon}_n(\lambda) \geq 0$ and $d \geq 0$. Finally, since $\bar{\epsilon}_n \to 0$ as $n \to \infty$, it says that $r_n \to 0$ as $n \to \infty$. This completes the proof. ∎

We define the function

$$U_n(\lambda) = L_n(\lambda) + \frac{d}{n}(1+\lambda) + r_n, \tag{56}$$

where d and r_n are defined in (54) and (55), respectively.

Lemma 4.3. *Suppose that the functions f_j and h_j satisfy the Lipschitz conditions for $1 \leq j \leq q$. The following statements hold true.*

- (i) *For each $n \in \mathbb{N}$, $U_n(\lambda)$ is a continuous, convex and piecewise linear function of λ. Moreover, if $n > d/\xi$, then $U_n(\lambda)$ is strictly decreasing.*
- (ii) *We have $Q(\lambda) \leq U_n(\lambda)$ for all $\lambda \geq \frac{\mu}{\xi}$.*
- (iii) *Suppose that $n \in \mathbb{N}$ with $n > d/\xi$. Let*

$$\eta_n^U = \frac{\mu + c_2 + (d/n) + r_n}{\xi - (d/n)}. \tag{57}$$

Then $\frac{\mu}{\xi} \leq \eta_n^U$ and there exists a unique $\lambda_n^U \in [\mu/\xi, \eta_n^U]$ such that $U_n(\lambda_n^U) = 0$. Moreover, we have

$$\eta_n^U \to \frac{\mu + c_2}{\xi} \tag{58}$$

as $n \to \infty$

Proof. To prove part (i), by (56) and part (i) of Lemma 4.1, it is easy to see that $U_n(\lambda)$ is convex, piecewise linear and continuous. Since

$$\begin{aligned} U_n(\lambda) &= \mu - \lambda\xi + \widehat{L}_n(\lambda) + \frac{d}{n} \cdot (1+\lambda) + r_n \text{ (by (56) and (43))} \\ &= \left(\frac{d}{n} - \xi\right)\lambda + \mu + \widehat{L}_n(\lambda) + \frac{d}{n} + r_n, \end{aligned}$$

we see that if $n > d/\xi$ then $U_n(\lambda)$ is strictly decreasing by Remark 4.1.

To prove part (ii), we have

$$\begin{aligned} Q(\lambda) &\leq V(\mathrm{P}_n^{(\lambda)}) + \varepsilon_n(\lambda) \text{ (by part (i) of Proposition 3.2)} \\ &\leq L_n(\lambda) + \widehat{\varepsilon}_n(\lambda) \text{ (by (46) and part (ii) of Lemma 4.1)} \\ &\leq L_n(\lambda) + \frac{d}{n} \cdot (1+\lambda) + r_n \text{ for all } \lambda \geq \frac{\mu}{\xi} \text{ (by (48))} \\ &= U_n(\lambda). \end{aligned}$$

To prove part (iii), since $n > d/\xi$, i.e., $\xi - (d/n) > 0$, it follows that

$$\eta_n^U \geq \frac{\mu + c_2 + (d/n) + r_n}{\xi} \geq \frac{\mu}{\xi}.$$

On the other hand, by (34) and (39), we have

$$L_n(\lambda) \leq \mu - \lambda\xi + c_2 \text{ for all } \lambda \geq \frac{\mu}{\xi}. \tag{59}$$

Hence, by (56), we have

$$U_n(\lambda) \leq \mu - \lambda\xi + c_2 + \frac{d}{n} \cdot (1 + \lambda) + r_n \text{ for all } \lambda \geq \frac{\mu}{\xi},$$

which implies

$$U_n(\eta_n^U) \leq \mu - \eta_n^U \xi + c_2 + \frac{d}{n} \cdot (1 + \eta_n^U) + r_n = 0.$$

From (32) and (48), we see that $L_n(\mu/\xi) \geq 0$ and $\frac{d}{n} \cdot (1+\lambda) + r_n \geq 0$. Therefore, from (56), we have $U_n(\mu/\xi) \geq 0$. By the continuity of $U_n(\lambda)$, there exists $\lambda_n^U \in [\mu/\xi, \eta_n^U]$ such that $U_n(\lambda_n^U) = 0$. Finally, by part (i), the strictly decreasing property of U_n shows the uniqueness of root λ_n^U. Besides, since $r_n \to 0$ as $n \to \infty$ by Lemma 4.2, from (57), we can obtain (58). This completes the proof. ∎

5. Approximate Solutions

Next, we are going to demonstrate the solvability of (CLFP) and show that it is possible to generate an approximate solution of (CLFP) according to a pre-determined error bound.

Lemma 5.1. *Let λ_n^L and λ_n^U be the roots of equations $L_n(\lambda) = 0$ and $U_n(\lambda) = 0$, respectively. Then the following statements hold true.*

(i) *The sequences $\{\lambda_n^L\}_{n=1}^\infty$ and $\{\lambda_n^U\}_{n=1}^\infty$ are bounded and*

$$\frac{d}{n}\left(1 + \eta_n^U\right) + r_n \to 0 \tag{60}$$

as $n \to \infty$, where η_n^U is given in (57).

(ii) *We have*

$$0 \leq \lambda_n^U - \lambda_n^L \leq \frac{1}{\xi}\left[\frac{d}{n}\left(1 + \eta_n^U\right) + r_n\right] \tag{61}$$

and $\lambda_n^U - \lambda_n^L \to 0$ as $n \to \infty$.

Proof. To prove part (i), since $\lambda_n^L \in [\mu/\xi, \eta^L]$ for all n and η^L is independent on n by Lemma 4.1, the sequence $\{\lambda_n^L\}_{n=1}^{\infty}$ is bounded. Similarly, since $\lambda_n^U \in [\mu/\xi, \eta_n^U]$ and the sequence $\{\eta_n^U\}$ is convergent by Lemma 4.3, the sequence $\{\lambda_n^U\}_{n=1}^{\infty}$ is also bounded. Since $r_n \to 0$ as $n \to \infty$ by Lemma 4.2, and the sequence $\{\eta_n^U\}_{n=1}^{\infty}$ is bounded by (58), we obtain (60).

To prove part (ii), by Lemma 4.1 (ii) and Lemma 4.3 (ii), we have $L_n(\lambda) \leq U_n(\lambda)$ for all $\lambda \geq \frac{\mu}{\xi}$. Since $L_n(\lambda_n^L) = U_n(\lambda_n^U) = 0$, we have

$$L_n(\lambda_n^U) \leq U_n(\lambda_n^U) = L_n(\lambda_n^L) = 0,$$

and this implies $\lambda_n^U \geq \lambda_n^L$, since $L_n(\lambda)$ is strictly decreasing. By (43), we have

$$\mu - \lambda_n^L \xi + \widehat{L}_n(\lambda_n^L) = 0 \tag{62}$$

and

$$\mu - \lambda_n^U \xi + \widehat{L}_n(\lambda_n^U) + \frac{d}{n}\left(1 + \lambda_n^U\right) + r_n = 0. \tag{63}$$

Let $\triangle_n = \widehat{L}_n(\lambda_n^L) - \widehat{L}_n(\lambda_n^U)$, then we have $\triangle_n \geq 0$ by Remark 4.1. By subtracting (63) from (62), we obtain

$$\left(\lambda_n^U - \lambda_n^L\right)\xi + \Delta_n - \frac{d}{n}\left(1 + \lambda_n^U\right) - r_n = 0,$$

which implies

$$\left(\lambda_n^U - \lambda_n^L\right)\xi - \frac{d}{n}\left(1 + \lambda_n^U\right) - r_n \leq 0.$$

Since $\lambda_n^U \leq \eta_n^U$ by part (iii) of Lemma 4.3, we obtain the desired inequalities (61). Finally, from (60), we have $\lambda_n^U - \lambda_n^L \to 0$ as $n \to \infty$. This completes the proof. ■

The following results show that the problem (CLFP) is solvable, and they are very useful for designing a practical algorithm.

Theorem 5.1. *Suppose that functions f_j and h_j satisfy the Lipschitz conditions for $1 \leq j \leq q$. Then, the following statements hold true.*

(i) *Given any $n \in \mathbb{N}$ with $n > d/\xi$, we have*

$$-\frac{d}{n}\left(1 + \lambda_n^U\right) - r_n \leq Q(\lambda_n^U) \leq 0 \leq Q(\lambda_n^L) \leq \frac{d}{n}\left(1 + \lambda_n^L\right) + r_n. \tag{64}$$

(ii) *Given any $n \in \mathbb{N}$ with $n > d/\xi$, there exists $\lambda^* \in [\lambda_n^L, \lambda_n^U]$ such that $Q(\lambda^*) = 0$. Moreover, if $\mathbf{x}_{\lambda^*}$ is an optimal solution of $(\mathrm{CLP}_{\lambda^*})$, then $\mathbf{x}_{\lambda^*}$ is an optimal solution of (CLFP) with $V(\mathrm{CP}) = V(\mathrm{CLFP}) = \lambda^*$.*

(iii) *We consider the sequence* $\{\lambda_n^*\}_{n=1}^{\infty}$ *defined by*

(65)
$$\lambda_n^* = \frac{1}{2}\left(\lambda_n^L + \lambda_n^U\right).$$

Then

$$|\lambda_n^* - \lambda^*| \le \frac{1}{2}\left(\lambda_n^U - \lambda_n^L\right) \to 0 \text{ as } n \to \infty,$$

i.e., $\lambda_n^* \to \lambda^*$ *as* $n \to \infty$.

(iv) *Let* $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ *be the natural solution of problem* $(\text{CLP}_{\lambda_n^*})$ *constructed from the optimal solution of* $(\Psi\text{P}_n^{(\lambda_n^*)})$. *Then* $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ *is also a feasible solution of problem* (CLFP). *Let*

(66)
$$\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) = \mu - \lambda_n^*\xi + \int_0^T \left(\mathbf{f}(t) - \lambda_n^*\mathbf{h}(t)\right)^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt$$

be the objective value of $(\text{CLP}_{\lambda_n^*})$ *of the feasible solution* $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$, *and let*

$$\theta\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) = \frac{\mu + \int_0^T (\mathbf{f}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt}$$

be the objective value of (CLFP) *of the feasible solution* $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$. *Then*

(67)
$$0 \le V(\text{CLFP}) - \theta\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) \le Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right),$$

where

(68)
$$Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) = \frac{1}{2}\left(\lambda_n^U - \lambda_n^L\right) + \frac{\left|\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)\right|}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt}.$$

Moreover, we have

(69)
$$0 \le Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) \le \frac{1}{2}\left(\lambda_n^U - \lambda_n^L\right) + \frac{\left|\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)\right|}{\xi} \le \frac{1}{\xi}\left[\frac{d}{n}\left(1 + \eta_n^U\right) + r_n\right]$$

and $Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) \to 0$ *as* $n \to \infty$. *In other words, the natural solution* $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ *is an approximate solution of* (CLFP) *with error bound* $Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)$.

Proof. To prove part (i), since $L_n(\lambda_n^L) = 0$, by Proposition 3.2, (46) and (48), we have

$$0 \le Q(\lambda_n^L) = Q(\lambda_n^L) - L_n(\lambda_n^L) \le \varepsilon_n(\lambda_n^L) \le \widehat{\varepsilon}_n(\lambda_n^L) \le \frac{d}{n}(1+\lambda_n^L) + r_n. \tag{70}$$

Similarly, we also have

$$0 \le Q(\lambda_n^U) - L_n(\lambda_n^U) \le \widehat{\varepsilon}_n(\lambda_n^U) \le \frac{d}{n}(1+\lambda_n^U) + r_n,$$

which implies

$$L_n(\lambda_n^U) \le Q(\lambda_n^U) \le L_n(\lambda_n^U) + \frac{d}{n}(1+\lambda_n^U) + r_n. \tag{71}$$

Since $L_n(\lambda_n^U) + \frac{d}{n}(1+\lambda_n^U) + r_n = U_n(\lambda_n^U) = 0$, from (71), we have

$$-\frac{d}{n}(1+\lambda_n^U) - r_n \le Q(\lambda_n^U) \le 0.$$

Therefore, from (70), we obtain the desired inequalities (64).

To prove part (ii), since $Q(\lambda)$ is continuous by Proposition 2.1, using part (i), there exists $\lambda^* \in [\lambda_n^L, \lambda_n^U]$ such that $Q(\lambda^*) = 0$. The remaining properties follow from Proposition 2.2.

To prove part (iii), since $\lambda_n^L \le \lambda^*$ for $n > d/\xi$ by part (ii), by part (ii) of Lemma 5.1, we obtain

$$|\lambda_n^* - \lambda^*| = \left|\frac{1}{2}\left(\lambda_n^L + \lambda_n^U\right) - \lambda^*\right| \le \frac{1}{2}\left(\lambda_n^L + \lambda_n^U\right) - \lambda_n^L = \frac{1}{2}\left(\lambda_n^U - \lambda_n^L\right) \to 0 \text{ as } n \to \infty.$$

To prove part (iv), it is obvious that $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ is a feasible solution of (CLFP). Since

$$\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) = \mu - \lambda_n^*\xi + \int_0^T (\mathbf{f}(t) - \lambda_n^*\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt,$$

we obtain

$$\mu + \int_0^T (\mathbf{f}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt = \hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) + \lambda_n^*\left(\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt\right),$$

which implies

$$\frac{\mu + \int_0^T (\mathbf{f}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt} = \lambda_n^* + \frac{\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt},$$

i.e.,

$$\lambda^* \geq \theta\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) = \lambda_n^* + \frac{\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt}.$$

Therefore, we obtain

$$\begin{aligned} 0 \leq \lambda^* - \theta\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) &= (\lambda^* - \lambda_n^*) - \frac{\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt} \\ &\leq |\lambda^* - \lambda_n^*| + \frac{\left|\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)\right|}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt} \\ &\leq \frac{1}{2}\left(\lambda_n^U - \lambda_n^L\right) + \frac{\left|\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)\right|}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)dt} = Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right). \end{aligned}$$

We remain to show that $Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) \to 0$ as $n \to \infty$. Since $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ is the natural solution constructed from the optimal solution of $(\Psi\mathrm{P}_n^{(\lambda_n^*)})$, from the expression (66), we see that $\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) = V(\Psi\mathrm{P}_n^{(\lambda_n^*)}) = L_n(\lambda_n^*)$. We have

$$\begin{aligned} \left|\hat{\theta}\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)\right| &= |L_n(\lambda_n^*)| = \left|L_n\left(\frac{\lambda_n^L + \lambda_n^U}{2}\right)\right| \\ &\leq \frac{1}{2}\left|L_n(\lambda_n^L)\right| + \frac{1}{2}\left|L_n(\lambda_n^U)\right| \text{ (by the convexity of } L_n(\lambda)) \\ &= \frac{1}{2}\left[\frac{d}{n}(1+\lambda_n^U) + r_n\right] \\ &\quad \text{(since } L_n(\lambda_n^L)=0 \text{ and } 0=U_n(\lambda_n^U)=L_n(\lambda_n^U) + \tfrac{d}{n}(1+\lambda_n^U) + r_n) \\ &\leq \frac{1}{2}\left[\frac{d}{n}\left(1+\eta_n^U\right) + r_n\right] \text{ (since } \lambda_n^U \leq \eta_n^U). \end{aligned}$$

By Lemma 5.1, we obtain the inequalities (69). Finally, using (60) and (69), we have $Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) \to 0$ as $n \to \infty$. This completes the proof. ∎

Now, we shall demonstrate the convergent property of the sequence $\{\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\}$ that are natural solutions of $(\mathrm{CLP}_{\lambda_n^*})$ constructed from the optimal solutions of problems $(\Psi\mathrm{P}_n^{(\lambda_n^*)})$. We recall that the dual space of the separable Banach space $L^1[0,T]$ can be identified with $L^\infty[0,T]$. The following lemmas are very useful for further discussion.

Lemma 5.2. (Friedman [9]). *Let* $\{f_k\}$ *be a sequence in* $L^\infty([0,T],\mathbb{R})$. *If the sequence* $\{f_k\}$ *is uniformly bounded with respect to* $\|\cdot\|_\infty$, *then there exists a subsequence* $\{f_{k_j}\}$ *which weakly-star converges to* $f_0 \in L^\infty([0,T],\mathbb{R})$. *In other words, for any* $g \in L^1([0,T],\mathbb{R})$, *we have*

$$\lim_{k_j\to\infty}\int_0^T f_{k_j}(t)g(t)dt = \int_0^T f_0(t)g(t)dt.$$

Lemma 5.3. *If the sequence* $\{f_k\}_{k=1}^\infty$ *is uniformly bounded on* $[0,T]$ *with respect to* $\|\cdot\|_\infty$, *and weakly-star converges to* $f_0 \in L^\infty([0,T],\mathbb{R})$, *then*

$$f_0(t) \le \limsup_{k\to\infty} f_k(t) \text{ a.e. in } [0,T]$$

and

$$f_0(t) \ge \liminf_{k\to\infty} f_k(t) \text{ a.e. in } [0,T].$$

Proof. The results follow from the similar arguments of Levinson [14, Lemma 2.1]. ∎

Theorem 5.2. *We consider the sequence* $\{\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\}$ *that is obtained according to part (iv) of Theorem 5.1. Then the sequence* $\{\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\}$ *has a subsequence* $\{\bar{\mathbf{x}}^{(n_\nu,\lambda_{n_\nu}^*)}(t)\}$ *which weakly-star converges to an optimal solution* $\bar{\mathbf{x}}^{(*,\lambda^*)}(t)$ *of* (CLFP).

Proof. According to the previous formulas for constructing the feasible solutions $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$, we see that the sequence $\{\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\}$ of vector-valued functions are uniformly bounded with respect to $\|\cdot\|_\infty$ in which the bounds are independent of n. Using Lemma 5.2, there exists a subsequence $\{\bar{\mathbf{x}}^{(n_\nu,\lambda_{n_\nu}^*)}(t)\}$ which weakly-star converges to $\mathbf{x}^{(*,\lambda^*)}(t)$. Since $\bar{x}_j^{(n_\nu,\lambda_{n_\nu}^*)}(t) \ge 0$ for all $t \in [0,T]$ and $j = 1,\cdots,q$, using Lemma 5.3, it follows that

$$x_j^{(*,\lambda^*)}(t) \ge \liminf_{n_v\to\infty} \bar{x}_j^{(n_\nu,\lambda_{n_\nu}^*)}(t) \ge 0 \text{ a.e. in } [0,T],$$

i.e., $\mathbf{x}^{(*,\lambda^*)}(t) \ge \mathbf{0}$ a.e. in $[0,T]$. Considering the feasibility of $\bar{\mathbf{x}}^{(n_\nu,\lambda_{n_\nu}^*)}(t)$, we have

$$B\bar{\mathbf{x}}^{(n_\nu,\lambda_{n_\nu}^*)}(t) \le \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(n_\nu,\lambda_{n_\nu}^*)}(s)ds \text{ for all } t\in[0,T] \tag{72}$$

From (72), since B is nonnegative, by taking the limit superior and applying Lemma 5.3, it follows that

$$B\mathbf{x}^{(*,\lambda^*)}(t) \le \limsup_{n_v\to\infty} B\bar{\mathbf{x}}^{(n_\nu,\lambda_{n_\nu}^*)}(t) \le \int_0^t K\mathbf{x}^{(*,\lambda^*)}(s)ds + \mathbf{g}(t) \text{ a.e. in } [0,T] \tag{73}$$

Let $\mathcal{N}_0$ be the subset of $[0,T]$ such that the inequality of (73) is violated and let $\mathcal{N}_1$ be the subset of $[0,T]$ such that $\mathbf{x}^{(*,\lambda^*)}(t) \not\geq \mathbf{0}$. Then, we define $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$ and

$$\bar{\mathbf{x}}^{(*,\lambda^*)}(t) = \begin{cases} \mathbf{x}^{(*,\lambda^*)}(t) & \text{if } t \notin \mathcal{N} \\ \mathbf{0} & \text{if } t \in \mathcal{N}, \end{cases}$$

where the set $\mathcal{N}$ has measure zero. We see that the subsequence $\{\bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)\}$ is also weakly-star converges to $\bar{\mathbf{x}}^{(*,\lambda^*)}(t)$. We remain to show that $\bar{\mathbf{x}}^{(*,\lambda^*)}(t)$ is an optimal solution of (CLFP). It is obvious that $\bar{\mathbf{x}}^{(*,\lambda^*)}(t) \geq \mathbf{0}$ for all $t \in [0,T]$ and $\bar{\mathbf{x}}^{(*,\lambda^*)}(t) = \mathbf{x}^{(*,\lambda^*)}(t)$ a.e. in $[0,T]$. We consider the following cases.

- For $t \notin \mathcal{N}$, from (73), we have

$$B\bar{\mathbf{x}}^{(*,\lambda^*)}(t) = B\mathbf{x}^{(*,\lambda^*)}(t) \leq \mathbf{g}(t) + \int_0^t K\mathbf{x}^{(*,\lambda^*)}(s)ds = \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(*,\lambda^*)}(s)ds.$$

- For $t \in \mathcal{N}$, since B is nonnegative, using (72) and weak-star convergence, we also have

$$\begin{aligned} B\bar{\mathbf{x}}^{(*,\lambda^*)}(t) &= \mathbf{0} \leq \limsup_{n_v\to\infty} B\mathbf{x}^{(n_v,\lambda^*_{n_\nu})}(t) \\ &\leq \mathbf{g}(t) + \int_0^t K\mathbf{x}^{(*,\lambda^*)}(s)ds = \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(*,\lambda^*)}(s)ds. \end{aligned}$$

Therefore, we obtain

$$B\bar{\mathbf{x}}^{(*,\lambda^*)}(t) \leq \mathbf{g}(t) + \int_0^t K\bar{\mathbf{x}}^{(*,\lambda^*)}(s)ds \text{ for all } t \in [0,T],$$

which says that $\bar{\mathbf{x}}^{(*,\lambda^*)}(t)$ is a feasible solution of (CLFP). By (67), we have

$$0 \leq V(\text{CLFP}) - \theta\left(\bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)\right) \leq Er\left(\bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)\right), \tag{74}$$

where

$$\theta\left(\bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)\right) = \frac{\mu + \int_0^T (\mathbf{f}(t))^\top \bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)dt}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)dt}. \tag{75}$$

By considering the weak-star convergence on (75), it follows that

$$\lim_{n_\nu\to\infty} \theta\left(\bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)\right) = \theta\left(\bar{\mathbf{x}}^{(*,\lambda^*)}(t)\right). \tag{76}$$

Since $Er\left(\bar{\mathbf{x}}^{(n_\nu,\lambda^*_{n_\nu})}(t)\right) \to 0$ as $n_\nu \to \infty$, by taking the limit on both sides of (74) and using (76), we obtain

$$V(\text{CLFP}) = \theta\left(\bar{\mathbf{x}}^{(*,\lambda^*)}(t)\right) = \frac{\mu + \int_0^T (\mathbf{f}(t))^\top \bar{\mathbf{x}}^{(*,\lambda^*)}(t)dt}{\xi + \int_0^T (\mathbf{h}(t))^\top \bar{\mathbf{x}}^{(*,\lambda^*)}(t)dt},$$

which says that $\bar{\mathbf{x}}^{(*,\lambda^*)}(t)$ is an optimal solution of (CLFP), and the proof is complete. ∎

6. Interval-type Algorithm and Numerical Examples

Since $L_n(\lambda)$ and $U_n(\lambda)$ are convex and piecewise linear continuous functions of λ, we can easily find the root the equations $L_n(\lambda) = 0$ and $U_n(\lambda) = 0$ by the bisection method in a finite number steps. For example, in order to find the roots of the equation $L_n(\lambda) = 0$, this method starts with an interval $[\beta^L, \beta^U]$ which contains the root λ_n^L of equation $L_n(\lambda) = 0$. We take the midpoint $\beta^M = (\beta^L + \beta^U)/2$ of the interval. Depending on whether $L_n(\beta^M) \geq 0$ or $L_n(\beta^M) < 0$, one considers the interval $[\beta^M, \beta^U]$ or the interval $[\beta^L, \beta^M]$ as the next interval containing λ_n^L. The more precise computational procedure is shown below.

Subroutine $ZERO(L_n)$ **(resp.** $ZERO(U_n)$**)**. Given any fixed $n \in \mathbb{N}$, find the roots of the equations $L_n(\lambda) = 0$ and $U_n(\lambda) = 0$.

- **Step 1**. Set $\beta^L = \mu/\xi$ and $\beta^U = \eta^L$ (resp. $\beta^U = \eta_n^U$).
- **Step 2**. Calculate

  $$\hat{\beta} = \beta^L - \frac{L_n(\beta^L)\cdot(\beta^L - \beta^U)}{L_n(\beta^L) - L_n(\beta^U)} \text{ (resp. } \hat{\beta} = \beta^L - \frac{U_n(\beta^L)\cdot(\beta^L - \beta^U)}{U_n(\beta^L) - U_n(\beta^U)}\text{)}.$$

  If $L_n(\hat{\beta}) = 0$ (resp. $U_n(\hat{\beta}) = 0$) then STOP and return $\lambda_n^L = \hat{\beta}$ (resp. $\lambda_n^U = \hat{\beta}$) as the root. Otherwise, set $\beta^M \leftarrow (\beta^L + \beta^U)/2$ and go to Step 3.
- **Step 3**. If $L_n(\beta^M) > 0$ (resp. $U_n(\beta^M) > 0$), then set $\beta^L \leftarrow \beta^M$, $\beta^U \leftarrow \hat{\beta}$ and go to Step 2. Otherwise, set $\beta^U \leftarrow \beta^M$ and go to Step 2.

According to Theorem 5.1, we are in a position to provide a computational procedure to obtain the approximate solution of (CLFP). For $n > d/\xi$, we define

$$\omega_n = \frac{1}{\xi}\left[\frac{d}{n}(1 + \eta_n^U) + r_n\right], \tag{77}$$

where d, r_n and η_n^U are defined in (54), (55) and (57), respectively. By (69), we have

$$0 \leq Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) \leq \omega_n.$$

Suppose that the error tolerance ϵ is pre-determined by the decision-makers. By calculating ω_n according to (77), we can determine the natural number $n \in \mathbb{N}$ such that

$$\omega_n \leq \epsilon \text{ and } n > \frac{d}{\xi},$$

which also says that

$$0 \leq Er\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right) \leq \epsilon.$$

This also means that the corresponding approximate solution $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ is acceptable, since the error tolerance ϵ is attained. Now, the main computational procedure is given below.

Main Program.

- **Step 1**. Set the error tolerance ϵ and the initial number n such that $n > d/\xi$.
- **Step 2**. Evaluate ω_n as defined in (77).
- **Step 3**. If $\omega_n > \epsilon$ then set $n \leftarrow n+1$ and go to Step 2; otherwise go to Step 4.
- **Step 4**. Find the roots λ_n^L and λ_n^U by invoking the subroutine $ZERO(L_n)$ and $ZERO(U_n)$ described above. Set $\lambda_n^* \leftarrow \frac{1}{2}(\lambda_n^L + \lambda_n^U)$.
- **Step 5**. Find the optimal solution of finite-dimensional linear programming problem $(\Psi\text{P}_n^{(\lambda_n^*)})$ using well-known efficient algorithms. Use this optimal solution to construct the natural solution $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ according to (20). Evaluate the error bound $Er(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t))$ defined in (68).
- **Step 6**. Return $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ as an approximate optimal solution of the original problem (CLFP) with error bound $Er(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t))$.

We have to mention that the evaluations of Step 2 are independent of Step 4 and Step 5, i.e., we can estimate the rough error bound ω_n of the desired approximate solution $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ without using the results of Step 4. It also means that we can save the computational time, since the main successive iterations occur in Steps 1-3, where the workload does not need the heavy computation.

In the sequel, we provide two numerical examples to demonstrate the usefulness of the numerical method established in this paper. For the given error tolerance ϵ, we first determine $n \in \mathbb{N}$ such that $\omega_n \leq \epsilon$ by using Steps 1-3. And then, by Steps 4-6, we can obtain the corresponding approximate solution $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)$ of (CLFP) with error bound $Er(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)) \leq \omega_n \leq \epsilon$.

Example 6.1. We consider the following problem

$$\text{maximize} \quad \frac{\dfrac{1}{3}+\displaystyle\int_0^1 \left[\ln\left(t+\dfrac{1}{2}\right)\cdot x_1(t)+t^2\cdot x_2(t)\right]dt}{\dfrac{1}{2}+\displaystyle\int_0^1 [\cos(t)\cdot x_1(t)+\sin(1-t)\cdot x_2(t)]\,dt}$$

$$\text{subject to} \quad 6\,x_1(t) \le t + \int_0^t [x_1(s)+2\,x_2(s)]\,ds \text{ for all } t\in[0,1]$$

$$5\,x_2(t) \le 2\,t + \int_0^t [3\,x_1(s)+x_2(s)]\,ds \text{ for all } t\in[0,1]$$

$$x_j(t)\in L_+^\infty[0,1] \text{ for } j=1,2.$$

Using the proposed computational procedure, the numerical results are shown in the following table.

ϵ	λ_n^L	λ_n^U	λ_n^*	$\theta\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)$	$Er(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t))$
0.05	0.810441805	0.832076481	0.821259143	0.810439836	0.021636646
0.01	0.810510648	0.815905402	0.813208025	0.810510531	0.005394871
0.005	0.810522108	0.813218329	0.811870219	0.810522073	0.002696256
0.001	0.810530704	0.811204541	0.810867622	0.810530700	0.000673841
0.0005	0.810532136	0.810869037	0.810700586	0.810532135	0.000336902
0.0001	0.810533389	0.810575500	0.810554444	0.810533389	0.000042111

The approximate optimal solution $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t) = (\bar{x}_1^{(n,\lambda_n^*)}(t), \bar{x}_2^{(n,\lambda_n^*)}(t))^\top$ of the above problem can be obtained by (20) with approximate objective value $\theta\left(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t)\right)$ and error $Er(\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t))$.

Example 6.2. We consider the following problem

$$\text{maximize} \quad \frac{\dfrac{1}{3}+\displaystyle\int_0^1 \left[\ln\left(t+\dfrac{1}{2}\right)\cdot x_1(t)+t^2\cdot x_2(t)+x_3(t)\right]dt}{\dfrac{1}{2}+\displaystyle\int_0^1 [\cos(t)\cdot x_1(t)+\sin(1-t)\cdot x_2(t)+t\cdot x_3(t)]\,dt}$$

$$\text{subject to} \quad 7\,x_1(t) \le t + \int_0^t [x_1(s)+x_2(s)]\,ds \text{ for all } t\in[0,1]$$

$$5\,x_2(t) \le 3\,t + \int_0^t [2\,x_2(s)+x_3(s)]\,ds \text{ for all } t\in[0,1]$$

$$6\,x_3(t) \le 2\,t + \int_0^t [x_1(s)+x_2(s)+3\,x_3(s)]ds \text{ for all } t\in[0,1]$$

$$x_j(t)\in L_+^\infty[0,1] \text{ for } j=1,2,3.$$

Using the proposed computational procedure, the numerical results are shown in the following table.

ϵ	λ_n^L	λ_n^U	λ_n^*	$\theta\left(\bar{x}^{(n,\lambda_n^*)}(t)\right)$	$Er(\bar{x}^{(n,\lambda_n^*)}(t))$
0.05	1.013402519	1.037701512	1.025552015	1.013352019	0.024349493
0.01	1.013682268	1.016691382	1.015186825	1.013681550	0.003009832
0.005	1.013702240	1.015205787	1.014454014	1.013702048	0.001503740
0.001	1.013717220	1.014092917	1.013905069	1.013717209	0.000375709
0.0005	1.013719717	1.013907549	1.013813633	1.013719714	0.000187835
0.0001	1.013721589	1.013768544	1.013745067	1.013721589	0.000046955

The approximate optimal solution $\bar{\mathbf{x}}^{(n,\lambda_n^*)}(t) = (\bar{x}_1^{(n,\lambda_n^*)}(t), \bar{x}_2^{(n,\lambda_n^*)}(t), \bar{x}_3^{(n,\lambda_n^*)}(t))^\top$ of the above problem can be obtained by (20) with approximate objective value $\theta\left(\bar{x}^{(n,\lambda_n^*)}(t)\right)$ and error $Er(\bar{x}^{(n,\lambda_n^*)}(t))$.

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