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COUPLING EXTRA-GRADIENT METHODS WITH KM'S METHODS FOR VARIATIONAL INEQUALITIES AND FIXED POINTS

Yonghong Yao, Yeong-Cheng Liou* and Pei-Xia Yang

Abstract. In this paper, we suggest and analyze a new method which couple extra-gradient methods with KM's methods for solving some variational inequality problem and fixed points problem. It is shown that the proposed method has strong convergence in a general Hilbert space.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear operator. It is well-known that the variational inequality problem VI(C, A) is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0, \quad \forall v \in C.$$

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [1]-[17], [20] and the references therein. In order to solve variational inequality $VI(C, A)$, Korpelevich [6] introduced a so-called extra-gradient method

$$\begin{cases} y_n = P_C[x_n - \lambda A x_n], \\ x_{n+1} = P_C[x_n - \lambda A y_n], n \geq 0, \end{cases} \tag{1.1}$$

where P_C is the metric projection from R^n onto C. However, the algorithm (1.1) fails, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces.

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*Corresponding author.

The purpose of this paper is to modify the above extra-gradient method such that the strong convergence is obtained in the setting of infinite-dimensional Hilbert spaces.

On the other hand, in the present paper we also concern on the algorithm construction for non-expansive mappings. This is main due to many practical problems can be formulated as a fixed point problem

$$x = Sx,$$

where S is a non-expansive mapping defined on a closed convex subset C of a Hilbert space. For instance, some problems in signal processing, e.g., phase retrieval [21],[22] and design of a nonlinear synthetic discriminant filter for optical pattern recognition ([23]) can be formulated as a split feasibility problem of finding a point x^* with the property:

$$x^* \in C \; and \; Ax^* \in Q,$$

where C and Q are closed convex subsets of R^n and R^m, respectively, and $A : R^n \to R^m$ is a linear operator can equivalently be rewritten as a fixed point problem

$$x* = Sx^* = P_C[I - \gamma A^*(I - P_Q)Ax^*],$$

where P_C and P_Q are the (nearest point) projections onto C and Q, respectively, γ is any positive parameter, and A^* is the adjoint of A. It is known that for sufficiently small $\gamma > 0$, the mapping $S = P_C[I-\gamma A^*(I-P_Q)A]$ is non-expansive. Another example is the intensity-modulated radiation therapy which have received a great deal of attention recently; see [24]-[27] and references therein. Therefore, it is an interesting topic of finding some algorithms for approximating fixed point of a non-expansive mapping. Some related works can be found in [28]-[41].

It is worth mentioning that some algorithms in signal processing and image reconstruction may be written as the well-known KM iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n \tag{1.2}$$

and that the main feature of its corresponding convergence theorems provided a unified frame for analyzing various concrete algorithms. For details, see [32], [43]-[44]. Although the above KM algorithm (1.2) solve some practical problems, we only obtain some weak convergence theorems. A natural question rises: could we obtain a strong convergence result by using the well-known KM's algorithm? In this connection, in 1975, Genel and Lindenstrass [42] gave a counterexample. For more details, please see [42]. This fact implies that some modifications associated with KM algorithms are needed.

In this paper, we suggest and analyze a new method which couple extra-gradient method (1.1) with KM's method (1.2) for solving some variational inequality problem and fixed points problem. It is shown that the proposed method has strong convergence in a general Hilbert space.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. A mapping $A : C \to H$ is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \forall u, v \in C.$$

Recall that a mapping $S : C \to C$ is said to be non-expansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

Denote by $Fix(S)$ the set of fixed points of S; that is, $Fix(S) = \{x \in C : Sx = x\}$.

It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}.$$

We denote u_0 by $P_C u$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$;

(ii) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ for every $x, y \in H$;

(iii) $\langle x - P_C x, y - P_C x\rangle \leq 0$ for all $x \in H$, $y \in C$.

We need the following well-known lemmas for proving our main results.

Lemma 2.1. (Demiclosedness principle). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a non-expansive mapping with $Fix(S) \neq \emptyset$. Then S is demiclosed on C, i.e., if $y_n \to z \in C$ weakly and $y_n - Sy_n \to y$ strongly, then $(I - S)z = y$.*

Lemma 2.2. (Suzuki's Lemma [35]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3. (Xu's Lemma [18]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section we will state and prove our main result. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. Set $\Omega = VI(C, A) \cap Fix(S)$.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly-monotone mapping and $S : C \to C$ be a non-expansive mapping with $\Omega \neq \emptyset$. For given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by*

$$\begin{cases} z_n = P_C[x_n - \lambda_n A x_n], \\ y_n = P_C[(1-\alpha_n)(z_n - \lambda_n A z_n)], \\ x_{n+1} = \beta x_n + (1-\beta) S y_n, \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence in $[a, b] \subset (0, 2\alpha)$ and $\beta \in (0, 1)$ is a constant. Assume the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $z_0 = P_\Omega(0)$.

We will divide our detail proofs into several conclusions. In the sequel, we assume all conditions in Theorem 3.1 are satisfied. In order to prove our main result, we first need some facts:

Fact 1. If $\tilde{x} \in VI(C, A)$, the we have

$$\tilde{x} = P_C(\tilde{x} - \gamma A\tilde{x}), \text{ for all } \gamma > 0.$$

In particular, if we choose $\gamma = \lambda_n(1 - \alpha_n)$, then we have

$$\tilde{x} = P_C[\tilde{x} - \lambda_n(1-\alpha_n)A\tilde{x}] = P_C[\alpha_n\tilde{x} + (1-\alpha_n)(\tilde{x} - \lambda_n A\tilde{x})], \forall n \geq 0. \tag{3.2}$$

Fact 2. (see [12]) $I - \lambda_n A$ is non-expansive and for all $x, y \in C$

$$\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2. \tag{3.3}$$

Conclusion 3.2. The sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{Ax_n\}, \{Az_n\}$ and $\{Sy_n\}$ are all bounded.

Proof. Take $x^* \in \Omega$. By (3.1), (3.2) and the non-expansivity of $I - \lambda_n A$, we have

$$\begin{aligned} \|z_n - x^*\| &= \|P_C[x_n - \lambda_n A x_n] - P_C[x^* - \lambda_n A x^*]\| \\ &\leq \|(x_n - x^*) - \lambda_n(Ax_n - \lambda A x^*)\| \\ &= \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\| \\ &\leq \|x_n - x^*\|, \end{aligned}$$

and

$$\begin{aligned} &\|y_n - x^*\| \\ &= \|P_C[(1-\alpha_n)(z_n - \lambda_n A z_n)] - P_C[\alpha_n x^* + (1-\alpha_n)(x^* - \lambda_n A x^*)]\| \\ &\leq \|\alpha_n(-x^*) + (1-\alpha_n)[(z_n - \lambda_n A z_n) - (x^* - \lambda_n A x^*)]\| \\ &\leq \alpha_n\|x^*\| + (1-\alpha_n)\|(I - \lambda_n A)z_n - (I - \lambda_n A)x^*\| \\ &\leq \alpha_n\|x^*\| + (1-\alpha_n)\|z_n - x^*\| \\ &\leq \alpha_n\|x^*\| + (1-\alpha_n)\|x_n - x^*\|. \end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta(x_n - x^*) + (1-\beta)(Sy_n - x^*)\| \\ &\leq \beta\|x_n - x^*\| + (1-\beta)\|Sy_n - x^*\| \\ &\leq \beta\|x_n - x^*\| + (1-\beta)\|y_n - x^*\| \\ &\leq \beta\|x_n - x^*\| + (1-\beta)[\alpha_n\|x^*\| + (1-\alpha_n)\|x_n - x^*\|] \\ &= (1-\beta)\alpha_n\|x^*\| + [1 - (1-\beta)\alpha_n]\|x_n - x^*\| \\ &\leq \max\{\|x^*\|, \|x_0 - x^*\|\}. \end{aligned} \tag{3.5}$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}, \{z_n\}, \{Ax_n\}, \{Az_n\}$ and $\{Sy_n\}$. ■

Conclusion 3.3. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|Ax_n - Ax^*\| = \lim_{n\to\infty} \|Az_n - Ax^*\| = 0$.

Proof. From (3.1), we have

$$\begin{aligned} &\|Sy_n - Sy_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| \\ &= \|P_C[(1-\alpha_n)(z_n - \lambda_n A z_n)] - P_C[(1-\alpha_{n-1})(z_{n-1} - \lambda_{n-1} A z_{n-1})]\| \\ &\leq \|(z_n - \lambda_n A z_n) - (z_{n-1} - \lambda_{n-1} A z_{n-1})\| + \alpha_n\|z_n - \lambda_n A z_n\| \\ &\quad + \alpha_{n-1}\|z_{n-1} - \lambda_{n-1} A z_{n-1}\| \\ &= \|(z_n - \lambda_n A z_n) - (z_{n-1} - \lambda_n A z_{n-1}) + (\lambda_{n-1} - \lambda_n) A z_{n-1}\| \\ &\quad + \alpha_n\|z_n - \lambda_n A z_n\| + \alpha_{n-1}\|z_{n-1} - \lambda_{n-1} A z_{n-1}\| \end{aligned}$$

$$
\begin{aligned}
&\le \|z_n - z_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| + \alpha_n\|z_n - \lambda_n Az_n\| \\
&\quad +\alpha_{n-1}\|z_{n-1} - \lambda_{n-1}Az_{n-1}\| \\
&= \|P_C[x_n - \lambda_n Ax_n] - P_C[x_{n-1} - \lambda_{n-1}Ax_{n-1}]\| + |\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| \\
&\quad +\alpha_n\|z_n - \lambda_n Az_n\| + \alpha_{n-1}\|z_{n-1} - \lambda_{n-1}Az_{n-1}\| \\
&\le \|(x_n - \lambda_n Ax_n) - (x_{n-1} - \lambda_n Ax_{n-1}) + (\lambda_{n-1} - \lambda_n)Ax_{n-1}\| \\
&\quad +|\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| + \alpha_n\|z_n - \lambda_n Az_n\| + \alpha_{n-1}\|z_{n-1} - \lambda_{n-1}Az_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| \\
&\quad +\alpha_n\|z_n - \lambda_n Az_n\| + \alpha_{n-1}\|z_{n-1} - \lambda_{n-1}Az_{n-1}\|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|Sy_n - Sy_{n-1}\| - \|x_n - x_{n-1}\| \\
\le\ & |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| \\
&+\alpha_n\|z_n - \lambda_n Az_n\| + \alpha_{n-1}\|z_{n-1} - \lambda_{n-1}Az_{n-1}\|.
\end{aligned}
$$

It follows that

$$\limsup_{n\to\infty}(\|Sy_n - Sy_{n-1}\| - \|x_n - x_{n-1}\|) \le 0.$$

Thus, by Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|Sy_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta)\|Sy_n - x_n\| = 0.$$

From (3.1) and (3.3), we get

$$
\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C[x_n - \lambda_n Ax_n] - P_C[x^* - \lambda_n Ax^*]\|^2 \\
&\le \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|^2 \\
&\le \|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ax^*\|^2 \\
&\le \|x_n - x^*\|^2 + a(b - 2\alpha)\|Ax_n - Ax^*\|^2.
\end{aligned}
\tag{3.6}
$$

From (3.3), (3.4), (3.6) and the convexity of the norm, we deduce

$$
\begin{aligned}
&\|y_n - x^*\|^2 \\
\le\ & \|\alpha_n(-x^*) + (1-\alpha_n)[(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*)]\|^2 \\
\le\ & \alpha_n\|x^*\|^2 + (1-\alpha_n)\|(I - \lambda_n A)z_n - (I - \lambda_n A)x^*\|^2 \\
\le\ & \alpha_n\|x^*\|^2 + (1-\alpha_n)[\|z_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Az_n - Ax^*\|^2] \\
\le\ & \alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 + (1-\alpha_n)a(b - 2\alpha)\|Az_n - Ax^*\|^2 \\
&+(1-\alpha_n)a(b - 2\alpha)\|Ax_n - Ax^*\|^2.
\end{aligned}
\tag{3.7}
$$

By the convexity of the norm, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\beta(x_n - x^*) + (1-\beta)(Sy_n - x^*)\|^2 \\ &\le \beta\|x_n - x^*\|^2 + (1-\beta)\|Sy_n - x^*\|^2 \\ &\le \beta\|x_n - x^*\|^2 + (1-\beta)\|y_n - x^*\|^2. \end{aligned} \tag{3.8}$$

Substitute (3.7) into (3.8) to obtain

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ \le\ &\beta\|x_n - x^*\|^2 + (1-\beta)[\alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 \\ &+(1-\alpha_n)a(b-2\alpha)\|Az_n - Ax^*\|^2 + (1-\alpha_n)a(b-2\alpha)\|Ax_n - Ax^*\|^2] \\ =\ &(1-\beta)\alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 + (1-\beta)(1-\alpha_n)a(b-2\alpha)\|Az_n - Ax^*\|^2 \\ &+(1-\beta)(1-\alpha_n)a(b-2\alpha)\|Ax_n - Ax^*\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &(1-\beta)(1-\alpha_n)a(2\alpha - b)(\|Ax_n - Ax^*\|^2 + \|Az_n - Ax^*\|^2) \\ \le\ &(1-\beta)\alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ \le\ &(1-\beta)\alpha_n\|x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \times \|x_n - x_{n+1}\|. \end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain $\|Ax_n - Ax^*\| \to 0$ and $\|Az_n - Ax^*\| \to 0$ as $n \to \infty$. ■

Conclusion 3.4. $\lim_{n\to\infty} \|Sy_n - y_n\| = 0.$

Proof. By the firmly non-expansivity of the metric projection P_C (see (ii)), we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|P_C[x_n - \lambda_n Ax_n] - P_C[x^* - \lambda_n Ax^*]\|^2 \\ &\le \langle (x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*), z_n - x^* \rangle \\ &= \frac{1}{2}\Big\{\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 + \|z_n - x^*\|^2 \\ &\quad -\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*) - (z_n - x^*)\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Ax^*)\|^2\Big\} \\ &= \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\ &\quad +2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle - \|\lambda_n(Ax_n - Ax^*)\|^2\Big\} \end{aligned}$$

$$\leq \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|\Big\},$$

and

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C[(1-\alpha_n)(z_n - \lambda_n Az_n)] - P_C(x^* - \lambda_n Ax^*)\|^2 \\
&\leq \langle (1-\alpha_n)(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*), y_n - x^*\rangle \\
&= \frac{1}{2}\Big\{\|(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*) - \alpha_n(I - \lambda_n A)z_n\|^2 + \|y_n - x^*\|^2 \\
&\quad - \|(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*) - (y_n - x^*) - \alpha_n(I - \lambda_n A)z_n\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*)\|^2 + \alpha_n M + \|y_n - x^*\|^2 \\
&\quad - \|(z_n - y_n) - \lambda_n(Az_n - Ax^*) - \alpha_n(I - \lambda_n A)z_n\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|z_n - x^*\|^2 + \alpha_n M + \|y_n - x^*\|^2 - \|z_n - y_n\|^2 \\
&\quad + 2\lambda_n\langle z_n - y_n, Az_n - Ax^*\rangle + 2\alpha_n\langle (I - \lambda_n A)z_n, z_n - y_n\rangle \\
&\quad - \|\lambda_n(Az_n - Ax^*) + \alpha_n(I - \lambda_n A)z_n\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|z_n - x^*\|^2 + \alpha_n M + \|y_n - x^*\|^2 - \|z_n - y_n\|^2 \\
&\quad + 2\lambda_n\|z_n - y_n\|\|Az_n - Ax^*\| + 2\alpha_n\|(I - \lambda_n A)z_n\|\|z_n - y_n\|\Big\},
\end{aligned}$$

where $M > 0$ is some constant.

It follows that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|,$$

and

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|z_n - x^*\|^2 + \alpha_n M - \|z_n - y_n\|^2 \\
&\quad + 2\lambda_n\|z_n - y_n\|\|Az_n - Ax^*\| + 2\alpha_n\|(I - \lambda_n A)z_n\|\|z_n - y_n\| \\
&\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - \|z_n - y_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\| \\
&\quad + \alpha_n M + 2\lambda_n\|z_n - y_n\|\|Az_n - Ax^*\| + 2\alpha_n\|(I - \lambda_n A)z_n\|\|z_n - y_n\|.
\end{aligned}$$

This together with (3.8) imply that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta\|x_n - x^*\|^2 + (1-\beta)\|y_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - (1-\beta)\|x_n - z_n\|^2 - (1-\beta)\|z_n - y_n\|^2 \\
&\quad + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\| + \alpha_n M \\
&\quad + 2\lambda_n\|z_n - y_n\|\|Az_n - Ax^*\| + 2\alpha_n\|(I - \lambda_n A)z_n\|\|z_n - y_n\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(1-\beta)\|x_n - z_n\|^2 + (1-\beta)\|z_n - y_n\|^2 \\
\leq\ &(\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&+2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\| + \alpha_n M \\
&+2\lambda_n\|z_n - y_n\|\|Az_n - Ax^*\| + 2\alpha_n\|(I - \lambda_n A)z_n\|\|z_n - y_n\|.
\end{aligned}$$

Since $\alpha_n \to 0, \|x_n - x_{n+1}\| \to 0$, $\|Ax_n - Ax^*\| \to 0$ and $\|Az_n - Ax^*\| \to 0$, we derive $\|x_n - z_n\| \to 0$ and $\|z_n - y_n\| \to 0$. Note that

$$\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\|.$$

Hence,

$$\|y_n - Sy_n\| \to 0.$$ ■

Conclusion 3.5.

$$\limsup_{n\to\infty}\langle z_0, z_0 - y_n\rangle \leq 0,$$

where $z_0 = P_\Omega(0)$.

Proof. We choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle z_0, z_0 - y_n\rangle = \lim_{i\to\infty}\langle z_0, z_0 - y_{n_i}\rangle.$$

As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $y_{n_i} \rightharpoonup z$. Since $\|Sy_n - y_n\| \to 0$, we obtain $Sy_{n_i} \rightharpoonup z$ as $i \to \infty$. This together with the demi-closedness principle (see Lemma 2.1), we have immediately $z \in Fix(S)$. Next, we only need to prove $z \in VI(C, A)$.

Define

$$Tv = \begin{cases} Av + N_C v, v \in C, \\ \emptyset, \qquad\qquad v \notin C. \end{cases}$$

Then T is maximal monotone (see [19]). Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Av\rangle \geq 0$. On the other hand, from $y_n = P_C[(1-\alpha_n)(z_n - \lambda_n Az_n)]$, we have

$$\langle v - y_n, y_n - (1-\alpha_n)(z_n - \lambda_n Az_n)\rangle \geq 0,$$

that is,

$$\langle v - y_n, \frac{y_n - z_n}{\lambda_n} + Az_n + \frac{\alpha_n}{\lambda_n}(I - \lambda_n A)z_n\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\lambda_{n_i}} + Az_{n_i} + \frac{\alpha_{n_i}}{\lambda_{n_i}}(I - \lambda_{n_i}A)z_{n_i}\rangle \\
&= \langle v - y_{n_i}, Av - Az_{n_i} - \frac{y_{n_i} - z_{n_i}}{\lambda_{n_i}} - \frac{\alpha_{n_i}}{\lambda_{n_i}}(I - \lambda_{n_i}A)z_{n_i}\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Az_{n_i}\rangle \\
&\quad -\langle v - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\lambda_{n_i}} + \frac{\alpha_{n_i}}{\lambda_{n_i}}(I - \lambda_{n_i}A)z_{n_i}\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Az_{n_i}\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\lambda_{n_i}} + \frac{\alpha_{n_i}}{\lambda_{n_i}}(I - \lambda_{n_i}A)z_{n_i}\rangle.
\end{aligned}$$

Noting that $\alpha_{n_i} \to 0$, $\|y_{n_i} - z_{n_i}\| \to 0$ and A is Lipschitz continuous, we obtain $\langle v - z, w\rangle \geq 0$. Since T is maximal monotone, we have $z \in T^{-1}(0)$ and hence $z \in VI(C, A)$. Thus, we obtain $z \in \Omega$. Therefore,

$$\limsup_{n\to\infty}\langle z_0, z_0 - y_n\rangle = \lim_{i\to\infty}\langle z_0, z_0 - y_{n_i}\rangle = \langle z_0, z_0 - z\rangle \leq 0. \qquad \blacksquare$$

Next, we prove Theorem 3.1. *Proof.* Finally, we prove $x_n \to z_0$. By the property (ii) of metric projection P_C, we have

$$\begin{aligned}
&\|y_n - z_0\|^2 \\
&= \|P_C[(1 - \alpha_n)(z_n - \lambda_n Az_n)] - P_C[\alpha_n z_0 + (1 - \alpha_n)(z_0 - \lambda_n Az_0)]\|^2 \\
&\leq \langle \alpha_n(-z_0) + (1 - \alpha_n)[(z_n - \lambda_n Az_n) - (z_0 - \lambda_n Az_0)], y_n - z_0\rangle \\
&\leq \alpha_n\langle z_0, z_0 - y_n\rangle + (1 - \alpha_n)\|(z_n - \lambda_n Az_n) - (z_0 - \lambda_n Az_0)\|\|y_n - z_0\| \\
&\leq \alpha_n\langle z_0, z_0 - y_n\rangle + (1 - \alpha_n)\|z_n - z_0\|\|y_n - z_0\| \\
&\leq \alpha_n\langle z_0, z_0 - y_n\rangle + \frac{1 - \alpha_n}{2}(\|z_n - z_0\|^2 + \|y_n - z_0\|^2).
\end{aligned}$$

Hence

$$\begin{aligned}
\|y_n - z_0\|^2 &\leq (1 - \alpha_n)\|z_n - z_0\|^2 + 2\alpha_n\langle z_0, z_0 - y_n\rangle \\
&\leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle z_0, z_0 - y_n\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \beta\|x_n - z_0\|^2 + (1 - \beta)\|y_n - z_0\|^2 \\
&\leq [1 - (1 - \beta)\alpha_n]\|x_n - z_0\|^2 + 2(1 - \beta)\alpha_n\langle z_0, z_0 - y_n\rangle.
\end{aligned}$$

We apply Lemma 2.3 to the last inequality to deduce that $x_0 \to z_0$. This completes the proof. $\blacksquare$

Remark 3.6. It is clear that Korpelevich's extra-gradient method has strong convergence in a finite-dimensional Hilbert space (as a matter of fact, in Euclid space R^n) and has only weak convergence in the setting of infinite-dimensional Hilbert spaces. It is well-known that the KM's method also has only weak convergence in Hilbert space. However our method which couple Korpelevich's extra-gradient method with KM's method has strong convergence in a general Hilbert space.

Remark 3.7. We not only prove the proposed algorithm (3.1) converges strongly to a common element of a solution of the variational inequality $VI(C, A)$ and the fixed point of a non-expansive mapping S, but also note that this common element $z_0 = P_\Omega(0)$ is the minimum norm element in Ω. This is an additional interesting point.

4. Application

A mapping $T : C \to C$ is called strictly pseudo-contractive if there exists k with $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,$$

for all $x, y \in C$. Put $A = I - T$, then we have

$$\|(I - A)x - (I - A)y\|^2 \le \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand,

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \ge \frac{1 - k}{2}\|Ax - Ay\|^2.$$

As an application of Theorem 3.1, we have the following.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. Let T be a k-strictly pseudo-contractive mapping of C into itself and let S be a non-expansive mapping of C into itself such that $Fix(T) \cap Fix(S) \ne \emptyset$. For given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by*

$$\begin{cases} z_n = (1 - \lambda_n)x_n + \lambda_n T x_n, \\ y_n = P_C[(1 - \alpha_n)(z_n - \lambda_n(I - T)z_n)], \\ x_{n+1} = \beta x_n + (1 - \beta)S y_n, \quad n \ge 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence in $[a, b] \subset (0, 1 - k)$ and $\beta \in (0, 1)$ is a constant. Assume the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ *generated by* (4.1) *converges strongly to* $z_0 = P_{Fix(T)\cap Fix(S)}(0)$.

Proof. Put $A = I - T$. Then A is $(1-k)/2$-inverse-strongly monotone. We have $Fix(T) = VI(C, A)$. So, by Theorem 3.1, we can obtain the desired result. This completes the proof. ∎

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Yonghong Yao
Department of Mathematics
Tianjin Polytechnic University
Tianjin 300387
P. R. China
E-mail: yaoyonghong@yahoo.cn

Yeong-Cheng Liou
Department of Information Management
Cheng Shiu University
Kaohsiung 833, Taiwan
E-mail: simplex_liou@hotmail.com

Pei-Xia Yang
Department of Mathematics
Tianjin Polytechnic University
Tianjin 300387
P. R. China
E-mail: yang_peixia@126.com