TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 4, pp. 1345-1361, August 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

POSITIVE SOLUTIONS FOR THE PERIODIC SCALAR p-LAPLACIAN: EXISTENCE AND UNIQUENESS

Sophia Th. Kyritsi and Nikolaos S. Papageorgiou

Abstract. We study a nonlinear periodic problem driven by the scalar p-Laplacian. The reaction term is a Carathéodory function $f(t,x)$ which satisfies only a unilateral growth condition in the x-variable. Assuming strict monotonicity for the quotient $f(t,x)/x^{p-1}$ and using variational methods coupled with suitable truncation techniques, we produce necessary and sufficient conditions for the existence and uniqueness of positive solutions.

1. INTRODUCTION

In this paper we study the following nonlinear periodic problem driven by the scalar p-Laplacian:

$$(1)\qquad \left\{ \begin{array}{l} -\big(|u'(t)|^{p-2}u'(t)\big)' = f\big(t,u(t)\big) \quad \text{a.e. on } T=[0,b], \\ u(0)=u(b),\ u'(0)=u'(b),\ u\geq 0,\ 1<p<\infty \end{array} \right\}.$$

The reaction term $f(t,x)$ is a Carathéodory function, i.e., for all $x\in\mathbb{R}$, $t\longrightarrow f(t,x)$ is measurable, and for a.a. $t\in T$, $x\longrightarrow f(t,x)$ is continuous.

The aim of this work is to establish the existence and uniqueness of positive solutions, when the nonlinearity $f(t,\cdot)$ is only unilaterally restricted (only from above). In fact, we produce necessary and sufficient conditions for such problems to have a unique positive solution.

There have been papers dealing with the existence and multiplicity of positive solutions for the periodic scalar p-Laplacian. We mention the works of Aizicovici-Papageorgiou-Staicu [1], Binding-Rynne [3, 4], del Pino-Manásevich-Murúa [5], Drabek-Manásevich [6], Kyritsi-Papageorgiou [7], Motreanu-Motreanu-Papageorgiou [8], Yang [11], and Zhang [12]. In all these works a bilateral polynomial growth is

Received April 4, 2011, accepted August 29, 2011.
Communicated by Biagio Ricceri.
2010 *Mathematics Subject Classification*: 34B15, 34B18, 34C25.
Key words and phrases: Scalar p-Laplacian, Unilateral growth, Existence and uniqueness of positive solutions, Weighted eigenvalues.

imposed on the right-hand side. To the best of our knowledge, the question of existence and uniqueness of periodic solutions for the scalar p-Laplacian has not been addressed in this generality.

Our approach is variational with suitable truncation techniques.

2. Mathematical Background

We start by considering the following weighted nonlinear eigenvalue problem:

$$\left\{\begin{array}{l} -\big(|u'(t)|^{p-2}u'(t)\big)' = \big(\widehat{\lambda}+\beta(t)\big)|u(t)|^{p-2}u(t) \quad \text{a.e. on } T=[0,b], \\ u(0)=u(b),\ u'(0)=u'(b),\ 1<p<\infty,\ \widehat{\lambda}\in\mathbb{R},\ \beta\in L^1(T) \end{array}\right\}. \tag{2}$$

This eigenvalue problem was first investigated by Zhang [12] and later Binding-Rynne [3, 4] answered important questions left open by Zhang and produced a more definitive picture for the spectrum of problem (2). In particular, from Binding-Rynne [4], we know that problem (2) admits a smallest eigenvalue $\widehat{\lambda}_0(\beta)\in\mathbb{R}$ which is simple and has the following variational characterization:

$$\widehat{\lambda}_0(\beta)=\inf\Big[\|u'\|_p^p-\int_0^b\beta(t)|u(t)|^p\mathrm{d}t : u\in W^{1,p}_{\text{per}}(0,b), \|u\|_p=1\Big], \tag{3}$$

where $W^{1,p}_{\text{per}}(0,b)=\{u\in W^{1,p}(0,b):u(0)=u(b)\}$. Recall that $W^{1,p}(0,b)$ is embedded continuously, (in fact compactly), into $C(T)$, therefore the evaluations at $t=0$ and $t=b$ in the definition of $W^{1,p}_{\text{per}}(0,b)$ make sense. Every eigenfunction $u\in W^{1,p}_{\text{per}}(0,b)$ corresponding to $\widehat{\lambda}_0(\beta)$ satisfies

$$u\in C^1_0(Z)1(T) \text{ and } |u(t)|>0 \quad \text{for all } t\in T.$$

So, an eigenfunction corresponding to $\widehat{\lambda}_0(\beta)$ has constant sign and we can always assume that it is positive. An eigenfunction corresponding to an eigenvalue $\widehat{\lambda}\neq\widehat{\lambda}_1(\beta)$ is necessarily nodal (i.e., sign changing).

We can rewrite (3) as follows:

$$\widehat{\lambda}_0(\beta)=\inf\Big[\|u'\|_p^p-\int_{\{u\neq 0\}}\beta(t)|u(t)|^p\mathrm{d}t : u\in W^{1,p}_{\text{per}}(0,b), \|u\|_p=1\Big]. \tag{4}$$

We observe that in (4), the integral $\int_{\{u\neq0\}}\beta|u|^p\mathrm{d}t$ makes sense even when $\beta(\cdot)$ is only a measurable function and there exists $\widehat{c}\in L^1(T)_+$ such that

$$\beta(t)\le\widehat{c}(t) \text{ a.e. on } T \text{ or } \beta(t)\ge-\widehat{c}(t) \text{ a.e. on } T.$$

In the first case $\widehat{\lambda}_0(\beta)\in(-\infty,+\infty]$ and in the second case $\widehat{\lambda}_0(\beta)\in[-\infty,+\infty)$.

In addition to the Sobolev space $W^{1,p}_{\text{per}}(0,b)$, we shall also use the Banach space $\widehat{C}^1(T) = C^1(T) \cap W^{1,p}_{\text{per}}(0,b) = \{u \in C^1(T) : u(0) = u(b)\}$. This is an ordered Banach space with an order cone given by

$$\widehat{C}_+ = \{u \in \widehat{C}^1(T) : u(t) \geq 0 \text{ for all } t \in T\}.$$

This cone has an nonempty interior given by

$$\text{int}\widehat{C}_+ = \{u \in \widehat{C}_+ : u(t) > 0 \text{ for all } t \in T\}.$$

Let $u, v \in \text{int}\widehat{C}_+$ and set

$$R(u,v) = |u'(t)|^p - |v'(t)|^{p-2} v'(t) \Big(\frac{u(t)^p}{v(t)^{p-1}}\Big)'.$$

From Allegretto-Huang [2] we know that $R(u,v)(t) \geq 0$ for all $t \in T$.
Let $A : W^{1,p}_{\text{per}}(0,b) \longrightarrow W^{1,p}_{\text{per}}(0,b)^*$ be the nonlinear operator defined by

$$\langle A(u), y\rangle = \int_0^b |u'|^{p-2} u' y' \mathrm{d}t \quad \text{for all } \; u, y \in W^{1,p}_{\text{per}}(0,b)$$

(by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $\big(W^{1,p}_{\text{per}}(0,b)^*, W^{1,p}_{\text{per}}(0,b)\big)$). This map is continuous and monotone (see, for example, Papageorgiou-Kyritsi [9]).

Throughout this work by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}_{\text{per}}(0,b)$ and for $p \in [1,\infty]$, by $\|\cdot\|_p$ we denote the norm of the Lebesgue space $L^p(T)$. Finally, for every $r \in \mathbb{R}$, we set $r^{\pm} = \max\{\pm r, 0\}$ and by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$.

3. Existence of Positive Solutions

The hypotheses on the reaction term $f(t,x)$ are the following:

<u>H</u>: $f : T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that

- **(i)** for all $x \geq 0$, $f(\cdot, x) \in L^1(T)$ and there exists $\alpha \in L^1(T)_+$ such that

$$f(t,x) \leq \alpha(t)(1 + x^{p-1}) \text{ for a.a } t \in T, \text{ all } x \geq 0;$$

- **(ii)** for a.a. $t \in T$, the function $x \longrightarrow \frac{f(t,x)}{x^{p-1}}$ is strictly decreasing on $(0, +\infty)$;
- **(iii)** if $\vartheta(t) = \lim\limits_{x \to +\infty} \frac{f(t,x)}{x^{p-1}}$, then $\widehat{\lambda}_0(\vartheta) > 0$;
- **(iv)** if $\vartheta_0(t) = \lim\limits_{x \to 0^+} \frac{f(t,x)}{x^{p-1}}$, then $\widehat{\lambda}_0(\vartheta_0) < 0$;

Remark 3.1. Because we are looking for positive solutions and the hypotheses on $f(t,\cdot)$ concern only the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, by truncating $f(t,\cdot)$ if necessary, we may (and will) assume that $f(t,x) = f(t,0)$ for a.a. $t \in T$, all $x \leq 0$. By virtue of hypothesis H(ii) we see that the limits $\vartheta(t), \vartheta_0(t)$ in hypotheses H(iii), (iv) exist and are measurable functions. We have

$$\frac{f(t,x)}{x^{p-1}} \leq f(t,1) \text{ for a.a. } t \in T, \text{ all } x \geq 1 \text{ and } f(\cdot,1) \in L^1(T),$$
$$\Rightarrow \vartheta(t) \leq f(t,1) \text{ for a.a. } t \in T,$$
$$\Rightarrow \widehat{\lambda}_0(\vartheta) \in (-\infty,+\infty].$$

Similarly, we have

$$\frac{f(t,x)}{x^{p-1}} \geq f(t,1) \text{ for a.a. } t \in T, \text{ all } x \in (0,1] \text{ and } f(\cdot,1) \in L^1(T),$$
$$\Rightarrow \vartheta_0(t) \geq f(t,1) \text{ for a.a. } t \in T,$$
$$\Rightarrow \widehat{\lambda}_0(\vartheta_0) \in [-\infty,+\infty).$$

If $\vartheta, \vartheta_0 \in L^1(T)$, then $\widehat{\lambda}_0(\vartheta), \widehat{\lambda}_0(\vartheta_0) \in \mathbb{R}$ and are the principal eigenvalues of (2) when $\beta = \vartheta$ and $\beta = \vartheta_0$ respectively. In the autonomous case, i.e., when $f(t,x) = f\o$, hypotheses H(iii), (iv) reduce to

$$\vartheta_0 < 0 < \vartheta$$

(recall that 0 is the principal eigenvalue of the negative periodic scalar p-Laplacian, i.e., for problem (2) when $\beta \equiv 0$).

Example 3.2. The function $f\o = \lambda(x^{r-1} - x^{q-1})$ for all $x \geq 0$ with $\lambda > 0$, $1 < r \leq p \leq q < \infty$, and $r \neq p$ or $q \neq p$, satisfies hypotheses H. In this case $\vartheta = -\infty$ if $q > p$ and $\vartheta = -\lambda$ if $r < p = q$ (and thus $\widehat{\lambda}_0(\vartheta) = +\infty$ if $q > p$ and $\widehat{\lambda}_0(\vartheta) = \lambda$ if $r < p = q$) and $\vartheta_0 = +\infty$ if $r < p$ and $\vartheta_0 = \lambda$ if $r = p < q$ (and thus $\widehat{\lambda}_0(\vartheta_0) = -\infty$ if $r < p$ and $\widehat{\lambda}(\vartheta_0) = -\lambda$ if $r = p < q$). Another admissible nonlinearity is provided by the $f\o = x^{r-1} - \delta x^{p-1} e^x$ for all $x \geq 0$, $\delta > 0$, which does not exhibit a polynomial growth from below.

We introduce the following truncation-perturbation of $f(t,x)$:

$$\widehat{f}(t,x) = \begin{cases} f(t,0), & \text{if } x \leq 0 \\ f(t,x) + x^{p-1} & \text{if } x > 0 \end{cases}. \tag{5}$$

This is a Carathéodory function. Let $\widehat{F}(t,x) = \int_0^b \widehat{f}(t,s)\mathrm{d}s$. Hypothesis H(i) and (5) imply that

$$\widehat{F}(t,x) \leq \alpha_1(t)(1+x^p) \text{ for a.a. } t \in T, \text{ all } x \geq 0 \text{ with } \alpha_1 \in L^1(T)_+. \tag{6}$$

This growth restriction on $\widehat{F}(t,\cdot)$, the fact that $f(\cdot,x)\in L^1(T)$ for all $x\geq 0$, and hypothesis H(ii) permit the introduction of the functional $\widehat{\varphi}: W^{1,p}_{\text{per}}(0,b)\longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}(u)=\frac{1}{p}\|u'\|_p^p+\frac{1}{p}\|u\|_p^p-\int_0^b \widehat{F}\big(t,u(t)\big)\mathrm{d}t \quad \text{for all } u\in W^{1,p}_{\text{per}}(0,b).$$

Proposition 3.3. **If** *hypotheses H hold,* **then** $\widehat{\varphi}$ *is coercive, i.e.,* $\widehat{\varphi}(u)\longrightarrow +\infty$ *as* $\|u\|\to\infty$.

Proof. We argue indirectly. So, suppose that the result is not true. Then we can find $\{u_n\}_{n\geq 1}\subseteq W^{1,p}_{\text{per}}(0,b)$ such that

$$\|u_n\|\longrightarrow\infty \ \text{ and } \ \widehat{\varphi}(u_n)\leq M_1 \ \text{ for some } M_1>0, \text{ all } n\geq 1. \tag{7}$$

We have

$$\frac{1}{p}\big(\|u_n'\|_p^p+\|u_n\|_p^p\big)\leq M_1+\int_0^b \widehat{F}(t,u_n)\mathrm{d}t \quad \text{for all } n\geq 1. \tag{8}$$

Note that

$$\widehat{F}\big(t,u_n(t)\big)=\widehat{F}\big(t,u_n^+(t)\big)+f(t,0)\big(-u_n^-(t)\big) \ \text{ for a.a. } t\in T, \text{ all } n\geq 1$$

(see (5))

$$\text{and} \quad \frac{f(t,x)}{x^{p-1}}\geq f(t,1) \ \text{ for a.a. } t\in T, \text{ all } x\in(0,1],$$

$$\Rightarrow \ f(t,x)\geq f(t,1)x^{p-1} \ \text{ for a.a. } t\in T, \text{ all } x\in(0,1],$$

$$\Rightarrow \ f(t,0)\geq 0 \ \text{ for a.a. } t\in T. \tag{9}$$

It follows that

$$\widehat{F}\big(t,u_n(t)\big)\leq \widehat{F}\big(t,u_n^+(t)\big)\leq \alpha(t)\big(1+|u_n(t)|^p\big) \text{ a.a. } t\in T,$$

(see [6]).

So, if we use this fact in (8), then

$$\begin{aligned}\frac{1}{p}\big(\|u_n'\|_p^p+\|u_n\|_p^p\big) &\leq M_1+\int_0^b \widehat{F}(t,u_n^+)\mathrm{d}t\\ &\leq c_1(1+\|u_n\|_\infty^p) \ \text{ for some } c_1>0, \text{ all } n\geq 1.\end{aligned} \tag{10}$$

From (10) and since $\|u_n\|\to\infty$ (see (7)), we see that $\|u_n\|_\infty\to\infty$. Let $y_n=\frac{u_n}{\|u_n\|_\infty}$, $n\geq 1$. Then $\|y_n\|_\infty=1$ for all $n\geq 1$ and from (10), we have

$$\frac{1}{p}\big(\|y_n'\|_p^p+\|y_n\|_p^p\big)\leq c_1\Big(\frac{1}{\|u_n\|_\infty^p}+1\Big) \quad \text{for all } n\geq 1,$$

$$\Rightarrow \ \{y_n\}_{n\geq 1}\subseteq W^{1,p}_{\text{per}}(0,b) \ \text{ is bounded.}$$

By passing to a suitable subsequence if necessary, we may assume that

(11) $$y_n \xrightarrow{w} y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } y_n \longrightarrow y \text{ in } C(T) \text{ with } \|y\|_\infty = 1.$$

If $F(t,x) = \int_0^x f(t,s)\mathrm{d}s$, then from (7) we have

(12)
$$\begin{aligned}
&\frac{1}{p}\left(\|y_n'\|_p^p + \|y_n\|_p^p\right) \le \frac{M_1}{\|u_n\|_\infty^p} + \int_0^b \frac{\widehat{F}(t,u_n)}{\|u_n\|_\infty^p}\mathrm{d}t \\
&= \frac{M_1}{\|u_n\|_\infty^p} + \int_{\{u_n>0\}}\Big(\frac{F(t,u_n)}{\|u_n\|_\infty^p} + \frac{1}{p}y_n^p\Big)\mathrm{d}t \\
&\quad + \int_{\{u_n\le 0\}}\frac{f(t,0)}{\|u_n\|_\infty^p}u_n\mathrm{d}t \quad \text{(see (5))} \\
&\le \frac{M_1}{\|u_n\|_\infty^p} + \frac{1}{p}\|y_n^+\|_\infty^p + \int_0^b \frac{F(t,u_n^+)}{\|u_n\|_\infty^p}\mathrm{d}t \quad \text{for all } n \ge 1
\end{aligned}$$

(see (9)).

First assume that $\{u_n^+\}_{n\ge 1} \subseteq C(T)$ is bounded. Then $y \le 0$. Hypothesis H(i) implies that

(13) $$F(t,x) \le \alpha_2(t)(1+x^p) \text{ for a.a. } t \in T, \text{ all } x \ge 0 \text{ with } \alpha_2 \in L^1(T)_+.$$

So, we have

$$\begin{aligned}
\int_0^b \frac{F(t,u_n^+)}{\|u_n\|_\infty^p}\mathrm{d}t &\le \int_0^b \alpha_2(t)\Big(\frac{1}{\|u_n\|_\infty^p} + (y_n^+)^p\Big)\mathrm{d}t \quad \text{(see(13))} \\
&\le c_2\Big(\frac{1}{\|u_n\|_\infty^p} + \|y_n^+\|_\infty^p\Big) \quad \text{for some } c_2>0, \text{ all } n \ge 1, \\
\Rightarrow \limsup_{n\to\infty} \int_0^b \frac{F(t,u_n^+)}{\|u_n\|_\infty^p}\mathrm{d}t &\le 0 \quad \text{(see (11) and recall } y \le 0).
\end{aligned}$$

Then passing to the limit as $n \to \infty$ in (12) and using (11), we obtain

$$\begin{aligned}
&\frac{1}{p}\left(\|y_n'\|_p^p + \|y\|_p^p\right) \le 0, \\
\Rightarrow\ & y = 0, \text{ which contradicts (11).}
\end{aligned}$$

Hence we may assume that $\|u_n^+\| \to \infty$. From (5) we have

(14) $$\frac{1}{p}\|(y_n^+)'\|_p^p \le \frac{M_1}{\|u_n^+\|_\infty^p} + \int_0^b \frac{F(t,u_n^+)}{\|u_n^+\|_\infty^p}\mathrm{d}t \quad \text{for all } n \ge 1 \quad \text{(see (9))}.$$

Since $F(t,0)=0$, we have

$$\int_0^b \frac{F(t,u_n^+)}{\|u_n^+\|_\infty^p}\mathrm{d}t = \int_{\{y^+=0\}} \frac{F(t,u_n^+)}{\|u_n^+\|_\infty^p}\mathrm{d}t + \int_{\{y>0\}\cap\{y_n>0\}} \frac{F(t,u_n^+)}{(u_n^+)^p}(y_n^+)^p\mathrm{d}t \tag{15}$$

for all $n \geq 1$.

We know that $y_n^+ \longrightarrow y^+$ in $C(T)$ (see (11)). Therefore, using (13) we have

$$\Big|\int_{\{y^+=0\}} \frac{F(t,u_n^+)}{\|u_n^+\|_\infty^p}\mathrm{d}t\Big| \leq \int_{\{y^+=0\}} \alpha_2(t)\Big(\frac{1}{\|u_n^+\|_\infty^p} + (y_n^+)^p\Big)\mathrm{d}t \longrightarrow 0 \text{ as } n\to\infty. \tag{16}$$

Note that

$$u_n^+(t) \longrightarrow +\infty \quad \text{for all } t \in \{y>0\} \text{ and } \chi_{\{y>0\}\cap\{y_n>0\}}(t) \longrightarrow \chi_{\{y>0\}}(t)$$

for a.a. $t\in T$.

Here by χ_A we indicate the characteristic function of a set $A \subseteq T$, i.e.,

$$\chi_A(t) = \begin{cases} 1 & \text{if } t\in A \\ 0 & \text{if } t\in T\setminus A \end{cases}.$$

Let $t \in \{\vartheta > -\infty\}\setminus N$, $|N|_1 = 0$ be such that $\frac{f(t,x)}{x^{p-1}} \longrightarrow \vartheta(t)^+$ as $x\to+\infty$ (see hypotheses H(ii) and (iii)). Then given any $\varepsilon>0$ we can find $M_2 = M_2(\varepsilon,t)>0$ such that

$$\begin{aligned} & f(t,x) \leq \big(\vartheta(t)+\varepsilon\big)x^{p-1} \quad \text{for all } x\geq M_2,\\ \Rightarrow\ & F(t,x) \leq \frac{1}{p}\big(\vartheta(t)+\varepsilon\big)x^p \quad \text{for all } x\geq M_2,\\ \Rightarrow\ & \frac{F(t,x)}{x^p} \leq \frac{1}{p}\big(\vartheta(t)+\varepsilon\big) \quad \text{for all } x\geq M_2,\\ \Rightarrow\ & \limsup_{x\to+\infty}\frac{F(t,x)}{x^p} \leq \frac{1}{p}\big(\vartheta(t)+\varepsilon\big). \end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we let $\varepsilon \longrightarrow 0^+$ to infer that

$$\limsup_{x\to+\infty}\frac{F(t,x)}{x^p} \leq \frac{1}{p}\,\vartheta(t) \text{ for a.a. } t\in\{\vartheta>-\infty\}.$$

Also, if $t\in\{\vartheta=-\infty\}\setminus N$, $|N|_1=0$ is such that $\frac{f(t,x)}{x^{p-1}} \longrightarrow -\infty = \vartheta(t)$ as $x\to+\infty$, then for every $\xi>0$, we can find $M_3 = M_3(\xi,t)>0$ such that

$$\begin{aligned} & f(t,x) \leq -\xi x^{p-1} \quad \text{for all } x\geq M_3,\\ \Rightarrow\ & \frac{F(t,x)}{x^p} \leq -\frac{\xi}{p} \quad \text{for all } x\geq M_3,\\ \Rightarrow\ & \limsup_{x\to+\infty}\frac{F(t,x)}{x^p} \leq -\frac{\xi}{p} \text{ for a.a. } t\in\{\vartheta=-\infty\}. \end{aligned}$$

Since ξ was arbitrary, we let $\xi \to +\infty$ and have

$$\lim_{x\to+\infty} \frac{F(t,x)}{x^p} = -\infty = \frac{\vartheta(t)}{p} \text{ for a.a. } t \in \{\vartheta = -\infty\}.$$

Therefore, finally we have

$$\limsup_{x\to+\infty} \frac{F(t,x)}{x^p} \leq \frac{1}{p}\vartheta(t) \text{ a.e. on } T. \tag{17}$$

Because of (13) we can use Fatou's lemma and (17) and obtain

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\{y>0\}\cap\{y_n>0\}} \frac{F(t,u_n^+)}{(u_n^+)^p}(y_n^+)^p \mathrm{d}t \\ &\leq \frac{1}{p}\int_{\{y>0\}} \vartheta y^p \mathrm{d}t = \frac{1}{p}\int_{\{y^+\neq 0\}} \vartheta (y^+)^p \mathrm{d}t. \end{aligned} \tag{18}$$

So, if in (15) we pass to the limit as $n \to \infty$ and use (16) and (18), we have

$$\limsup_{n\to\infty} \int_0^b \frac{F(t,u_n^+)}{\|u_n^+\|_\infty^p} \mathrm{d}t \leq \frac{1}{p}\int_{\{y^+\neq 0\}} \vartheta (y^+)^p \mathrm{d}t. \tag{19}$$

We return to (14), take limits as $n \to \infty$ and use (11) and (19). We obtain

$$\frac{1}{p}\|(y^+)'\|_p^p \leq \frac{1}{p}\int_{\{y^+\neq 0\}} \vartheta (y^+)^p \mathrm{d}t. \tag{20}$$

If $y^+ = 0$, then from (12) in the limit as $n \to \infty$, we have

$$\frac{1}{p}\|y^-\|^p \leq 0, \text{ i.e., } y^- = 0.$$

Therefore $y = 0$, a contradiction to (11).

So, $y^+ \neq 0$. Then from (20) and since in (4) the minimized function is p-homogeneous, it follows that $\widehat{\lambda}_0(\vartheta) \leq 0$, a contradiction to hypothesis H(iii). This proves that $\widehat{\varphi}$ is coercive. ■

Proposition 3.4. **If** *hypotheses H hold,* **then** $\widehat{\varphi}$ *is sequentially weakly lower semicontinuous.*

Proof. Recall that in a Banach space the norm functional is sequentially weakly lower semicontinuous and by the Sobolev embedding theorem $W^{1,p}_{\text{per}}(0,b)$ is embedded compactly into $C(T)$. Therefore, in order to show the sequential weak lower semicontinuity of $\widehat{\varphi}$, it suffices to show that the integral functional $I_{\widehat{F}} : W^{1,p}_{\text{per}}(0,b) \longrightarrow \mathbb{R}$ defined by $I_{\widehat{F}}(u) = -\int_0^b \widehat{F}\big(t,u(t)\big)\mathrm{d}t$ for all $u \in W^{1,p}_{\text{per}}(0,b)$, is sequentially weakly lower semicontinuous. To this end, let $u_n \xrightarrow{w} u$ in $W^{1,p}_{\text{per}}(0,b)$. Then $u_n \longrightarrow u$ in $C(T)$ and so $u_n^{\pm} \longrightarrow u^{\pm}$ in $C(T)$. We have:

$$\begin{aligned}&-\int_0^b \widehat{F}(t,u_n)\mathrm{d}t\\ &= -\int_0^b F(t,u_n^+)\mathrm{d}t - \frac{1}{p}\|u_n^+\|_p^p - \int_0^b f(t,0)(-u_n^-)\mathrm{d}t \quad \text{for all } n\geq 1\end{aligned} \tag{21}$$

(see (5)).

Note that

$$\frac{1}{p}\|u_n^+\|_p^p \longrightarrow \frac{1}{p}\|u^+\|_p^p \text{ and } \int_0^b f(t,0)(-u_n^-)\mathrm{d}t \longrightarrow \int_0^b f(t,0)(-u^-)\mathrm{d}t. \tag{22}$$

Also, (13) permits the use of Fatou's lemma and we have

$$\begin{aligned}\liminf_{n\to\infty}\Big(-\int_0^b F(t,u_n^+)\mathrm{d}t\Big) &= -\limsup_{n\to\infty}\int_0^b F(t,u_n^+)\mathrm{d}t\\ &\geq -\int_0^b \limsup_{n\to\infty} F(t,u_n^+)\mathrm{d}t\\ &= -\int_0^b F(t,u^+)\mathrm{d}t.\end{aligned} \tag{23}$$

From (21) through (23) it follows that

$$\begin{aligned}\liminf_{n\to\infty} I_{\widehat{F}}(u_n) \ &\geq \ -\int_0^b F(t,u^+)\mathrm{d}t - \frac{1}{p}\|u^+\|_p^p - \int_0^b f(t,0)(-u^-)\mathrm{d}t = I_{\widehat{F}}(u),\\ &\Rightarrow I_{\widehat{F}}(\cdot) \text{ is sequentially weakly lower semicontinuous},\\ &\Rightarrow \widehat{\varphi} \text{ is sequentially weakly lower semicontinuous.}\end{aligned}$$

■

Next we prove the differentiability of the functional $\widehat{\varphi}$.

Proposition 3.5. If *hypotheses H hold,* **then** $\widehat{\varphi} \in C^1(W^{1,p}_{per}(0,b))$.

Proof. From the definition of $\widehat{\varphi}$ it is clear that it suffices to show that the functional $u \longrightarrow \int_0^b \widehat{F}\big(t,u(t)\big)\mathrm{d}t$, $u \in W^{1,p}_{\text{per}}(0,b)$ is C^1. To this end let $u,h \in W^{1,p}_{\text{per}}(0,b)$ and let

$$w(h) = \int_0^b \big(\widehat{F}(t,u+h) - \widehat{F}(t,u) - \widehat{f}(t,u)h\big)\mathrm{d}t.$$

We have

$$\begin{aligned}\widehat{F}\big(t,u(t)+h(t)\big) - \widehat{F}\big(t,u(t)\big) \ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\widehat{F}\big(t,u(t)+rh(t)\big)\mathrm{d}r\\ &= \int_0^1 \widehat{f}\big(t,u(t)+rh(t)\big)h(t)\mathrm{d}r.\end{aligned}$$

Therefore

$$|w(h)| \leq \int_0^b \int_0^1 \big|\widehat{f}\big(t, u(t) + rh(t)\big) - \widehat{f}\big(t, u(t)\big)\big||h(t)|\mathrm{d}r\mathrm{d}t. \tag{24}$$

Because of (5), we have

$$\widehat{f}\big(t, u(t){+}rh(t)\big){=}\begin{cases} f(t,0) & \text{if } t{\in}\{u+rh\leq 0\} \\ f\big(t, u(t){+}rh(t)\big){+}\big(u(t)+rh(t)\big)^{p-1} & \text{if } t{\in}\{u+rh{>}0\} \end{cases}. \tag{25}$$

By virtue of hypothesis H(ii) for a.a. $t{\in}\{u+rh{>}0\}$ and all $r \in [0,1]$, we have

$$\begin{aligned} f\big(t, u(t)+rh(t)\big) \;&\geq\; f(t, \|u+rh\|_\infty)\frac{\big(u(t)+rh(t)\big)^{p-1}}{\|u+rh\|_\infty^{p-1}} \\ &\geq\; \frac{f(t, \|u\|_\infty{+}\|h\|_\infty)}{\|u\|_\infty{+}\|h\|_\infty}\big(u(t){+}rh(t)\big)^{p-1} \text{ (recall } r{\in}[0,1]) \\ &\geq\; -2^{p-2}\frac{\big|f(t, \|u\|_\infty+\|h\|_\infty)\big|}{\|u\|_\infty+\|h\|_\infty}(\|u\|_\infty^{p-1}+\|h\|_\infty^{p-1}), \\ &\Rightarrow\; f\big(t, u(t)+rh(t)\big) \geq \alpha_3(t) \end{aligned} \tag{26}$$

for a.a. $t{\in}\{u+rh{>}0\}$, all $r \in [0,1]$, with $\alpha_3 \in L^1(T)$.

In addition, hypothesis H(i) implies that for a.a. $t{\in}\{u{+}rh{>}0\}$, all $r \in [0,1]$, we have

$$\begin{aligned} f\big(t, u(t)+rh(t)\big) \;&\leq\; \alpha(t)\Big(1+\big(u(t)+rh(t)\big)^{p-1}\Big) \\ &\leq\; \alpha(t)\big(1+2^{p-2}(\|u\|_\infty+\|h\|_\infty)\big), \\ &\Rightarrow\; f\big(t, u(t)+rh(t)\big) \leq \alpha_4(t) \end{aligned} \tag{27}$$

for a.a. $t{\in}\{u+rh{>}0\}$, all $r \in [0,1]$, with $\alpha_4 \in L^1(T)$.

Recalling that $f(\cdot,0) \in L^1(T)$, from (25) through (27), we infer that

$$\big|\widehat{f}\big(t, u(t){+}rh(t)\big)\big| \leq \alpha_5(t) \;\text{ for a.a. } t{\in}T, \text{ all } r \in [0,1], \text{ with } \alpha_5 \in L^1(T). \tag{28}$$

From (24) we have

$$\begin{aligned} |w(h)| &\leq \int_0^b \int_0^1 \big|\widehat{f}\big(t, u(t)+rh(t)\big) - \widehat{f}\big(t, u(t)\big)\big|\|h\|_\infty\mathrm{d}r\mathrm{d}t \\ &\leq c_3\int_0^1 \int_0^b \big|\widehat{f}\big(t, u(t)+rh(t)\big) - \widehat{f}\big(t, u(t)\big)\big|\mathrm{d}t\mathrm{d}r\|h\| \\ &\qquad \text{for some } c_3{>}0 \quad \text{(by Fubini's theorem)} \\ &\leq c_3\int_0^1 \|N_{\widehat{f}}(u+rh) - N_{\widehat{f}}(u)\|_1\mathrm{d}r\|h\| \end{aligned}$$

where $N_{\widehat{f}}(v)(\cdot) = \widehat{f}\big(\cdot, v(\cdot)\big)$ for all $v \in W^{1,p}_{\text{per}}(0,b)$.

From (28) and the dominated convergence theorem, we see that

$$\begin{aligned} &\int_0^1 \|N_{\widehat{f}}(u+rh) - N_{\widehat{f}}(u)\|_1 \mathrm{d}r \longrightarrow 0 \ \text{ as } \|h\| \to \infty, \\ \Rightarrow\ &\frac{|w(h)|}{\|h\|} \longrightarrow 0 \ \text{ as } \|h\| \to \infty, \\ \Rightarrow\ &\widehat{\varphi}'(u) = A(u) - N_{\widehat{f}}(u). \end{aligned}$$

But $A(\cdot)$ is continuous and from the above argument it is clear that $N_{\widehat{f}}(\cdot)$ is continuous too. Therefore $\widehat{\varphi} \in C^1(W^{1,p}_{\text{per}}(0,b))$. ■

Now we are ready to produce nontrivial positive solutions for problem problem (1).

Proposition 3.6. If *hypotheses H hold,* **then** *problem* (1) *has a solution* $u_0 \in \widehat{C}_+ \setminus \{0\}$.

Proof. Propositions 3.3, 3.4, and the Weierstrass theorem imply that there is a $u_0 \in W^{1,p}_{\text{per}}(0,b)$ such that

$$\widehat{\varphi}(u_0) = \inf\big[\widehat{\varphi}(u) : u \in W^{1,p}_{\text{per}}(0,b)\big] = \widehat{m}. \tag{29}$$

Note that, if $u_0^- \neq 0$, then

$$\begin{aligned} \widehat{\varphi}(u_0^+) &= \frac{1}{p}\|(u_0^+)'\|_p^p + \frac{1}{p}\|u_0^+\|_p^p - \int_0^b \widehat{F}(t,u_0^+)\mathrm{d}t \\ &= \frac{1}{p}\|(u_0^+)'\|_p^p - \int_0^b F(t,u_0^+)\mathrm{d}t \quad \text{(see (5))} \\ &< \frac{1}{p}\|u_0'\|_p^p + \frac{1}{p}\|u_0^-\|_p^p - \int_0^b F(t,u_0^+)\mathrm{d}t - \int_0^b f(t,0)(-u_0^-)\mathrm{d}t \\ &\quad \text{(see (9))} \\ &= \widehat{\varphi}(u_0), \end{aligned}$$

which contradicts (29) (recall $u_0^+ \in W^{1,p}_{\text{per}}(0,b)$). Therefore $u_0^- = 0$ and so $u_0 \geq 0$.

Next we show that $u_0 \neq 0$. By virtue of hypothesis H(iv) and the definition of $\widehat{\lambda}_1(\vartheta_0)$ (see (4)), we see that we can find $\widehat{u} \in W^{1,p}_{\text{per}}(0,b)$ such that

$$\|\widehat{u}'\|_p^p - \int_{\{\widehat{u}\neq 0\}} \vartheta_0|\widehat{u}|^p \mathrm{d}t < 0 \ \text{ and } \ \|\widehat{u}\|_p = 1. \tag{30}$$

Replacing $\widehat{u}$ with $|\widehat{u}| \in W^{1,p}_{\text{per}}(0,b)$ if necessary, we may assume that $\widehat{u} \geq 0$, $\widehat{u} \neq 0$ (see (30)).

For $x>0$, we have

$$F(t,x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}F(t,rx)\mathrm{d}r = \int_0^1 f(t,rx)x\mathrm{d}r,$$

(31)
$$\Rightarrow \frac{F(t,x)}{x^p} = \int_0^1 \frac{f(t,rx)}{x^{p-1}}\mathrm{d}r \geq \frac{f(t,x)}{x^{p-1}}\int_0^1 r^{p-1}\mathrm{d}r = \frac{1}{p}\frac{f(t,x)}{x^{p-1}}$$
(see hypothesis H(ii)),

$$\Rightarrow \liminf_{x\to 0^+}\frac{F(t,x)}{x^p} \geq \frac{1}{p}\vartheta_0(t) \text{ for a.a. } t\in T.$$

For $r\in(0,1]$ small, we will have $r\widehat{u}(t)\in[0,1]$ for all $t\in T$. Then

(32)
$$\frac{F\big(t,r\widehat{u}(t)\big)}{r^p} = \frac{1}{r^p}\int_0^{r\widehat{u}(t)} f(t,s)\mathrm{d}s \geq \frac{1}{r^p}\int_0^{r\widehat{u}(t)} f(t,1)s^{p-1}\mathrm{d}s$$
(see hypothesis H(ii)),
$$\geq \frac{1}{p}f(t,1)\widehat{u}(t)^p$$
$$\geq -\frac{1}{p}f(t,1)\|\widehat{u}\|_\infty^p.$$

By hypothesis $-\frac{1}{p}f(\cdot,1)\|\widehat{u}\|_\infty^p \in L^1(T)$. Because of (32), we can apply Fatou's lemma and using (31), we obtain

$$\liminf_{r\to 0^+}\int_{\{\widehat{u}\neq 0\}}\frac{F(t,r\widehat{u})}{r^p}\mathrm{d}t \geq \frac{1}{p}\int_{\{\widehat{u}\neq 0\}}\vartheta_0\widehat{u}^p\mathrm{d}t$$
$$\Rightarrow \frac{1}{p}\|\widehat{u}'\|_p^p - \int_0^b \frac{F(t,r\widehat{u})}{r^p}\mathrm{d}t < 0 \text{ for } r\in(0,1) \text{ small} \quad \text{(see (30))},$$
$$\Rightarrow \widehat{\varphi}(r\widehat{u})<0 \text{ for } r\in(0,1) \text{ small} \quad (\text{recall } \widehat{u}\geq 0),$$
$$\Rightarrow \widehat{m}=\widehat{\varphi}(u_0)<0=\widehat{\varphi}(0) \quad \text{(see (29))},$$
$$\Rightarrow u_0\neq 0.$$

From (29) and Proposition 3.5, we have

(33)
$$\widehat{\varphi}'(u_0)=0,$$
$$\Rightarrow A(u_0)=N_f(u_0) \text{ with } N_f(u)(\cdot)=f\big(\cdot,u(\cdot)\big) \quad \text{for all } u\in W^{1,p}_{\mathrm{per}}(0,b)$$
(recall $u_0\geq 0$ and see (5)),
$$\Rightarrow \left\{\begin{array}{l} -\big(|u_0'(t)|^{p-2}u_0'(t)\big)'=f\big(t,u_0(t)\big) \text{ a.e. on } T,\\ u_0(0)=u_0(b),\ u_0'(0)=u_0'(b) \text{ with } u_0\in C^1(T)\end{array}\right\}$$
(see Kyritsi-Papageorgiou [7]). ■

In fact we can improve the conclusion of this proposition, by strengthening a little hypothesis H(i). So, the new hypotheses on $f(t,x)$ are:

H′: $f: T\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that

(i) for all $x \geq 0$, there exists $\widehat{M} > 0$ such that for all $x \geq \widehat{M}$, $f(\cdot, x) \in L^\infty(T)$ and

$$f(t,x) \leq \alpha(t)(1 + x^{p-1}) \text{ for a.a } t \in T, \text{ all } x \geq 0, \text{ with } \alpha \in L^1(T)_+;$$

hypotheses H′(ii), (iii), (iv) are the same as the corresponding hypotheses H(ii), (iii), (iv).

Proposition 3.7. If *hypotheses H′ hold,* **then** *problem* (1) *has a solution* $u_0 \in$ *int*$\widehat{C}_+$.

Proof. From Proposition 3.6 we already have a positive solution $u_0 \in \widehat{C}_+ \setminus \{0\}$. By virtue of hypothesis H′(ii), for $\xi \geq \max\{\widehat{M}, \|u_0\|_\infty\}$ and for a.a. $t \in \{u_0 > 0\}$ we have

$$f\big(t, u_0(t)\big) \geq \frac{f(t,\xi)}{\xi^{p-1}}\, u_0(t)^{p-1} \geq -c_3 u_0(t)^{p-1} \text{ for some } c_3 > 0$$

(see hypothesis H′(ii)).

Therefore from (33) we have

$$\begin{aligned} &\big(|u_0'(t)|^{p-2}u_0'(t)\big)' \leq c_3 u_0'(t)^{p-1} \text{ a.e. on } T \\ &(\text{recall } u_0'(t) = 0 \text{ a.e. on } T\{u_0 = 0\}, \text{ see [9]}), \\ \Rightarrow\ &u_0 \in \text{int}\widehat{C}_+ \quad (\text{by Vazquez [10]}). \end{aligned}$$ ■

4. Uniqueness of Positive Solutions

In this section we establish the uniqueness of the positive solution. In fact we show that hypotheses H′(iii) and (iv) are both necessary and sufficient for the existence and uniqueness of a positive solution for problem (1).

Proposition 4.1. If *hypotheses H′ hold,* **then** *problem* (1) *has a unique positive solution* $u_0 \in$ *int*$\widehat{C}_+$.

Proof. Let $u, v \in \widehat{C}_+ \setminus \{0\}$ be two positive solutions for problem (1). From the proof of Proposition 3.7, we have that $u, v \in \text{int}\widehat{C}_+$. So,

$$\begin{aligned} &\int_0^b \frac{f(t,u)}{u^{p-1}}(u^p - v^p)\mathrm{d}t \\ &= -\int_0^b \left(|u'|^{p-2}u'\right)'\Big(u - \frac{v^p}{u^{p-1}}\Big)\mathrm{d}t \\ &= \int_0^b |u'|^{p-2}u'\Big(u' - \Big(\frac{v^p}{u^{p-1}}\Big)'\Big)\mathrm{d}t \quad \text{(by integration by parts)} \\ &= \|u'\|_p^p - \int_0^b |u'|^{p-2}u'\Big(\frac{v^p}{u^{p-1}}\Big)'\mathrm{d}t \\ &= \|u'\|_p^p - \|v'\|_p^p + \int_0^b R(v,u)\mathrm{d}t \quad \text{(see Section 2).} \end{aligned} \tag{34}$$

Similarly interchanging the roles of u and v, we obtain

$$\int_0^b \frac{f(t,v)}{v^p}(v^p - u^p)\mathrm{d}t = \|v'\|_p^p - \|u'\|_p^p + \int_0^b R(u,v)\mathrm{d}t. \tag{35}$$

Adding (34) and (35), we have

$$0 \geq \int_0^b \Big(\frac{f(t,u)}{u^{p-1}} - \frac{f(t,v)}{v^{p-1}}\Big)(u^p - v^p)\mathrm{d}t = \int_0^b [R(v,u) + R(u,v)]\mathrm{d}t \geq 0$$

(see hypothesis H$'$(ii) and recall $R(u,v), R(v,u) \geq 0$).

It follows that $R(u,v) = R(v,u) = 0$ and so $u = kv$ for some $k > 0$ (see Allegretto-Huang [2]). The fact that for a.a. $t \in T$, $x \longrightarrow \frac{f(t,x)}{x^{p-1}}$ is strictly decreasing, (see hypothesis H$'$(ii)), implies that $k = 1$ and so $u = v$. This proves the uniqueness of the positive solution $u_0 \in \mathrm{int}\widehat{C}_+$. ■

As we already mentioned, hypotheses H$'$(iii) and (iv) are also necessary for the uniqueness of the positive solution $u_0 \in \mathrm{int}\widehat{C}_+$.

Proposition 4.2. **If** $f : T \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a Carathéodory function satisfying hypotheses H $'$(i), (ii), and problem* (1) *has a unique positive solution* $u_0 \in \widehat{C}_+ \setminus \{0\}$, **then** $\widehat{\lambda}_0(\vartheta_0) < 0 < \widehat{\lambda}_0(\vartheta)$ *where*

$$\vartheta_0(t) = \lim_{x \to 0^+} \frac{f(t,x)}{x^{p-1}} \ \textit{and} \ \vartheta(t) = \lim_{x \to +\infty} \frac{f(t,x)}{x^{p-1}}.$$

Proof. From Proposition 3.7, we know that $u_0 \in \mathrm{int}\widehat{C}_+$. We have

$$\widehat{\lambda}_0(\vartheta_0) \leq \frac{\|u_0'\|_p^p - \int_0^b \vartheta_0 u_0^p \mathrm{d}t}{\|u_0\|_p^p} \quad \text{(see (4) and recall } u_0(t) > 0 \quad \text{for all } t \in T)$$

$$= \frac{\int_0^b f(t,u_0)u_0 \mathrm{d}t - \int_0^b \vartheta_0 u_0^p \mathrm{d}t}{\|u_0\|_p^p}$$

$$< \frac{\int_0^b \vartheta_0 u_0^p \mathrm{d}t - \int_0^b \vartheta_0 u_0^p \mathrm{d}t}{\|u_0\|_p^p} = 0 \quad \text{(see hypothesis H}'\text{(ii))}.$$

So, we have proved that $\widehat{\lambda}_0(\vartheta_0) < 0$.

Let $\beta(t) = \frac{f(t,\|u_0\|_\infty+1)}{(\|u_0\|_\infty+1)^{p-1}}$. Then $\beta \in L^1(T)$ (see hypothesis H′(i)). Let $\widehat{u}_1 \in \mathrm{int}\widehat{C}_+$ be the L^p-normalized eigenfunction corresponding to the eigenvalue $\widehat{\lambda}_0(\beta)$ (see Binding-Rynne [4]). For $k > 0$ large enough we will have $u_0 < k\widehat{u}_1 = \widetilde{u}_1$. As in the proof of Proposition 4.1, we show that

$$\int_0^b \frac{f(t,u_0)}{u_0^{p-1}}(u_0^p - \widetilde{u}_1^p)\mathrm{d}t = \|u_0'\|_p^p - \|\widetilde{u}_1'\|_p^p + \int_0^b R(\widetilde{u}_1,u_0)\mathrm{d}t. \tag{36}$$

$$\int_0^b \big(\widehat{\lambda}_0(\beta)+\beta\big)(\widetilde{u}_1^p - u_0^p)\mathrm{d}t = \|\widetilde{u}_1'\|_p^p - \|u_0'\|_p^p + \int_0^b R(u_0,\widetilde{u}_1)\mathrm{d}t. \tag{37}$$

We add (36) and (37). Then

$$\int_0^b \Big(\frac{f(t,u_0)}{u_0^{p-1}} - \big(\widehat{\lambda}_0(\beta)+\beta\big)\Big)(u_0^p - \widetilde{u}_1^p)\mathrm{d}t = \int_0^b [R(\widetilde{u}_1,u_0)+R(u_0,\widetilde{u}_1)]\mathrm{d}t \geq 0. \tag{38}$$

By virtue of hypothesis H′(ii) we have

$$\begin{aligned} \frac{f(t,u_0)}{u_0^{p-1}} > &\ \frac{f(t,\|u_0\|_\infty+1)}{(\|u_0\|_\infty+1)^{p-1}} = \beta(t) \text{ a.e. on } T, \\ \Rightarrow &\ \frac{f(t,u_0)}{u_0^{p-1}} - \beta(t) > 0 \text{ a.e. on } T. \end{aligned} \tag{39}$$

Also since $u_0 < \widetilde{u}_1$, we have

$$(u_0^p - \widetilde{u}_1^p)(t) < 0 \quad \text{for all } \ t \in T. \tag{40}$$

Using (39) and (40) in (38), we infer that $\widehat{\lambda}_0(\beta) > 0$. But note that $\beta \geq \vartheta$ (see hypothesis H′(ii)) and so from (4) we have that $0 < \widehat{\lambda}_0(\beta) \leq \widehat{\lambda}_0(\vartheta)$. ∎

Summarizing the situation, we have the following definitive existence and uniqueness theorem for problem (1).

Theorem 4.3. **If** $f : T\times\mathbb{R} \longrightarrow \mathbb{R}$ *is a Carathéodory function satisfying hypotheses* H'*(i), (ii),* **then** *problem* (1) *has a unique positive solution* $u_0 \in int\widehat{C}_+$

if and only if

$$\widehat{\lambda}_0(\vartheta_0) < 0 < \widehat{\lambda}_0(\vartheta),$$

where $\vartheta_0(t) = \lim\limits_{x\to 0^+} \frac{f(t,x)}{x^{p-1}}$ *and* $\vartheta(t) = \lim\limits_{x\to +\infty} \frac{f(t,x)}{x^{p-1}}$.

ACKNOWLEDGMENTS

The authors wish to thank a very knowledgeable for his/her corrections and remarks.

REFERENCES

1. S. Aizicovici, N. S. Papageorgiou and V. Staicu, Multiple nontrival solutions for nonlinear periodic problems with the p-Laplacian, *J. Diff. Eqns.*, **243** (2007), 504-535.
2. W. Allegretto and Y. Huang, A Picone's identity for the p-Laplacian and applications, *Nonlinear Analysis*, **32** (1998), 819-830.
3. P. Binding and B. P. Rynne, The spectrum of the periodic p-Laplacian, *J. Differential Equations*, **235** (2007), 199-218.
4. P. Binding and B. P. Rynne, Variational and nonvariational eigenvalues of the p-Laplacian, *J. Differential Equations*, **244** (2008), 24-39.
5. M. del Pino, M. A.-R. Manásevich and A. Murúa, Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE, *Nonlinear Analysis*, **18** (1992), 79-92.
6. P. Drabek and R. Manásevich, On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian, *Differential Integral Eqns.*, **12** (1999), 773-788.
7. S. Th. Kyritsi and N. S. Papageorgiou, On the multiplicity of solutions for nonlinear periodic problems with the nonlinearity crossing several eigenvalues, *Glasgow Math. J.*, **52** (2010), 271-302.
8. D. Motreanu, V. Motreanu and N. S. Papageorgiou, Multiple solutions for resonant periodic equations, *Nonlinear Diff. Equ. Appl.*, **17** (2010), 535-557.
9. N. S. Papageorgiou and S. Th. Kyritsi-Yiallourou, *Handbook of Applied Analysis*, Springer, New York, 2009.
10. J. Vazquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.*, **12** (1984), 191-202.
11. X. Yang, Multiple periodic solutions of a class of p-Laplacian, *J. Math. Anal. Appl.*, **314** (2006), 17-29.
12. M. Zhang, The roatation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, *J. London Math. Soc.*, **64** (2001), 125-143.

Sophia Th. Kyritsi
Department of Mathematics
Hellenic Naval Academy
Piraeus 18539
Greece
E-mail: skyrits@math.ntua.gr

Nikolaos S. Papageorgiou
National Technical University
Department of Mathematics
Zografou Campus
Athens 15780
Greece
E-mail: npapg@math.ntua.gr