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THE HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENT AND THEIR APPLICATIONS

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Abstract. In this paper, a certain Herz-type Hardy spaces with variable exponent are introduced, and characterizations of these spaces are established in terms of atomic and molecular decompositions. Using these decompositions, the authors obtain the boundedness of some operators on the Herz-type Hardy spaces with variable exponent.

1. Introduction

In recent years, the theory of function spaces with variable exponents has developed since the paper [8] of Kováčik and Rákosník appeared in 1991. Lebesgue and Sobolev spaces with integrability exponent have been extensively investigated, see [5] and the references therein. Many applications of these spaces were given, see [6]. Very recently, Izuki [7] introduced the Herz spaces with variable exponent and proved the boundedness of some sublinear operators on these spaces.

Inspired by [9, 10], we introduce a certain Herz-type Hardy spaces with variable exponent which is a generalization of classical Herz-type Hardy spaces, and establish the atomic and molecular decompositions. Using these decompositions, we obtain the boundedness of some operators on the Herz-type Hardy spaces with variable exponent.

To be precise, we first briefly recall some standard notations in the remainder of this section. In Section 2, we will define the Herz-type Hardy spaces with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, and give their atomic characterizations. In Section 3, we will present the molecular characterizations of $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \to [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

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This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is a constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The variable L^p spaces are a special case of Musielak-Orlicz spaces. For all compact subsets $E \subset \Omega$, the space $L^{p(\cdot)}_{\rm loc}(\Omega)$ is defined by $L^{p(\cdot)}_{\rm loc}(\Omega) := \{f : f \in L^{p(\cdot)}(E)\}$. Define $\mathcal{P}(\Omega)$ to be set of $p(\cdot) : \Omega \to [1, \infty)$ such that

$$1 < p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} \le \operatorname{ess\,sup}\{p(x) : x \in \Omega\} = p^+ < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$.

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. There exist some sufficient conditions on $p(\cdot)$ such that the maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, see [1-4, 11, 12]. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ as the set of positive integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_{A_k} is the characteristic function of A_k.

Definition 1.1. ([7]). Let $\alpha \in \mathbb{R}, 0 < p \le \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p} \|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

In the proof of our main result, we will use the following lemmas.

Lemma 1.1. ([8]). *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. *If* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *and* $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, *then* fg *is integrable on* $\mathbb{R}^n$ *and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2. ([7]). *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then there exists a positive constant* C *such that for all balls* B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}$$

$$\text{and} \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 *are constants with* $0 < \delta_1, \delta_2 < 1$.

Throughout this paper δ_2 is the same as in Lemma 1.2.

Lemma 1.3. ([7]). *Suppose* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then there exists a constant* $C > 0$ *such that for all balls* B *in* $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

In [13], we establish the following boundedness theorem on the Herz spaces with variable exponent for a class of sublinear operators.

Lemma 1.4. ([13]). *Let* $0 < \alpha < n\delta_2, 0 < p < \infty$ *and* $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *If a sublinear operator* T *satisfies*

$$(1.1) \qquad |Tf(x)| \le C\|f\|_1/|x|^n, \quad \text{if} \quad \text{dist}(x, \text{supp}f) > |x|/2,$$

for any integrable function f *with a compact support and* T *is bounded on* $L^{q(\cdot)}(\mathbb{R}^n)$, *then* T *is bounded on* $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, *respectively.*

2. The Atomic Characterizations and Their Applications

In this section, we will give the definition of Herz-type Hardy spaces with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N f(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N f(x) = \sup_{\phi \in \mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \le N} |x^\alpha D^\beta \phi(x)| \le 1\}$ and $N > n+1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 2.1. Let $\alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n+1$.

(i) The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Herz-type Hardy space with variable exponent $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

It is obvious that $G_N f$ satisfies (1.1). Thus, by Lemma 1.4, we can easily prove that if $0 < \alpha < n\delta_2, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \cap L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) = \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$$

and

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \cap L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n) = K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n).$$

If $n\delta_2 \le \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \cap L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \subsetneqq \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$$

and

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \cap L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) \subsetneqq K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n).$$

Thus we are interested in the case $\alpha \geq n\delta_2$. In this case, we establish characterizations of the spaces $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ in terms of central atomic decompositions. For $x \in \mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x.

Definition 2.2. Let $n\delta_2 \leq \alpha < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and non-negative integer $s \geq [\alpha - n\delta_2]$.

(i) A function a on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom, if it satisfies

(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq |B(0,r)|^{-\alpha/n}$.

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, |\beta| \leq s$.

(ii) A function a on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type, if it satisfies the conditions (2), (3) above and

$(1)'$ $\operatorname{supp} a \subset B(0,r), r \geq 1$.

Remark 2.1. If $q(x) = q$ is a constant, then taking $\delta_2 = 1 - 1/q$ we can get the classical case.

Theorem 2.1. *Let $n\delta_2 \leq \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have* (i) $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n), \tag{2.1}$$

where each a_k is a central $(\alpha, q(\cdot))$-atom with support contained in B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

(ii) $f \in HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *if and only if*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha, q(\cdot))$-atom of restricted type with support contained in B_k and $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

Proof. We only prove (i), and (ii) can be proved in the similar way. To prove the necessity, choose $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi \geq 0, \int_{\mathbb{R}^n} \phi(x)dx = 1, \operatorname{supp}\phi \subset \{x : |x| \leq 1\}$. For $j \in \mathbb{Z}_+$, let

$$\phi_{(j)}(x) = 2^{jn}\phi(2^j x).$$

For each $f \in \mathcal{S}'(\mathbb{R}^n)$, set

$$f^{(j)}(x) = f * \phi_{(j)}(x).$$

It is obvious that $f^{(j)} \in C^\infty(\mathbb{R}^n)$ and $\lim_{j\to\infty} f^{(j)} = f$ in the sense of distribution. Let ψ be a radial smooth function such that $\operatorname{supp}\psi \subset \{x : 1/2 - \varepsilon \leq |x| \leq 1 + \varepsilon\}$ with $0 < \varepsilon < 1/4, \psi(x) = 1$ for $1/2 \leq |x| \leq 1$. Let $\psi_k(x) = \psi(2^{-k}x)$ for $k \in \mathbb{Z}$ and

$$\tilde{A}_{k,\varepsilon} = \{x : 2^{k-1} - 2^k\varepsilon \leq |x| \leq 2^k + 2^k\varepsilon\}.$$

Observe that $\operatorname{supp}\psi_k \subset \tilde{A}_{k,\varepsilon}$ and $\psi_k(x) = 1$ for $x \in A_k = \{x : 2^{k-1} < |x| \leq 2^k\}$. Obviously, $1 \leq \sum_{k=-\infty}^{\infty} \psi_k(x) \leq 2$, $|x| > 0$. Let

$$\Phi_k(x) = \begin{cases} \psi_k(x) \Big/ \sum_{l=-\infty}^{\infty} \psi_l(x), & x \neq 0, \\ 0, \quad x = 0, \end{cases}$$

then $\sum_{k=-\infty}^{\infty} \Phi_k(x) = 1$ for $x \neq 0$. For some $m \in \mathbb{N}$, we denote by $\mathcal{P}_m$ the class of all the real polynomials with the degree less than m. Let $P_k^{(j)}(x) = P_{\tilde{A}_{k,\varepsilon}}(f^{(j)}\Phi_k)(x)\chi_{\tilde{A}_{k,\varepsilon}}(x) \in \mathcal{P}_m(\mathbb{R}^n)$ be the unique polynomial satisfying

$$\int_{\tilde{A}_{k,\varepsilon}} \left(f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)\right) x^\beta dx = 0, \ \ |\beta| \leq m = [\alpha - n\delta_2].$$

Write

$$f^{(j)}(x) = \sum_{k=-\infty}^{\infty} \left(f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)\right) + \sum_{k=-\infty}^{\infty} P_k^{(j)}(x) = \sum_I^{(j)} + \sum_{II}^{(j)}.$$

For the term $\sum_I^{(j)}$, let $g_k^{(j)}(x) = f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)$ and $a_k^{(j)}(x) = g_k^{(j)}(x)/\lambda_k$, where $\lambda_k = b|B_{k+1}|^{\alpha/n} \sum_{l=k-1}^{k+1} \|(G_N f)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}$, and b is a constant which will be chosen later. Note that $\operatorname{supp} a_k^{(j)} \subset B_{k+1}$, $\sum_I^{(j)} = \sum_{k=-\infty}^{\infty} \lambda_k a_k^{(j)}(x)$.

Now we estimate $\|g_k^{(j)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$. To do this, let $\{\phi_d^k : |d| \le m\}$ be the orthogonal polynomials restricted to $\tilde{A}_{k,\varepsilon}$ with respect to the weight $1/|\tilde{A}_{k,\varepsilon}|$, which are obtained from $\{x^\beta : |\beta| \le m\}$ by Gram-Schmidt's method, that is

$$\langle \phi_\nu^k, \phi_\mu^k \rangle = \frac{1}{|\tilde{A}_{k,\varepsilon}|} \int_{\tilde{A}_{k,\varepsilon}} \phi_\nu^k(x)\phi_\mu^k(x)dx = \delta_{\nu\mu}.$$

It is readily to see that $P_k^{(j)}(x) = \sum_{|d|\le m} \langle f^{(j)}\Phi_k, \phi_d^k \rangle \phi_d^k(x)$ for $x \in \tilde{A}_{k,\varepsilon}$. On the other hand, from $\frac{1}{|\tilde{A}_{k,\varepsilon}|}\int_{\tilde{A}_{k,\varepsilon}} \phi_\nu^k(x)\phi_\mu^k(x)dx = \delta_{\nu\mu}$ we infer that

$$\frac{1}{|\tilde{A}_{1,\varepsilon}|} \int_{\tilde{A}_{1,\varepsilon}} \phi_\nu^k(2^{k-1}y)\phi_\mu^k(2^{k-1}y)dy = \delta_{\nu\mu}.$$

We can get $\phi_\nu^k(2^{k-1}y) = \phi_\nu^1(y)\ a.e.$. That is $\phi_\nu^k(x) = \phi_\nu^1(2^{1-k}x)\ a.e.$ for $x \in \tilde{A}_{k,\varepsilon}$. Thus $|\phi_\nu^k(x)| \le C$, and for $x \in \tilde{A}_{k,\varepsilon}$, by the generalized Hölder inequality we have

$$\begin{aligned}|P_k^{(j)}(x)| &\le \frac{C}{|\tilde{A}_{k,\varepsilon}|} \int_{\tilde{A}_{k,\varepsilon}} |f^{(j)}(x)\Phi_k(x)|dx \\ &\le \frac{C}{|\tilde{A}_{k,\varepsilon}|} \|f^{(j)}\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, by Lemma 1.3 we have

$$\begin{aligned}&\|g_k^{(j)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le \|f^{(j)}\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \|P_k^{(j)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le \|f^{(j)}\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \frac{C}{|\tilde{A}_{k,\varepsilon}|}\|f^{(j)}\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le \|f^{(j)}\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C\|f^{(j)}\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\|(f * \phi_{(j)})\Phi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C' \sum_{l=k-1}^{k+1} \|(G_N f)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Choose $b = C'$, then $\|a_k^{(j)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B_{k+1}|^{-\alpha/n}$ and each $a_k^{(j)}$ is a central $(\alpha, q(\cdot))$-atom with support contained in B_{k+1}. Furthermore,

$$\sum_{k=-\infty}^{\infty} |\lambda_k|^p \le C \sum_{k=-\infty}^{\infty} |B_{k+1}|^{p\alpha/n} \left(\sum_{l=k-1}^{k+1} \|(G_N f)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \le C\|G_N f\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)},$$

where C is independent of j and f.

It remains to estimate $\sum_{II}^{(j)}$. Let $\{\psi_d^k : |d| \le m\}$ be the dual basis of $\{x^\beta : |\beta| \le m\}$ with respect to the weight $1/|\tilde{A}_{k,\varepsilon}|$ on $\tilde{A}_{k,\varepsilon}$, that is

$$\langle \psi_d^k, x^\beta \rangle = \frac{1}{|\tilde{A}_{k,\varepsilon}|} \int_{\tilde{A}_{k,\varepsilon}} x^\beta \psi_d^k(x) dx = \delta_{\beta d}.$$

Similar to the method of [9], let

$$h_{k,d}^{(j)}(x) = \sum_{l=-\infty}^{k} \left(\frac{\psi_d^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} - \frac{\psi_d^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|} \right) \int_{\mathbb{R}^n} f^{(j)}(x)\Phi_l(x)x^d dx.$$

We can write

$$\begin{aligned}
\sum_{II}^{(j)} &= \sum_{k=-\infty}^{\infty} \sum_{|d|\le m} \langle f^{(j)}\Phi_k, x^d \rangle \psi_d^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x) \\
&= \sum_{|d|\le m} \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}^n} f^{(j)}\Phi_k x^d dx \right) \frac{\psi_d^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} \\
&= \sum_{|d|\le m} \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k} \int_{\mathbb{R}^n} f^{(j)}(x)\Phi_l(x)x^d dx \right) \\
&\quad \times \left(\frac{\psi_d^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} - \frac{\psi_d^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|} \right) \\
&= \sum_{|d|\le m} \sum_{k=-\infty}^{\infty} \alpha_{k,d} h_{k,d}^{(j)}(x)/\alpha_{k,d} = \sum_{|d|\le m} \sum_{k=-\infty}^{\infty} \alpha_{k,d} a_{k,d}^{(j)}(x),
\end{aligned}$$

where

$$\alpha_{k,d} = \tilde{b} \sum_{l=k-1}^{k+2} \|(G_N f)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_{k+2}|^{\alpha/n}$$

and $\tilde{b}$ is a constant which will be chosen later. Note that

$$\int_{\mathbb{R}^n} \sum_{l=-\infty}^{k} |\Phi_l(x)x^d| dx = \sum_{l=-\infty}^{k} \int_{\tilde{A}_{k,\varepsilon}} |\Phi_l(x)x^d| dx \le C2^{k(n+|d|)}.$$

By a computation we have

$$\left| \int_{\mathbb{R}^n} f^{(j)}(y) \sum_{l=-\infty}^{k} \Phi_l(y) y^d dy \right| \le C2^{k(n+|d|)} G_N f(x), \ x \in B_{k+2}.$$

This together with the inequality that

$$\left(\frac{\psi_d^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|}-\frac{\psi_d^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|}\right)\le C2^{-k(n+|d|)}\sum_{l=k-1}^{k+1}\chi_l(x),$$

shows that

$$\|h_{k,d}^{(j)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C''\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Take $\tilde{b}=C''$. It is readily to verify that each $a_{k,d}^{(j)}$ is a central $(\alpha,q(\cdot))$-atom with support contained in $\tilde{A}_{k,\varepsilon}\cup\tilde{A}_{k+1,\varepsilon}\subset B_{k+2}$, and $\alpha_{k,d}=C''\sum_{l=k-1}^{k+2}\|(Gf)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_{k+2}|^{\alpha/n}$, where C'' is a constant independent of j,f,k and d. Moreover,

$$\begin{aligned}\sum_{k,d}|\alpha_{k,d}|^p&\le C\sum_{k=-\infty}^{\infty}|B_{k+2}|^{\alpha p/n}\left(\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\\&\le C\|G_Nf\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty,\end{aligned}$$

where C is independent of j and f.

Thus we obtain that

$$f^{(j)}(x)=\sum_{d=-\infty}^{\infty}\lambda_d a_d^{(j)}(x),$$

where each $a_d^{(j)}$ is a central $(\alpha,q(\cdot))$-atom with support contained in $\tilde{A}_{d,\varepsilon}\cup\tilde{A}_{d+1,\varepsilon}\subset B_{d+2}$, λ_d is independent of j and

$$\left(\sum_{d=-\infty}^{\infty}|\lambda_d|^p\right)^{1/p}\le C\|G_Nf\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty,$$

where C is independent of j and f.

Since

$$\sup_{j\in\mathbb{Z}_+}\|a_0^{(j)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le|B_2|^{-\alpha/n},$$

by the Banach-Alaoglu theorem we can obtain a subsequence $\{a_0^{(j_{n_0})}\}$ of $\{a_0^{(j)}\}$ converging in the weak* topology of $L^{q(\cdot)}(\mathbb{R}^n)$ to some $a_0\in L^{q(\cdot)}(\mathbb{R}^n)$. It is readily to verify that a_0 is a central $(\alpha,q(\cdot))$-atom supported on B_2. Next, since

$$\sup_{j_{n_0}\in\mathbb{Z}_+}\|a_1^{(j_{n_0})}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le|B_3|^{-\alpha/n},$$

another application of the Banach-Alaoglu theorem yields a subsequence $\{a_1^{(j_{n_1})}\}$ of $\{a_1^{(j_{n_0})}\}$ which converges weak* in $L^{q(\cdot)}(\mathbb{R}^n)$ to a central $(\alpha, q(\cdot))$-atom a_1 with support in B_3. Furthermore,

$$\sup_{j_{n_1}\in\mathbb{Z}_+} \|a_{-1}^{(j_{n_1})}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq |B_1|^{-\alpha/n}.$$

Similarly, there exists a subsequence $\{a_{-1}^{(j_{n_{-1}})}\}$ of $\{a_{-1}^{(j_{n_1})}\}$ which converges weak* in $L^{q(\cdot)}(\mathbb{R}^n)$ to some $a_{-1} \in L^{q(\cdot)}(\mathbb{R}^n)$, and a_{-1} is a central $(\alpha, q(\cdot))$-atom supported on B_1. Repeating the above procedure for each $d \in \mathbb{Z}$, we can find a subsequence $\{a_d^{(j_{n_d})}\}$ of $\{a_d^{(j)}\}$ converging weak* in $L^{q(\cdot)}(\mathbb{R}^n)$ to some $a_d \in L^{q(\cdot)}(\mathbb{R}^n)$ which is a central $(\alpha, q(\cdot))$-atom supported on B_{d+2}. By the usual diagonal method we obtain a subsequence $\{j_\nu\}$ of $\mathbb{Z}_+$ such that for each $d \in \mathbb{Z}$, $\lim_{\nu\to\infty} a_d^{(j_\nu)} = a_d$ in the weak* topology of $L^{q(\cdot)}(\mathbb{R}^n)$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$.

Now our proof is reduced to prove that

$$f = \sum_{d=-\infty}^{\infty} \lambda_d a_d, \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n). \tag{2.2}$$

For each $\varphi \in \mathcal{S}(\mathbb{R}^n)$, note that $\operatorname{supp} a_d^{(j_\nu)} \subset (\tilde{A}_{d,\varepsilon} \cup \tilde{A}_{d+1,\varepsilon}) \subset (A_{d-1} \cup A_d \cup A_{d+1} \cup A_{d+2})$. We have

$$\langle f, \varphi\rangle = \lim_{\nu\to\infty} \sum_{d=-\infty}^{\infty} \lambda_d \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)dx.$$

See [9] for the details.

Recall that $m = [\alpha - n\delta_2]$. If $d \leq 0$, then by Lemma 1.2 and the generalized Hölder inequality we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)dx\right| &= \left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\left(\varphi(x) - \sum_{|\beta|\leq m}\frac{D^\beta\varphi(0)}{\beta!}x^\beta\right)dx\right| \\
&\leq C\int_{\mathbb{R}^n} |a_d^{(j_\nu)}(x)|\cdot|x|^{m+1}dx \\
&\leq C2^{d(m+1)}\int_{\mathbb{R}^n} |a_d^{(j_\nu)}(x)|dx \\
&\leq C2^{d(m+1-\alpha)}\|\chi_{B_{d+2}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C2^{d(m+1-\alpha)}\left(\frac{|B_{d+2}|}{|B_2|}\right)^{\delta_2}\|\chi_{B_2}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{d(m+1-\alpha+n\delta_2)}\frac{|B_2|}{|B_0|}\|\chi_{B_0}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{d(m+1-\alpha+n\delta_2)}\inf\left\{\lambda>0:\int_{B_0}\lambda^{-q'(x)}dx\leq 1\right\}\\
&\leq C2^{d(m+1-\alpha+n\delta_2)}\inf\left\{1\geq\lambda>0:\int_{B_0}\lambda^{-(q')^+}dx\leq 1\right\}\\
&= C2^{d(m+1-\alpha+n\delta_2)}|B_0|^{1/(q')^+}\\
&= C2^{d(m+1-\alpha+n\delta_2)},
\end{aligned}
$$

where C is independent of d.

If $d>0$, let $k_0\in\mathbb{Z}_+$ such that $k_0+\alpha-n>0$, then by Lemma 1.2 and the generalized Hölder inequality we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx\right| &\leq C\int_{\mathbb{R}^n}|a_d^{(j_\nu)}(x)||x|^{-k_0}dx\\
&\leq C2^{-d(k_0+\alpha)}\|\chi_{B_{d+2}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-d(k_0+\alpha)}\frac{|B_{d+2}|}{|B_0|}\|\chi_{B_0}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-d(k_0+\alpha-n)}\inf\left\{\lambda>0:\int_{B_0}\lambda^{-q'(x)}dx\leq 1\right\}\\
&\leq C2^{-d(k_0+\alpha-n)}\inf\left\{1\geq\lambda>0:\int_{B_0}\lambda^{-(q')^+}dx\leq 1\right\}\\
&= C2^{-d(k_0+\alpha-n)},
\end{aligned}
$$

where C is independent of d.

Let

$$
\mu_d=\begin{cases}|\lambda_d|2^{d(m+1-\alpha+n\delta_2)}, & d\leq 0,\\ |\lambda_d|2^{-d(k_0+\alpha-n)}, & d>0,\end{cases}
$$

Then

$$
\sum_{d=-\infty}^{\infty}|\mu_d|\leq C\left(\sum_{d=-\infty}^{\infty}|\lambda_d|^p\right)^{1/p}\leq C\|G_Nf\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty
$$

and

$$
|\lambda_d|\left|\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx\right|\leq C|\mu_d|,
$$

which implies that

$$
\langle f,\varphi\rangle=\sum_{d=-\infty}^{\infty}\lim_{\nu\to\infty}\lambda_d\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx=\sum_{d=-\infty}^{\infty}\lambda_d\int_{\mathbb{R}^n}a_d(x)\varphi(x)dx.
$$

This establishes the identity (2.2).

To prove the sufficiency, we consider the two cases $0 < p \le 1$ and $1 < p < \infty$.

If $0 < p \le 1$, it suffices to show that for each central $(\alpha, q(\cdot))$-atom a,

$$\|G_N a\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C,$$

with the constant $C > 0$ independent of a. For a fixed central $(\alpha, q(\cdot))$-atom a, with $\operatorname{supp} a(x) \subset B(0, 2^{k_0})$ for some $k_0 \in \mathbb{Z}$. Write

$$\begin{aligned}\|G_N a\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{k_0+3} |B_k|^{\alpha p/n} \|(G_N a)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \sum_{k=k_0+4}^{\infty} |B_k|^{\alpha p/n} \|(G_N a)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &= I + II.\end{aligned}$$

By the $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of the grand maximal operator G_N we have

$$I \le \|G_N a\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{k_0+3} |B_k|^{\alpha p/n} \le C\|a\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{k_0+3} |B_k|^{\alpha p/n} \le C.$$

To estimate II, we need a pointwise estimate for $G_N a(x)$ on A_k. Let $\phi \in \mathcal{A}_N, m \in \mathbb{N}$ such that $\alpha - n\delta_2 < m + 1$. Denote by P_m the m-th order Taylor series expansion. If $|x - y| < t$, then from the vanishing moment condition of a we have

$$\begin{aligned}|a * \phi_t(y)| &= t^{-n} \left| \int_{\mathbb{R}^n} a(z) \left(\phi\left(\frac{y-z}{t}\right) - P_m\left(\frac{y}{t}\right) \right) dz \right| \\ &\le Ct^{-n} \int_{\mathbb{R}^n} |a(z)| \left|\frac{z}{t}\right|^{m+1} (1 + |y - \theta z|/t)^{-(n+m+1)} dz \\ &\le C \int_{\mathbb{R}^n} |a(z)||z|^{m+1} (t + |y - \theta z|)^{-(n+m+1)} dz,\end{aligned}$$

where $0 < \theta < 1$. Since $x \in A_k$ for $k \ge k_0 + 4$, we have $|x| \ge 2 \cdot 2^{k_0+1}$. From $|x - y| < t$ and $|z| < 2^{k_0+1}$, we have

$$t + |y - \theta z| \ge |x - y| + |y - \theta z| \ge |x| - |z| \ge |x|/2.$$

Thus,

$$\begin{aligned}|a * \phi_t(y)| &\le C \int_{\mathbb{R}^n} |a(z)||z|^{m+1} (|x - y| + |y - \theta z|)^{-(n+m+1)} dz \\ &\le C 2^{k_0(m+1)} |x|^{-(n+m+1)} \int_{\mathbb{R}^n} |a(z)| dz \\ &\le C 2^{k_0(m+1)} |x|^{-(n+m+1)} |B_{k_0}|^{-\alpha/n} \|\chi_{B_{k_0}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, we have

$$G_N a(x) \le C 2^{k_0(m+1)-k(n+m+1)} |B_{k_0}|^{-\alpha/n} \|\chi_{B_{k_0}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}, \ x \in A_k, \ \text{and}\, k \ge k_0 + 4.$$

So by Lemma 1.2 and Lemma 1.3 we have

$$\begin{aligned} II &\le C \sum_{k=k_0+4}^{\infty} 2^{p[k_0(m+1)-k(n+m+1)]} \left(\frac{|B_k|}{|B_{k_0}|}\right)^{p\alpha/n} \\ &\quad \times \|\chi_{B_{k_0}}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C \sum_{k=k_0+4}^{\infty} 2^{p[k_0(m+1)-k(n+m+1)]} \left(\frac{|B_k|}{|B_{k_0}|}\right)^{p\alpha/n} \\ &\quad \times \|\chi_{B_{k_0}}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)} \left(|B_k| \|\chi_{B_k}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\right)^p \\ &= C \sum_{k=k_0+4}^{\infty} 2^{p(k_0-k)(m+1-\alpha)} \left(\frac{\|\chi_{B_{k_0}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \\ &\le C \sum_{k=k_0+4}^{\infty} 2^{p(k_0-k)(m+1-\alpha+n\delta_2)} \le C. \end{aligned}$$

This proves the desired estimate for the case $0 < p \le 1$.

If $1 < p < \infty$, write

$$\begin{aligned} \|G_N f\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\le \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=-\infty}^{\infty} |\lambda_l| \|(G_N a_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|a_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\quad + C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \|(G_N a_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &= III + IV. \end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned} III &\le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=k-1}^{\infty} |\lambda_l| |B_l|^{-\alpha/n}\right)^p \\ &\le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=k-1}^{\infty} |\lambda_l|^p |B_l|^{-\alpha p/(2n)}\right) \left(\sum_{l=k-1}^{\infty} |B_l|^{-\alpha p'/(2n)}\right)^{p/p'} \\ &\le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/(2n)} \sum_{l=k-1}^{\infty} |\lambda_l|^p |B_l|^{-\alpha p/(2n)} \\ &\le C \sum_{l=-\infty}^{\infty} |\lambda_l|^p. \end{aligned}$$

Now suppose $\alpha - n\delta_2 < m+1$. As in the argument for II, we can obtain that

$$
\begin{aligned}
& IV \\
& \le C \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| 2^{l(m+1)-k(n+m+1)} \left(\frac{|B_k|}{|B_l|} \right)^{\alpha/n} \|\chi_{B_l}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\
& \le C \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| 2^{(l-k)(m+1-\alpha+n\delta_2)} \right)^p \\
& \le C \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l|^p 2^{(l-k)(m+1-\alpha+n\delta_2)p/2} \right) \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)(m+1-\alpha+n\delta_2)p'/2} \right)^{p/p'} \\
& \le C \sum_{l=-\infty}^{\infty} |\lambda_l|^p.
\end{aligned}
$$

This finishes the proof of Theorem 2.1.

Remark 2.2. If $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$, we can replace $f^{(j)}$ by f in the proof of the necessity, and have

$$
f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x) + \sum_{k=-\infty}^{\infty} \mu_k b_k(x),
$$

where

$$
\begin{gathered}
\|a_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,2^{k+1})|^{-\alpha/n},\ \|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,2^{k+2})|^{-\alpha/n}, \\
\operatorname{supp} a_k \subset \tilde{A}_{k,\varepsilon},\ \operatorname{supp} b_k \subset \tilde{A}_{k,\varepsilon} \cup \tilde{A}_{k+1,\varepsilon},
\end{gathered}
$$

and

$$
0 \le \lambda_k, \mu_k \le C 2^{\alpha k} \sum_{j=k-1}^{k+1} \|G_N(f)\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)},
$$

with $N > n + \alpha + 1$.

Remark 2.3. For the case $0 < p \le 1$, if we remove the condition $\operatorname{supp} a_k \subset B_k$, then the conclusion of Theorem 2.1 is also true .

As an application of the atomic decomposition theorems, we shall extend Lemma 1.4 to the case of $\alpha \ge n\delta_2$.

Theorem 2.2. *Let* $n\delta_2 \le \alpha < \infty, 0 < p < \infty, q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *and the integer* $s = [\alpha - n\delta_2]$. *If a sublinear operator* T *satisfies that*

(i) T *is bounded on* $L^{q(\cdot)}(\mathbb{R}^n)$*;*

(ii) there exists a constant $\delta > 0$ such that $s + \delta > \alpha - n\delta_2$, and for any compact support function f with

$$\int_{\mathbb{R}^n} f(x)x^\beta \, dx = 0, \ \ |\beta| \le s,$$

Tf satisfies the size condition

$$\begin{aligned} |Tf(x)| &\le C(\text{diam}(\text{supp}\, f))^{s+\delta}|x|^{-(n+s+\delta)}\|f\|_1, \\ &\quad \text{if} \quad \text{dist}(x, \text{supp} f) \ge |x|/2. \end{aligned} \tag{2.3}$$

Then T can be extended to be a bounded operator from $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or bounded from $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ to $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$).

Proof. It suffices to prove homogeneous case. Suppose $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. By Theorem 2.1, $f = \sum_{j=-\infty}^{\infty} \lambda_j b_j$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each b_j is a central $(\alpha, q(\cdot))$-atom with support contained in B_j and

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p}.$$

Therefore, we get

$$\begin{aligned} \|Tf\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\Bigg[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Bigg(\sum_{j=-\infty}^{k-2} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^p \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Bigg(\sum_{j=k-1}^{\infty} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^p\Bigg] \\ &= C(I_1 + I_2). \end{aligned}$$

Let us first estimate I_1. By (2.3) and the generalized Hölder inequality, we get

$$\begin{aligned} |Tb_j(x)| &\le C|x|^{-(n+s+\delta)}2^{j(s+\delta)} \int_{B_j} |b_j(y)|dy \\ &\le C2^{-k(n+s+\delta)}2^{j(s+\delta)}\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{j(s+\delta-\alpha)-k(s+\delta+n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

So by Lemma 1.1 and Lemma 1.2, we have

$$
\begin{aligned}
&\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{j(s+\delta-\alpha)-k(s+\delta+n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{j(s+\delta-\alpha)-k(s+\delta)}2^{-kn}\big(|B_k|\|\chi_{B_k}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\big)\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&= C2^{j(s+\delta-\alpha)-k(s+\delta)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\le C2^{(s+\delta+n\delta_2)(j-k)-j\alpha}.
\end{aligned}
\tag{2.4}
$$

Therefore, when $0<p\le 1$, by $n\delta_2\le\alpha<s+\delta+n\delta_2$, we get

$$
\begin{aligned}
I_1 &= \sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^p2^{[(s+\delta+n\delta_2)(j-k)-j\alpha]p}\Big)\\
&= C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{(j-k)(s+\delta+n\delta_2-\alpha)p}\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
\tag{2.5}
$$

When $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_2\le\alpha<s+\delta+n\delta_2$, by (2.4) and the Hölder inequality, we have

$$
\begin{aligned}
I_1 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(s+\delta+n\delta_2)(j-k)-j\alpha}\Big)^p\\
&\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^p2^{(j-k)(s+\delta+n\delta_2-\alpha)p/2}\Big)\\
&\qquad\times\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(s+\delta+n\delta_2-\alpha)p'/2}\Big)^{p/p'}\\
&\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^p2^{(j-k)(s+\delta+n\delta_2-\alpha)p/2}\Big)\\
&= C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{(j-k)(s+\delta+n\delta_2-\alpha)p/2}\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
\tag{2.6}
$$

Let us now estimate I_2. When $0<p\le 1$, by $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T, we

have

$$
\begin{aligned}
I_2 &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p\|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big) \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p|B_j|^{-\alpha p/n}\Big) \\
&= C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+1}2^{(k-j)\alpha p} \le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
\tag{2.7}
$$

When $1 < p < \infty$, by $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T and the Hölder inequality, we have

$$
\begin{aligned}
I_2 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p\|(Tb_j)\chi_k\|^{p/2}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big) \\
&\quad\times\Big(\sum_{j=k-1}^{\infty}\|(Tb_j)\chi_k\|^{p'/2}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p/p'} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p\|b_j\|^{p/2}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\Big(\sum_{j=k-1}^{\infty}\|b_j\|^{p'/2}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p/p'} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p|B_j|^{-\alpha p/(2n)}\Big)\Big(\sum_{j=k-1}^{\infty}|B_j|^{-\alpha p'/(2n)}\Big)^{p/p'} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p/2}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p|B_j|^{-\alpha p/(2n)}\Big) \\
&= C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+1}2^{(k-j)\alpha p/2} \le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
\tag{2.8}
$$

Combining (2.5)-(2.8), we have

$$\|Tf\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Thus, the proof of Theorem 2.2 is completed.

Furthermore, we will consider the Calderón-Zygmund operator T of Coifman and Meyer with associated standard kernel K in the sense of

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \notin \operatorname{supp} f,$$

which is a $L^{q(\cdot)}(\mathbb{R}^n)$-bounded operator. See [2] for more details.

Theorem 2.3. *Suppose that T is an above Calderón-Zygmund operator, and that $0 < \delta \leq 1$ is the constant associated with the standard kernel K, then for $n\delta_2 \leq \alpha < n\delta_2 + \delta$ and $0 < p < \infty$, T is bounded from $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

Proof. Note that $n\delta_2 \leq \alpha < n\delta_2 + \delta$ implies that $s = [\alpha - n\delta_2] = 0$. Thus, the operator T considered here satisfies (2.3) with $s = 0$. The desired conclusion follows from Theorem 2.2 directly.

3. The Molecular Characterizations and Their Applications

In this section, we will consider the molecular decomposition of the Herz-type Hardy spaces with variable exponent. We first give the notation of molecule.

Definition 3.1. Let $n\delta_2 \leq \alpha < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $s \geq [\alpha - n\delta_2]$ be a non-negative integer. Set $\varepsilon > \max\{s/n, \alpha/n - \delta_2\}, a = \delta_2 - \alpha/n + \varepsilon$ and $b = \delta_2 + \varepsilon$. A function $M_l \in L^{q(\cdot)}(\mathbb{R}^n)$ with $l \in \mathbb{Z}$ (or $l \in \mathbb{N}$) is said to be a dyadic central $(\alpha, q(\cdot); s, \varepsilon)_l$-molecule (or dyadic central $(\alpha, q(\cdot); s, \varepsilon)_l$)-molecule of restricted type) if it satisfies

(1) $\|M_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq 2^{-l\alpha}$.

(2) $\mathcal{R}_{q(\cdot)}(M_l) = \|M_l\|^{a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} \left\| |\cdot|^{nb} M_l(\cdot) \right\|^{1-a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} < \infty$.

(3) $\int_{\mathbb{R}^n} M_l(x) x^\beta dx = 0$, for any β with $|\beta| \leq s$.

Definition 3.2. Let $n\delta_2 \leq \alpha < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $s \geq [\alpha - n\delta_2]$ be a non-negative integer. Set $\varepsilon > \max\{s/n, \alpha/n - \delta_2\}$, $a = \delta_2 - \alpha/n + \varepsilon$ and $b = \delta_2 + \varepsilon$.

(i) A function $M \in L^{q(\cdot)}(\mathbb{R}^n)$ is said to be a central $(\alpha, q(\cdot); s, \varepsilon)$-molecule if it satisfies

(1) $\mathcal{R}_{q(\cdot)}(M) = \|M\|^{a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} \left\| |\cdot|^{nb} M(\cdot) \right\|^{1-a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} < \infty$.

(2) $\int_{\mathbb{R}^n} M(x) x^\beta dx = 0$, for any β with $|\beta| \leq s$.

(ii) A function $M \in L^{q(\cdot)}(\mathbb{R}^n)$ is said to be a central $(\alpha, q(\cdot); s, \varepsilon)$-molecule of restricted type if it satisfies (1), (2) in (i) and

(3) $\|M\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq 1$.

The following lemma proves that molecule is a generalization of atom.

Lemma 3.1. *Let $n\delta_2 \leq \alpha < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $s \geq [\alpha - n\delta_2]$ be a non-negative integer, $\varepsilon > \max\{s/n, \alpha/n - \delta_2\}$, $a = \delta_2 - \alpha/n + \varepsilon$ and $b = \delta_2 + \varepsilon$. If M is a central $(\alpha, q(\cdot))$-atom (or $(\alpha, q(\cdot))$-atom of restricted type), M is also a central $(\alpha, q(\cdot); s, \varepsilon)$-molecule (or $(\alpha, q(\cdot); s, \varepsilon)$-molecule of restricted type) such that $\mathcal{R}_{q(\cdot)}(M) \leq C$ with C independent of M.*

Proof. We only need consider the case that a is a $(\alpha, q(\cdot))$-atom with support on a ball $B(0,r)$. A straightforward computation leads to that

$$\|M\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{a/b} \left\| |\cdot|^{nb} M(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{1-a/b} \leq r^{nb(1-a/b)} \|M\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C r^{\alpha} r^{-\alpha} \leq C.$$

Now we give the molecular decomposition of the Herz-type Hardy spaces with variable exponent.

Theorem 3.1. *Let $n\delta_2 \leq \alpha < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $s \geq [\alpha - n\delta_2]$ be a non-negative integer. Set $\varepsilon > \max\{s/n, \alpha/n - \delta_2\}, a = \delta_2 - \alpha/n + \varepsilon$ and $b = \delta_2 + \varepsilon$. Then we have*

(i) *$f \in H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ if and only if f can be represented as*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each M_k is a dyadic central $(\alpha, q(\cdot); s, \varepsilon)_k$-molecule, and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|_{H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

(ii) *$f \in HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ if and only if*

$$f = \sum_{k=0}^{\infty} \lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each M_k is a dyadic central $(\alpha, q(\cdot); s, \varepsilon)_k$-molecule of restricted type, and $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|_{HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

Theorem 3.2. *Let $n\delta_2 \leq \alpha < \infty$, $0 < p \leq 1$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $s \geq [\alpha - n\delta_2]$ be a non-negative integer. Set $\varepsilon > \max\{s/n, \alpha/n - \delta_2\}, a = \delta_2 - \alpha/n + \varepsilon$ and $b = \delta_2 + \varepsilon$. Then we have*

(i) *$f \in H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ if and only if f can be represented as*

$$f = \sum_{k=1}^{\infty} \lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each M_k *is a central* $(\alpha, q(\cdot); s, \varepsilon)$*-molecule, and* $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$. *Moreover,*

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=1}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

(ii) $f \in HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *if and only if*

$$f = \sum_{k=1}^{\infty} \lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathrm{\mathbb{R}^n}),$$

where each M_k *is a central* $(\alpha, q(\cdot); s, \varepsilon)$*-molecule of restricted type, and* $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$. *Moreover,*

$$\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=1}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

By Theorem 2.1, Remark 2.2 and Lemma 3.1, we know that Theorem 3.1 and Theorem 3.2 can be obtained from the following lemma.

Lemma 3.2. *Let* $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $s \ge [\alpha - n\delta_2]$ *be a non-negative integer,* $\varepsilon > \max\{s/n, \alpha/n - \delta_2\}$, $a = \delta_2 - \alpha/n + \varepsilon$ *and* $b = \delta_2 + \varepsilon$.

(i) *If* $0 < p \le 1$, *there exists a constant* C *such that for any central* $(\alpha, q(\cdot); s, \varepsilon)$*-molecule (or* $(\alpha, q(\cdot); s, \varepsilon)$*-molecule of restricted type)* M,

$$\|M\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C \, (\text{or } \|M\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C).$$

(ii) *There exists a constant* C *such that for any* $l \in \mathbb{Z}(\text{or } l \in \mathbb{N})$ *and dyadic central* $(\alpha, q(\cdot); s, \varepsilon)_l$*-molecule (or dyadic central* $(\alpha, q(\cdot); s, \varepsilon)_l$*-molecule of restricted type)* M_l,

$$\|M_l\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C \, (\text{or } \|M_l\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C).$$

Proof. We only prove (i) for homogeneous case, the proof for non-homogeneous case and the proof of (ii) are similar. Let M be a central $(\alpha, q(\cdot); s, \varepsilon)$-molecule. Let

$$\sigma = \|M\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{-1/\alpha}, \ E_0 = \{x : |x| \le \sigma\}$$

and

$$E_{k,\sigma} = \{x : 2^{k-1}\sigma < |x| \le 2^k \sigma\}, \ k \in \mathbb{Z}_+.$$

Set $B_{k,\sigma} = \{x : |x| \le 2^k\sigma\}$. Denote by $\chi_{k,\sigma}$ the characteristic function of $E_{k,\sigma}$. It follows that

$$M(x) = \sum_{k=0}^{\infty} M(x)\chi_{k,\sigma}(x).$$

Let $M_k(x) = M(x)\chi_{k,\sigma}(x)$. We denote by $\mathcal{P}_m$ the class of all real polynomials of degree m. Let $P_{E_{k,\sigma}} M_k \in \mathcal{P}_m$ be the unique polynomial satisfying

$$\int_{E_{k,\sigma}} \left(M_k(x) - P_{E_{k,\sigma}} M_k(x)\right) x^\beta dx = 0, \ \ |\beta| \le s. \tag{3.1}$$

Set $Q_k(x) = (P_{E_{k,\sigma}} M_k)(x)\chi_{k,\sigma}(x)$. If we can prove that

(a) there is a constant $C > 0$ and a sequences of numbers $\{\lambda_k\}_k$ such that

$$\sum_{k=0}^{\infty} |\lambda_k|^p < \infty, \ M_k - Q_k = \lambda_k a_k,$$

where each a_k is a $(\alpha, q(\cdot))$-atom;

(b) $\sum_{k=0}^{\infty} Q_k$ has a $(\alpha, q(\cdot))$-atom decomposition,

then our desired conclusion can be deduced directly.

We first show (a). Without loss of generality, we can suppose that $\mathcal{R}_{q(\cdot)}(M) = 1$, which implies that

$$\left\| |\cdot|^{nb} M(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \|M\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{-a/(b-a)} = \sigma^{na}.$$

Set $\{\varphi_l^k : |l| \le s\} \subset \mathcal{P}_s(\mathbb{R}^n)$ such that

$$\langle \varphi_\mu^k, \varphi_\nu^k \rangle_{E_{k,\sigma}} = \frac{1}{|E_{k,\sigma}|} \int_{E_{k,\sigma}} \varphi_\mu^k(x) \varphi_\nu^k(x) dx = \delta_{\mu\nu}.$$

It is easy to see that

$$Q_k(x) = \sum_{|l| \le s} \langle M_k, \varphi_l^k \rangle_{E_{k,\sigma}} \varphi_l^k(x), \ \text{if}\ x \in E_{k,\sigma} \tag{3.2}$$

and

$$|Q_k(x)| \le \frac{C}{|E_{k,\sigma}|} \int_{E_{k,\sigma}} |M_k(x)| dx.$$

Thus for any $k \in \mathbb{Z}_+$, by Lemma 1.3 and $b - a = \alpha/n$ we have

$$\begin{aligned}
&\|M_k - Q_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le \|M_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \|Q_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le \|M_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \frac{C}{|E_{k,\sigma}|}\|M_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{E_{k,\sigma}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{E_{k,\sigma}}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le \|M_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C\|M_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\|M_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\left\||\cdot|^{nb}M(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}|2^k\sigma|^{-nb} \\
&= C|2^k\sigma|^{-nb}\sigma^{na} = C2^{-kna}|B_{k,\sigma}|^{-\alpha}.
\end{aligned}$$

We see that $M_k - Q_k = \lambda_k a_k$, with $\lambda_k = C2^{-kn\alpha}$ and a_k a central $(\alpha, q(\cdot))$-atom supported in $B_{k,\sigma}$. Obviously, $\sum_{k=0}^{\infty}|\lambda_k|^p < \infty$.

Next we will show (b). Let $\{\psi_l^k : |l| \le s\} \subset \mathcal{P}_s(\mathbb{R}^n)$ be the dual basis of $\{x^\alpha : |\alpha| \le s\}$ with respect to the weight $1/|E_{k,\sigma}|$ on $E_{k,\sigma}$, that is

$$\langle \psi_l^k, x^\alpha\rangle = \frac{1}{|E_{k,\sigma}|}\int_{E_{k,\sigma}} \psi_l^k(x)x^\alpha dx = \delta_{l\alpha}.$$

If set $\varphi_l^k(x) = \sum_{|\nu|\le s}\beta_{l\nu}^k x^\nu$ and $\psi_l^k(x) = \sum_{|\nu|\le s}\tau_{\nu l}^k\varphi_\nu^k(x)$, then we have

$$\tau_{\nu l}^k = \langle \psi_l^k, \varphi_\nu^k\rangle = \sum_{|\gamma|\le s}\beta_{\nu\gamma}^k\langle\psi_l^k, x^\gamma\rangle = \sum_{|\gamma|\le s}\beta_{\nu\gamma}^k\delta_{l\gamma} = \beta_{\nu l}^k.$$

So $\psi_l^k(x) = \sum_{|\nu|\le s}\beta_{\nu l}^k\varphi_\nu^k(x)$. By a computation we deduce that for $x \in E_{k,\sigma}$,

$$\langle M_k, \varphi_l^k\rangle_{E_{k,\sigma}}\varphi_l^k(x) = \langle M_k, \sum_{|\nu|\le s}\beta_{\nu l}^k x^\nu\rangle_{E_{k,\sigma}}\varphi_l^k(x) = \sum_{|\nu|\le s}\langle M_k, x^\nu\rangle_{E_{k,\sigma}}\beta_{\nu l}^k\varphi_l^k(x),$$

which together with (3.2) implies that

$$Q_k(x) = \sum_{|l|\le s}\langle M_k, x^l\rangle_{E_{k,\sigma}}\psi_l^k(x), \text{ if } x \in E_{k,\sigma}. \tag{3.3}$$

Now we assert that there is a constant $C > 0$ such that

$$|\psi_l^k(x)| \le C(2^{k-1}\sigma)^{-|l|}. \tag{3.4}$$

We set $E = \{x \in \mathbb{R}^n : 1 \leq |x| \leq 2\},\ F = \{x \in \mathbb{R}^n : |x| \leq 1\},\ \{e_l : |l| \leq s\} \subset \mathcal{P}_s(\mathbb{R}^n)$ satisfying $\frac{1}{|E|}\int_E e_l(x)x^\alpha dx = \delta_{l\alpha}$, and $\{\tilde{e}_l : |l| \leq s\} \subset \mathcal{P}_s(\mathbb{R}^n)$ satisfying $\frac{1}{|F|}\int_F \tilde{e}_l(x)x^\alpha dx = \delta_{l\alpha}$.

Noting that

$$\delta_{l\alpha} = \frac{1}{|E_{k,\sigma}|}\int_{E_{k,\sigma}} \psi_l^k(x)x^\alpha dx = \frac{1}{|E|}\int_E (2^{k-1}\sigma)^{|\alpha|}\psi_l^k(2^{k-1}\sigma y)y^\alpha dy,$$

we get $e_l(y) = (2^{k-1}\sigma)^{|l|}\psi_l^k(2^{k-1}\sigma y)$. This in turn leads to that

$$\psi_l^k(y) = (2^{k-1}\sigma)^{-|l|}e_l\left(\frac{x}{2^{k-1}\sigma}\right),\ x \in E_{k,\sigma}.$$

Similarly, we have

$$\psi_l^k(x) = (2^{k-1}\sigma)^{-|l|}\tilde{e}_l\left(\frac{x}{\sigma}\right),\ x \in F.$$

Taking $C = \sup_{l:|l|\leq s}\{\|e_l\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{e}_l\|_{L^\infty(\mathbb{R}^n)}\}$. Then (3.4) follows directly.

Now we can conclude (b). Set

$$N_l^k = \sum_{j=k}^\infty |E_{j,\sigma}|\langle M_j, x^l\rangle_{E_{j,\sigma}},\ k \in \mathbb{N}.$$

It is readily to see that

$$N_l^0 = \sum_{j=0}^\infty |E_{j,\sigma}|\langle M_j, x^l\rangle_{E_{j,\sigma}} = \sum_{j=0}^\infty \int_{E_{j,\sigma}} M(x)x^l dx = \int_{\mathbb{R}^n} M(x)x^l dx = 0,$$

and for $k \in \mathbb{Z}_+$, there exists $E_\sigma \subset E_{j,\sigma}$ such that $|E_\sigma| = \min\{1, |E_{j,\sigma}|\}$, so we have

$$\begin{aligned}
|N_l^k| &\leq \sum_{j=k}^\infty \int_{E_{j,\sigma}} |M_j(x)x^l|dx \\
&\leq C\sum_{j=k}^\infty \left\|M_j(\cdot)|\cdot|^l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{E_{j,\sigma}}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leq C\sum_{j=k}^\infty (2^j\sigma)^{|l|-nb}\left\||\cdot|^{nb}M_j(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\frac{|E_{j,\sigma}|}{|E_\sigma|}\right)^{\delta_2}\|\chi_{E_\sigma}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leq C\sum_{j=k}^\infty (2^j\sigma)^{|l|-nb+n\delta_2}\left\||\cdot|^{nb}M_j(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}|E_\sigma|^{1/(q')^+} \\
&\leq C\sum_{j=k}^\infty \sigma^{|l|+na-nb+n\delta_2}2^{j(|l|-nb+n\delta_2)} \\
&\leq C\sigma^{|l|-\alpha+n\delta_2}2^{k(|l|-nb+n\delta_2)}.
\end{aligned}$$

This via (3.4) shows that

$$|E_{k,\sigma}|^{-1}|N_l^k\psi_l^k(x)\chi_{k,\sigma}(x)| \le C\sigma^{n\delta_2-n-\alpha}2^{-kn(b+1-\delta_2)} \to 0,\ \text{if}\, k\to\infty. \tag{3.5}$$

By Abel transform and (3.5) we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} Q_k(x)\\
&= \sum_{|l|\le s}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}|E_{j,\sigma}|\langle M_j, x^l\rangle_{E_{j,\sigma}}\right)\\
&\quad\times\left\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}\\
&= \sum_{|l|\le s}\sum_{k=0}^{\infty}(-N_l^{k+1})\left\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}.
\end{aligned}$$

Meanwhile, we also have

$$\begin{aligned}
&\left|N_l^{k+1}\left\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}\right|\\
&\le C|N_l^{k+1}||E_{k+1,\sigma}|^{-1}|\psi_l^{k+1}(x)|\\
&\le C2^{-kna}|E_{k+1,\sigma}|^{\delta_2-1-\alpha/n}.
\end{aligned}$$

Set $\lambda_{lk} = C2^{-kna}$ and

$$a_{lk} = \lambda_{lk}^{-1}(-N_l^{k+1})\left\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}.$$

Then we have

$$\sum_{k=0}^{\infty}Q_k(x) = \sum_{|l|\le s}\sum_{k=0}^{\infty}\lambda_{lk}a_{lk},$$

with a_{lk} a $(\alpha, q(\cdot))$-atom, and $\sum_{|l|\le s}\sum_{k=0}^{\infty}|\lambda_{lk}|^p < \infty$.

The conclusion (b) then holds.

Theorem 3.3. *For any central $(\alpha, q(\cdot))$-atom f, let*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy,\ \ x\notin \operatorname{supp} f$$

satisfying $\int_{\mathbb{R}^n} Tf(x)dx = 0$ be a bounded operator on $L^{q(\cdot)}(\mathbb{R}^n)$ for some $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, and the kernel K satisfies that there are constants $C' > 0$ and $0<\delta\le 1$ such that

$$|K(x,y)-K(x,0)| \le C'\frac{|y|^\delta}{|x-y|^{n+\delta}},\ \ |x|\ge 2|y|.$$

Then for any α and p with $n\delta_2 \leq \alpha < n\delta_2 + \delta$ and $0 < p < \infty$, there exists a constant C such that $\|Tf\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \leq C$ $\Big($or $\|Tf\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \leq C\Big)$.

Proof. We only prove homogeneous case. Let f be a central $(\alpha, q(\cdot))$-atom supporting in $B(0,r)(r>0)$. It suffices to show Tf is a central $(\alpha, q(\cdot); 0, \varepsilon)$-molecule for some $1+\delta/n-\delta_2 \geq \varepsilon > \alpha/n - \delta_2$. To this aim, let $a = \delta_2 - \alpha/n + \varepsilon$, $b = \delta_2 + \varepsilon$. Obviously, we only need to verify the size condition for molecules, that is

$$\mathcal{R}_{q(\cdot)}(Tf) = \|Tf\|^{a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big\| |\cdot|^{nb}(Tf)(\cdot)\Big\|^{1-a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C,$$

with C independent of f. To do this, we first estimate $\big\||\cdot|^{nb}(Tf)(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}$.

In fact, we have

$$\Big\||\cdot|^{nb}(Tf)(\cdot)\Big\|_{L^{q(\cdot)}(|\cdot|\leq 2r)} \leq Cr^{nb}\|Tf\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq Cr^{nb-\alpha}.$$

On the other hand, the vanishing moment of f and the regularity of K give us that for x with $|x| > 2r$,

$$\begin{aligned}
|Tf(x)| &= \left|\int_{\mathbb{R}^n} K(x,y)f(y)dy\right| \\
&= \left|\int_{\mathbb{R}^n} (K(x,y)-K(x,0))f(y)dy\right| \\
&\leq C\int_{\mathbb{R}^n} \frac{|y|^{\delta}}{|x-y|^{n+\delta}}|f(y)|dy \\
&\leq Cr^{n+\delta}|x|^{-(n+\delta)}\frac{1}{|B_{0,r}|}\int_{B_{0,r}} |f(y)|dy \\
&\leq Cr^{n+\delta}|x|^{-(n+\delta)}Mf(x)
\end{aligned}$$

and so by $nb - n - \delta \leq 0$ we have

$$\begin{aligned}
\Big\||\cdot|^{nb}(Tf)(\cdot)\Big\|_{L^{q(\cdot)}(|\cdot|>2r)} &\leq Cr^{n+\delta}\Big\||\cdot|^{nb-n-\delta}Mf(\cdot)\Big\|_{L^{q(\cdot)}(|\cdot|>2r)} \\
&\leq Cr^{n+\delta+nb-n-\delta}\|Mf\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq Cr^{nb}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq Cr^{nb-\alpha}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\mathcal{R}_{q(\cdot)}(Tf) &= \|Tf\|^{a/b}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\||\cdot|^{nb}(Tf)(\cdot)\Big\|^{1-a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|f\|^{a/b}_{L^{q(\cdot)}(\mathbb{R}^n)} r^{(nb-\alpha)(1-a/b)} \\
&\leq Cr^{-\alpha a/b+(nb-\alpha)(1-a/b)} = C,
\end{aligned}$$

where C is independent of f.

This completes the proof of Theorem 3.3.

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