TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 4, pp. 1293-1327, August 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

GEOMETRY OF $\mathcal{P}R$-WARPED PRODUCTS IN PARA-KÄHLER MANIFOLDS

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Abstract. In this paper, we initiate the study of $\mathcal{P}R$-warped products in para-Kähler manifolds and prove some fundamental results on such submanifolds. In particular, we establish a general optimal inequality for $\mathcal{P}R$-warped products in para-Kähler manifolds involving only the warping function and the second fundamental form. Moreover, we completely classify $\mathcal{P}R$-warped products in the flat para-Kähler manifold with least codimension which satisfy the equality case of the inequality. Our results provide an answer to the Open Problem (3) proposed in [19, Section 5].

1. Introduction

An almost para-Hermitian manifold is a manifold $\widetilde{M}$ equipped with an almost product structure $\mathcal{P} \neq \pm I$ and a pseudo-Riemannian metric $\widetilde{g}$ such that

$$\mathcal{P}^2 = I, \ \ \widetilde{g}(\mathcal{P}X, \mathcal{P}Y) = -\widetilde{g}(X, Y), \tag{1.1}$$

for vector fields X, Y tangent to $\widetilde{M}$, where I is the identity map. Clearly, it follows from (1.1) that the dimension of $\widetilde{M}$ is even and the metric $\widetilde{g}$ is neutral. An almost para-Hermitian manifold is called ***para-Kähler*** if it satisfies $\widetilde{\nabla}\mathcal{P} = 0$ identically, where $\widetilde{\nabla}$ denotes the Levi Civita connection of $\widetilde{M}$. We define $||X||_2$ associated with $\widetilde{g}$ on $\widetilde{M}$ by $||X||_2 = \widetilde{g}(X, X)$.

Properties of para-Kähler manifolds were first studied in 1948 by Rashevski who considered a neutral metric of signature (m, m) defined from a potential function on a locally product $2m$-manifold [27]. He called such manifolds stratified spaces. Para-Kähler manifolds were explicitly defined by Rozenfeld in 1949 [28]. Such manifolds were also defined by Ruse in 1949 [29] and studied by Libermann [23] in the context of G-structures.

Received August 23, 2011, accepted September 5, 2011.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 53B25, 53B30, 53C15, 53C20.
Key words and phrases: Para-Kähler manifold, $\mathcal{P}R$-Submanifold, Warped product.

There exist many para-Kähler manifolds, for instance, it was proved in [22] that a homogeneous manifold $\widetilde{M} = G/H$ of a semisimple Lie group G admits an invariant para-Kähler structure $(\widetilde{g}, \mathcal{P})$ if and only if it is a covering of the adjoint orbit $\mathrm{Ad}_G h$ of a semisimple element h. Para-Kähler manifolds have been applied in supersymmetric field theories as well as in string theory in recent years (see, for instance, [16, 17, 18]). (For a nice survey on para-Kähler manifolds, see [19].)

A pseudo-Riemannian submanifold M of a para-Kähler manifold $\widetilde{M}$ is called *invariant* if the tangent bundle of M is invariant under the action of $\mathcal{P}$. M is called *anti-invariant* if $\mathcal{P}$ maps each tangent space $T_pM,\ p \in M$, into the normal space $T_p^{\perp}M$. A *Lagrangian submanifold* M of a para-Kähler manifold $\widetilde{M}$ is an anti-invariant submanifold satisfying $\dim \widetilde{M} = 2\dim M$. Such submanifolds have been investigated recently in [12, 13, 14, 15].

A pseudo-Riemannian submanifold M of a para-Kähler manifold $\widetilde{M}$ is called a *$\mathcal{P}R$-submanifold* if the tangent bundle TM of M is the direct sum of an *invariant* distribution $\mathcal{D}$ and an *anti-invariant* distribution $\mathcal{D}^{\perp}$, i.e.,

$$T(M) = \mathcal{D} \oplus \mathcal{D}^{\perp},\ \ \mathcal{P}\mathcal{D} = \mathcal{D},\ \ \mathcal{P}\mathcal{D}^{\perp} \subseteq T_p^{\perp}(M).$$

A $\mathcal{P}R$-submanifold is called a *$\mathcal{P}R$-warped product* if it is a warped product $N_{\top} \times_f N_{\perp}$ of an invariant submanifold $N_{\top}$ and an anti-invariant submanifold $N_{\perp}$.

In this paper we initiate the study of $\mathcal{P}R$-warped products in para-Kähler manifolds. The basic properties of $\mathcal{P}R$-warped products are given in section 3. We establish in section 4 a general optimal inequality for $\mathcal{P}R$-warped products in para-Kähler manifolds involving only the warping function and the second fundamental form. In section 5, we provide the exact solutions of a PDE system associated with $\mathcal{P}R$-warped products. In the last section, we classify $\mathcal{P}R$-warped products $N_{\top} \times_f N_{\perp}$ with least codimension in the flat para-Kähler manifold which verify the equality case of the general inequality derived in section 4.

2. Preliminaries

2.1. Warped product manifolds

The notion of warped product (or, more generally warped bundle) was introduced by Bishop and O'Neill in [4] in order to construct a large variety of manifolds of negative curvature. For example, negative space forms can easily be constructed in this way from flat space forms. The interest of geometers was to extend the classical de Rham theorem to warped products. Hiepko proved a result in [21] which will be used in this paper.

Let us recall some basic results on warped products. Let B and F be two pseudo-Riemannian manifolds with pseudo-Riemannian metrics g_B and g_F respectively, and f

a positive function on B. Consider the product manifold $B \times F$. Let $\pi_1 : B \times F \longrightarrow B$ and $\pi_2 : B \times F \longrightarrow F$ be the canonical projections.

We define the manifold $M = B \times_f F$ and call it *warped product* if it is equipped with the following warped metric

$$g(X,Y) = g_B\big(\pi_{1*}(X), \pi_{1*}(Y)\big) + f^2(\pi_1(p))g_F\big(\pi_{2*}(X), \pi_{2*}(Y)\big) \tag{2.1}$$

for all $X, Y \in T_p(M)$, $p \in M$, or equivalently,

$$g = g_B + f^2\ g_F. \tag{2.2}$$

The function f is called *the warping function.* For the sake of simplicity we will identify a vector field X on B (respectively, a vector field Z on F) with its lift $\tilde{X}$ (respectively $\tilde{Z}$) on $B \times_f F$.

If ∇, ∇^B and ∇^F denote the Levi-Civita connections of M, B and F, respectively, then the following formulas hold

$$\begin{aligned} &\nabla_X Y = \nabla^B_X Y, \\ &\nabla_X Z = \nabla_Z X = X(\ln f)\ Z, \\ &\nabla_Z W = \nabla^F_Z W - g(Z,W)\ \nabla(\ln f) \end{aligned} \tag{2.3}$$

where X, Y are tangent to B and Z, W are tangent to F. Moreover, $\nabla(\ln f)$ is the gradient of $\ln f$ with respect to the metric g.

2.2. Geometry of submanifolds

Let M be an n-dimensional submanifold of $\widetilde{M}$. We need the Gauss and Weingarten formulas:

$$(\mathbf{G})\quad \widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \qquad (\mathbf{W})\quad \widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi\,,$$

for vector fields X, Y tangent to M and ξ normal to M, where ∇ is the induced connection, $\nabla^\perp$ is the normal connection on the normal bundle $T^\perp(M)$, σ is the second fundamental form, and A_ξ is the shape operator associated with the normal section ξ. The mean curvature vector H of M is defined by $H = \frac{1}{n}\mathrm{trace\ h}$.

For later use we recall the equations of Gauss and Codazzi:

$$(\mathbf{EG})\ \ g\big(R_{XY}Z, W\big) = \widetilde{g}\big(\widetilde{R}_{XY}Z, W\big) + \widetilde{g}\big(\sigma(Y,Z), \sigma(X,W)\big) - \widetilde{g}\big(\sigma(X,Z), \sigma(Y,W)\big),$$

$$(\mathbf{EC})\ \ (\widetilde{R}_{XY}Z)^\perp = (\bar{\nabla}_X \sigma)(Y,Z) - (\bar{\nabla}_Y \sigma)(X,Z)$$

for X, Y, Z and W tangent to M, where R, $\widetilde{R}$ are the curvature tensors on M and $\widetilde{M}$, respectively, $(\widetilde{R}_{XY}Z)^\perp$ is the normal component of $\widetilde{R}_{XY}Z$ and $\bar{\nabla}$ is the van der Waerden - Bortolotti connection defined as

$$(\bar{\nabla}_X \sigma)(Y,Z) = \nabla^\perp_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X, Z). \tag{2.4}$$

In this paper the curvature is defined by $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

A submanifold is called *totally geodesic* if its second fundamental form vanishes identically. For a normal vector field ξ on M, if $A_\xi = \lambda I$, for certain function λ on M, then ξ is called a *umbilical section* (or M is *umbilical with respect to* ξ). If M is umbilical with respect to every (local) normal vector field, then M is called a *totally umbilical submanifold*. A pseudo-Riemannian submanifold is called *minimal* if the mean curvature vector H vanishes identically. And it is called *quasi-minimal* if H is a light-like vector field.

Recall that for a warped product $M = B \times_f F$, B is totally geodesic and F is totally umbilical in M.

2.3. Para-Kähler n-plane

The simplest example of para-Kähler manifold is the para-Kähler n-plane $(\mathbb{E}^{2n}_n, \mathcal{P}, g_0)$ consisting of the pseudo-Euclidean $2n$-space $\mathbb{E}^{2n}_n$, the standard flat neutral metric

$$g_0 = -\sum_{j=1}^{n} dx_j^2 + \sum_{j=1}^{n} dy_j^2, \tag{2.5}$$

and the almost product structure

$$\mathcal{P} = \sum_{j=1}^{n} \frac{\partial}{\partial y_j} \otimes dx_j + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \otimes dy_j. \tag{2.6}$$

We simply denote the para-Kähler n-plane $(\mathbb{E}^{2n}_n, \mathcal{P}, g_0)$ by $\mathcal{P}^n$.

3. $\mathcal{P}R$-Submanifolds of Para-kähler Manifolds

For any vector field X tangent to M, we put $PX = tan(\mathcal{P}X)$ and $FX = nor(\mathcal{P}X)$, where tan_p and nor_p are the natural projections associated to the direct sum decomposition

$$T_p(\widetilde{M}) = T_p(M) \oplus T_p^\perp(M)\ ,\ p \in M.$$

Then P is an endomorphism of the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on M. Similarly, for a normal vector field ξ, we put $t\xi = tan(\mathcal{P}\xi)$ and $f\xi = nor(\mathcal{P}\xi)$ for the tangential and the normal part of $\mathcal{P}\xi$, respectively.

Let ν denote the orthogonal complement of $\mathcal{P}\mathcal{D}^\perp$ in $T^\perp(M)$. Then we have

$$T^\perp(M) = \mathcal{P}\mathcal{D}^\perp \oplus \nu.$$

Notice that ν is invariant, i.e., $\mathcal{P}\nu = \nu$.

The following proposition characterizes $\mathcal{P}R$-submanifolds of para-Kähler manifolds. A similar result is known for CR-submanifolds in Kählerian manifolds and contact CR-submanifolds in Sasakian manifolds. See e.g. [30].

Proposition 3.1. *Let $M \to \widetilde{M}$ be an isometric immersion of a pseudo-Riemannian manifold M into a para-Kähler manifold $\widetilde{M}$. Then a necessary and sufficient condition for M to be a $\mathcal{P}R$-submanifold is that $F \circ P = 0$.*

Proof. For U tangent to M we have the following decomposition

$$U = \mathcal{P}^2 U = P^2 U + FPU + tFU + fFU.$$

By identifying the tangent and the normal parts respectively, we find

$$P^2 + tF = I \quad \text{and} \quad FP + fF = 0.$$

Suppose that M is a $\mathcal{P}R$-submanifold. After we choose $U = X \in \mathcal{D}$ we have $\mathcal{P}X = PX$ and $FX = 0$. Hence $P^2 = I$ and $FP = 0$ on $\mathcal{D}$. On the other hand, if $U = Z = \mathcal{D}^\perp$, we have $PZ = 0$. Hence $FP = 0$ on $\mathcal{D}^\perp$ too.

Conversely, suppose that $FP = 0$. Put

$$\mathcal{D} = \{X \in T(M) : \mathcal{P}X \in T(M)\} \text{ and } \mathcal{D}^\perp = \{Z \in T(M) : \mathcal{P}Z \in T^\perp(M)\}.$$

Then by direct computations we conclude that $\mathcal{D}$ and $\mathcal{D}^\perp$ are orthogonal such that $T(M) = \mathcal{D} \oplus \mathcal{D}^\perp$. ∎

The following results from [15] are necessary for our further computations.

Proposition 3.2. *Let M be a $\mathcal{P}R$-submanifold of a para-Kähler manifold $\widetilde{M}$. Then*

(*i*) *the anti-invariant distribution $\mathcal{D}^\perp$ is a non-degenerate integrable distribution;*

(*ii*) *the invariant distribution $\mathcal{D}$ is a non-degenerate minimal distribution;*

(*iii*) *the invariant distribution $\mathcal{D}$ is integrable if and only if $\sigma(PX, Y) = \sigma(X, PY)$, for all $X, Y \in \mathcal{D}$;*

(*iv*) *$\mathcal{D}$ is integrable if and only if $\dot{\sigma}$ is symmetric, equivalently to $\dot{\sigma}(PX, Y) = \dot{\sigma}(X, PY)$. Here $\dot{\sigma}$ denotes the second fundamental form of $\mathcal{D}$ in M.*

Now, let us give some useful formulas.

Lemma 3.3. *If M is a $\mathcal{P}R$-submanifold of a para-Kähler manifold $\widetilde{M}$, then*

(a) *$\widetilde{g}(A_{FZ}U, PX) = g(\nabla_U Z, X)$,*

(b) *$A_{FZ}W = A_{FW}Z$ and $A_{f\xi}X = -A_\xi PX$,*

for all $X, Y \in \mathcal{D}$, $Z, W \in \mathcal{D}^\perp$, $U \in T(M)$ and $\xi \in \Gamma(\nu)$.

We need the following for later use.

Proposition 3.4. *Let M be a $\mathcal{P}R$-submanifold of a para-Kähler manifold $\widetilde{M}$. Then*

(i) *the distribution* $\mathcal{D}^{\perp}$ *is totally geodesic if and only if*

$$\widetilde{g}(\sigma(\mathcal{D},\mathcal{D}^{\perp}),\mathcal{P}\mathcal{D}^{\perp})=0 \tag{3.1}$$

(ii) *the distribution* $\mathcal{D}$ *is totally geodesic if and only if*

$$\widetilde{g}(\sigma(\mathcal{D},\mathcal{D}),\mathcal{P}\mathcal{D}^{\perp})=0 \tag{3.2}$$

(iii) $\mathcal{D}$ *is totally umbilical if and only if there exists* $Z_0 \in \mathcal{D}^{\perp}$ *such that*

$$\sigma(X,Y)=g(X,PY)\ FZ_0\ (\mathrm{mod}\ \nu)\ ,\ \forall\ X,Y\in\mathcal{D}. \tag{3.3}$$

Proof. This can be proved by classical computations: see e.g. [6] or [24]. ∎

3.1. $\mathcal{P}R$-products

A $\mathcal{P}R$-submanifold of a para-Kähler manifold is called a $\mathcal{P}R$*-product* if it is locally a direct product $N_\top \times N_\perp$ of an invariant submanifold $N_\top$ and an anti-invariant submanifold $N_\perp$.

The next result characterizes $\mathcal{P}R$-products in terms of the operator P.

Proposition 3.5. (Characterization). *A* $\mathcal{P}R$*-submanifold of a para-Kähler manifold is a* $\mathcal{P}R$*-product if and only if* P *is parallel.*

Proof. By straightforward computations (as in [6, Theorem 4.1] or [24, Theorem 2.2]) we may prove that

$$(\nabla_U P)V=\nabla_U(PV)-P\nabla_U V=0\ ,\ \forall\ U,V\in\chi(M),$$

which implies the desired result. ∎

The following result was proved in [15, page 224].

Proposition 3.6. *Let* $N_\top\times N_\perp$ *be a* $\mathcal{P}R$*-product of the para-Kähler* $(h+p)$*-plane* $\mathcal{P}^{h+p}$ *with* $h=\frac{1}{2}\dim N_\top$ *and* $p=\dim N_\perp$. *If* $N_\perp$ *is either spacelike or timelike, then the* $\mathcal{P}R$*-product is an open part of a direct product of a para-Kähler* h*-plane* $\mathcal{P}^h$ *and a Lagrangian submanifold* L *of* $\mathcal{P}^p$*, i.e.,*

$$N_\top\times N_\perp\subset\mathcal{P}^h\times L\subset\mathcal{P}^h\times\mathcal{P}^p=\mathcal{P}^{h+p}.$$

3.2. $\mathcal{P}R$-warped products

Let us begin with the following result.

Proposition 3.7. *If a $\mathcal{P}R$-submanifold M is a warped product $N_\perp \times_f N_\top$ of an anti-invariant submanifold $N_\perp$ and an invariant submanifold $N_\top$ with warping function $f : N_\perp \longrightarrow \mathbb{R}_+$, then M is a $\mathcal{P}R$ product $N_\perp \times N_\top^f$, where $N_\top^f$ is the manifold $N_\top$ endowed with the homothetic metric $g_\top^f = f^2 g_\top$.*

Proof. Consider $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. Compute

$$\widetilde{g}(\sigma(X,Y), FZ) = \widetilde{g}(\widetilde{\nabla}_X Y, \mathcal{P}Z) = -\widetilde{g}(Y, \mathcal{P}\widetilde{\nabla}_X Z) = g(PY, \nabla_X Z) = \\ = g(PY, Z(\ln f)\ X) = Z(\ln f)\ g(X, PY).$$

Since $\sigma(\cdot\ ,\ \cdot)$ is symmetric and $g(\cdot\ ,\ P\,\cdot)$ is skew-symmetric, it follows that $Z(\ln f)$ vanishes for all Z tangent to $N_\perp$. Consequently, f is a constant and thus the warped product is nothing but the product $N_\perp \times N_\top^f$. ∎

The previous result shows that there do not exist warped product $\mathcal{P}R$-submanifolds in para-Käehler manifolds of the form $N_\perp \times_f N_\top$, other than $\mathcal{P}R$-products. Thus, in view of Proposition 3.7 we give the following definition:

Definition 3.8. A $\mathcal{P}R$-submanifold of a para-Kähler manifold $\widetilde{M}$ is called a $\mathcal{P}R$-*warped product* if it is a warped product of the form: $N_\top \times_f N_\perp$, where $N_\top$ in an invariant submanifold, $N_\perp$ is an anti-invariant submanifold of $\widetilde{M}$ and f is a non-constant function $f : N_\top \to \mathbb{R}_+$.

Since the metric on N_T of a $\mathcal{P}R$-warped product $N_\top \times_f N_\perp$ is neutral, we simply called the $\mathcal{P}R$-warped product $N_\top \times_f N_\perp$ *space-like* or *time-like* depending on $N_\perp$ is space-like or time-like, respectively.

The next result characterizes $\mathcal{P}R$-warped products in para-Kähler manifolds.

Proposition 3.9. *Let M be a proper $\mathcal{P}R$-submanifold of a para-Kähler manifold. Then M is a $\mathcal{P}R$-warped product if and only if*

$$A_{FZ}X = (PX(\mu))\, Z\ ,\ \forall\ X \in \mathcal{D},\ Z \in \mathcal{D}^\perp, \tag{3.4}$$

for some smooth function μ on M satisfying $W(\mu) = 0$, $\forall\ W \in \mathcal{D}^\perp$.

The proof of this result is similar as in the case of Kähler or Sasakian ambient space. The key is the characterization of warped products given by Hiepko in [21].

4. An Optimal Inequality

Theorem 4.1. *Let $M = N_\top \times_f N_\perp$ be a $\mathcal{P}R$-warped product in a para-Kähler manifold $\widetilde{M}$. Suppose that $N_\perp$ is space-like and $\nabla^\perp(\mathcal{P}N_\perp) \subseteq \mathcal{P}N_\perp$. Then the second fundamental form of M satisfies*

$$S_\sigma \leq 2p||\nabla \ln f||_2 + ||\sigma_\nu^{\mathcal{D}}||_2, \tag{4.1}$$

where $p = \dim N_\perp$*,* $S_\sigma = \widetilde{g}(\sigma,\sigma)$*,* $\nabla \ln f$ *is the gradient of* $\ln f$ *with respect to the metric* g *and* $||\sigma_\nu^{\mathcal{D}}||_2 = \widetilde{g}\big(\sigma_\nu(\mathcal{D},\mathcal{D}),\sigma_\nu(\mathcal{D},\mathcal{D})\big)$*. Here the index* ν *represents the* ν*-component of that object.*

Proof. If we denote by $g_\top$ and $g_\perp$ the metrics on $N_\top$ and $N_\perp$, then the warped metric on M is $g = g_\top + f^2 g_\perp$. Let us consider

- on $N_\top$: an orthonormal basis $\{X_i, X_{i*} = PX_i\}$, $i = 1,\ldots,h$, where $h = \dim N_\top$; moreover, one can suppose that $\epsilon_i := g(X_i,X_i) = 1$ and hence $\epsilon_{i*} := g(X_{i*},X_{i*}) = -1$, for all i.
- on $N_\perp$: an orthonormal basis $\{\tilde{Z}_a\}$, $a = 1,\ldots,p$; we put $\epsilon_a := g_\perp(\tilde{Z}_a,\tilde{Z}_a) = 1$, for all a;
- in each point $(x,y)\in M$: $Z_a(x,y) = \frac{1}{f(x)}\,\tilde{Z}_a(y)$;
- in ν: an orthonormal basis $\{\xi_\alpha, \xi_{\alpha*} = f\xi_{\alpha*}\}$, $\alpha = 1,\ldots,q$; moreover, one can suppose that $\epsilon_\alpha := \widetilde{g}(\xi_\alpha,\xi_\alpha) = 1$ and hence $\epsilon_{\alpha*} := \widetilde{g}(\xi_{\alpha*},\xi_{\alpha*}) = -1$.

Now, we want to compute

$$\widetilde{g}(\sigma,\sigma) = \widetilde{g}\big(\sigma(\mathcal{D},D),\sigma(\mathcal{D},\mathcal{D})\big) + 2\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}^\perp),\sigma(\mathcal{D},\mathcal{D}^\perp)\big) + \widetilde{g}\big(\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp),\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp)\big),$$

where

$$\begin{aligned}(4.2)\qquad &\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}),\sigma(\mathcal{D},\mathcal{D})\big)\\ &= \sum_{i,j=1}^{h}\Big(\epsilon_i\epsilon_j\widetilde{g}\big(\sigma(X_i,X_j),\sigma(X_i,X_j)\big)\\ &\quad+\epsilon_{i*}\epsilon_j\widetilde{g}\big(\sigma(X_{i*},X_j),\sigma(X_{i*},X_j)\big)+\epsilon_i\epsilon_{j*}\widetilde{g}\big(\sigma(X_i,X_{j*}),\sigma(X_i,X_{j*})\big)\\ &\quad+\epsilon_{i*}\epsilon_{j*}\widetilde{g}\big(\sigma(X_{i*},X_{j*}),\sigma(X_{i*},X_{j*})\big)\Big),\end{aligned}$$

$$\begin{aligned}(4.3)\qquad \widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}^\perp),\sigma(\mathcal{D},\mathcal{D}^\perp)\big) &= \sum_{i=1}^{h}\sum_{a=1}^{p}\Big(\epsilon_i\epsilon_a\widetilde{g}\big(\sigma(X_i,Z_a),\sigma(X_i,Z_a)\big)\\ &\quad+\epsilon_{i*}\epsilon_a\widetilde{g}\big(\sigma(X_{i*},Z_a),\sigma(X_{i*},Z_a)\big)\Big)\end{aligned}$$

and

$$(4.4)\qquad \widetilde{g}\big(\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp),\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp)\big) = \sum_{a,b=1}^{p}\epsilon_a\epsilon_b\widetilde{g}\big(\sigma(Z_a,Z_b),\sigma(Z_a,Z_b)\big).$$

To do so, first we analyze $\sigma(\mathcal{D},\mathcal{D})$. Since $\mathcal{D}$ is totally geodesic, we have $\sigma(\mathcal{D},\mathcal{D})\in\nu$. Hence one can write the following

$$\begin{aligned}\sigma(X_i,X_j) &= \sigma_{ij}^{\alpha}\xi_\alpha+\sigma_{ij}^{\alpha*}\xi_{\alpha*}, &\sigma(X_{i*},X_j) &= \sigma_{i*j}^{\alpha}\xi_\alpha+\sigma_{i*j}^{\alpha*}\xi_{\alpha*},\\ \sigma(X_{i*},X_{j*}) &= \sigma_{i*j*}^{\alpha}\xi_\alpha+\sigma_{i*j*}^{\alpha*}\xi_{\alpha*}, &\sigma(X_i,X_{j*}) &= \sigma_{ij*}^{\alpha}\xi_\alpha+\sigma_{ij*}^{\alpha*}\xi_{\alpha*}.\end{aligned}$$

It follows that

$$\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}),\sigma(\mathcal{D},\mathcal{D})\big) = \sum_{i,j=1}^{h}\sum_{\alpha=1}^{q}\Big\{\big[(\sigma_{ij}^{\alpha})^2-(\sigma_{ij}^{\alpha*})^2\big]-\big[(\sigma_{i*j}^{\alpha})^2-(\sigma_{i*j}^{\alpha*})^2\big] -\big[(\sigma_{ij*}^{\alpha})^2-(\sigma_{ij*}^{\alpha*})^2\big]+\big[(\sigma_{i*j*}^{\alpha})^2-(\sigma_{i*j*}^{\alpha*})^2\big]\Big\}. \tag{4.5}$$

Due to the integrability of $\mathcal{D}$ we deduce that $\sigma_{i*j}^{\alpha}=\sigma_{ij*}^{\alpha}$, $\sigma_{i*j}^{\alpha*}=\sigma_{ij*}^{\alpha*}$, $\sigma_{i*j*}^{\alpha}=\sigma_{ij}^{\alpha}$, $\sigma_{i*j*}^{\alpha*}=\sigma_{ij}^{\alpha*}$. Furthermore, using Lemma 3.3, we may write

$$\widetilde{g}\big(\sigma(X,Y),\xi\big)=-\widetilde{g}\big(\sigma(X,PY),f\xi\big)\ ,\ \forall\ X,Y\in\mathcal{D},\ \xi\in\nu$$

and consequently we have

$$\begin{aligned}\sigma_{ij}^{\alpha} &= \widetilde{g}\big(\sigma(X_i,X_j),\xi_\alpha\big)=-\widetilde{g}\big(\sigma(X_i,X_{j*}),\xi_{\alpha*}\big)=\sigma_{ij*}^{\alpha*},\\ \sigma_{ij}^{\alpha*} &= -\widetilde{g}\big(\sigma(X_i,X_j),\xi_{\alpha*}\big)=\widetilde{g}\big(\sigma(X_i,X_{j*}),\xi_\alpha\big)=\sigma_{ij*}^{\alpha}.\end{aligned}$$

By replacing all these in (4.5), we obtain

$$\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}),\sigma(\mathcal{D},\mathcal{D})\big)=||\sigma_{\nu}^{\mathcal{D}}||_2=4\sum_{i,j=1}^{h}\sum_{\alpha=1}^{q}\big[(\sigma_{ij}^{\alpha})^2-(\sigma_{ij}^{\alpha*})^2\big]. \tag{4.6}$$

Let us focus now on $\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}^{\perp}),\sigma(\mathcal{D},\mathcal{D}^{\perp})\big)$. As before, we write

$$\begin{aligned}\sigma(X_i,Z_a) &= \sigma_{ia}^{b}FZ_b+\sigma_{ia}^{\alpha}\xi_\alpha+\sigma_{ia}^{\alpha*}\xi_{\alpha*},\\ \sigma(X_{i*},Z_a) &= \sigma_{i*a}^{b}FZ_b+\sigma_{i*a}^{\alpha}\xi_\alpha+\sigma_{i*a}^{\alpha*}\xi_{\alpha*}.\end{aligned}$$

It follows that

$$\begin{aligned}\widetilde{g}\big(\sigma(X_i,Z_a),\sigma(X_i,Z_a)\big) &= -\sum_{b=1}^{p}(\sigma_{ia}^{b})^2+\sum_{\alpha=1}^{q}\big[(\sigma_{ia}^{\alpha})^2-(\sigma_{ia}^{\alpha*})^2\big],\\ \widetilde{g}\big(\sigma(X_{i*},Z_a),\sigma(X_{i*},Z_a)\big) &= -\sum_{b=1}^{p}(\sigma_{i*a}^{b})^2+\sum_{\alpha=1}^{q}\big[(\sigma_{i*a}^{\alpha})^2-(\sigma_{i*a}^{\alpha*})^2\big].\end{aligned}$$

We obtain

$$\begin{aligned}&\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}^{\perp}),\sigma(\mathcal{D},\mathcal{D}^{\perp})\big)\\ &=-\sum_{i=1}^{h}\sum_{a,b=1}^{p}\big[(\sigma_{ia}^{b})^2-(\sigma_{i*a}^{b})^2\big]\\ &\quad+\sum_{i=1}^{h}\sum_{a=1}^{p}\sum_{\alpha=1}^{q}\big[(\sigma_{ia}^{\alpha})^2-(\sigma_{ia}^{\alpha*})^2-(\sigma_{i*a}^{\alpha})^2+(\sigma_{i*}^{\alpha*})^2\big].\end{aligned} \tag{4.7}$$

From Lemma 3.3 we have

$$\widetilde{g}\big(\sigma(PX,Z),f\xi\big)=-\widetilde{g}\big(\sigma(X,Z),\xi\big)$$

and consequently

$$\begin{aligned}\sigma^{\alpha}_{i*a}&=\widetilde{g}\big(\sigma(X_{i*},Z_a),\xi_\alpha\big)=-\widetilde{g}\big(\sigma(X_i,Z_a),\xi_{\alpha*}\big)=\sigma^{\alpha*}_{ia},\\ \sigma^{\alpha*}_{i*a}&=-\widetilde{g}\big(\sigma(X_{i*},Z_a),\xi_{\alpha*}\big)=\widetilde{g}\big(\sigma(X_i,Z_a),\xi_\alpha\big)=\sigma^{\alpha}_{ia}\ .\end{aligned}\tag{4.8}$$

Moreover we know that $\widetilde{g}\big(\sigma(PX,Z),FW\big)=-X(\ln f)g(Z,W)$. This yields

$$\sigma^b_{ia}=PX_i(\ln f)\ \delta_{ab}\text{ and }\sigma^b_{i*a}=X_i(\ln f)\ \delta_{ab}.\tag{4.9}$$

By combining (4.7), (4.8) and (4.9) we get

$$\begin{aligned}\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}^\perp),\sigma(\mathcal{D},\mathcal{D}^\perp)\big)&=p\sum_{i=1}^{h}\Big[\big(X_i(\ln f)\big)^2-\big(PX_i(\ln f)\big)^2\Big]\\ &\quad+2\sum_{i=1}^{h}\sum_{a=1}^{p}\sum_{\alpha=1}^{q}\big[(\sigma^\alpha_{ia})^2-(\sigma^{\alpha*}_{ia})^2\big].\end{aligned}\tag{4.10}$$

As $\widetilde{g}\big(\sigma(X,Z),f\xi\big)=-\widetilde{g}\big(\nabla^\perp_X FZ,\xi\big)$ and using the hypothesis $\nabla^\perp_{\mathcal{D}}\mathcal{PD}^\perp\subseteq\mathcal{PD}^\perp$ we get $\sigma(\mathcal{D},\mathcal{D}^\perp)\subseteq\mathcal{PD}^\perp$. Hence σ^α_{ia} and $\sigma^{\alpha*}_{ia}$ vanish. Thus

$$\widetilde{g}\big(\sigma(\mathcal{D},\mathcal{D}^\perp),\sigma(\mathcal{D},\mathcal{D}^\perp)\big)=p\ g\big(\nabla\ln f,\nabla\ln f\big).\tag{4.11}$$

Finally, we study $\widetilde{g}\big(\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp),\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp)\big)$. We write

$$\sigma(Z_a,Z_b)=\sigma^c_{ab}FZ_c+\sigma^\alpha_{ab}\xi_\alpha+\sigma^{\alpha*}_{ab}\xi_{\alpha*}$$

and hence

$$\widetilde{g}\big(\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp),\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp)\big)=-\sum_{a,b,c=1}^{p}(\sigma^c_{ab})^2+\sum_{a,b=1}^{p}\sum_{\alpha=1}^{q}\big[(\sigma^\alpha_{ab})^2-(\sigma^{\alpha*}_{ab})^2\big].$$

As $\widetilde{g}\big(\sigma(Z,W),f\xi\big)=-\widetilde{g}\big(\nabla^\perp_Z FW,\xi\big)$ and using the hypothesis $\nabla^\perp_{\mathcal{D}^\perp}\mathcal{PD}^\perp\subseteq\mathcal{PD}^\perp$ we get $\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp)\subseteq\mathcal{PD}^\perp$. Hence σ^α_{ab} and $\sigma^{\alpha*}_{ab}$ vanish. We conclude with

$$\widetilde{g}\big(\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp),\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp)\big)=-\sum_{a,b,c=1}^{p}(\sigma^c_{ab})^2\ .\tag{4.12}$$

From these we obtain the theorem. ∎

Remark 4.2. If the manifold $N_\perp$ in Theorem 4.1 is time-like, then (4.1) shall be replaced by

$$S_\sigma \geq 2p||\nabla \ln f||_2 + ||\sigma_\nu^{\mathcal{D}}||_2. \tag{4.13}$$

Remark 4.3. For every $\mathcal{P}R$-warped product $N_\top \times N_\perp$ in a para-Kähler manifold $\widetilde{M}$, $\dim \widetilde{M} \geq \dim N_\top + 2\dim N_\perp$ holds. Thus the smallest codimension is $\dim N_\perp$.

Theorem 4.4. *Let $N_\top \times_f N_\perp$ be a $\mathcal{P}R$-warped product in a para-Kähler manifold $\widetilde{M}$. If $N_\perp$ is space-like (respectively, time-like) and $\dim \widetilde{M} = \dim N_\top + 2\dim N_\perp$, then the second fundamental form of M satisfies*

$$S_\sigma \leq 2p||\nabla \ln f||_2 \quad \text{(respectively, } S_\sigma \geq 2p||\nabla \ln f||_2). \tag{4.14}$$

If the equality sign of (4.14) *holds identically, we have*

$$\sigma(\mathcal{D}, \mathcal{D}) = \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = \{0\}. \tag{4.15}$$

Proof. Inequality (4.14) follows from (4.1). When the equality sign holds, (4.15) follows from the proof of Theorem 4.1. ∎

5. Exact Solutions for a Special Pde's System

We need the exact solutions of the following PDE system for later use.

Proposition 5.1. *The non-constant solutions $\psi = \psi(s_1, \ldots, s_h, t_1, \ldots, t_h)$ of the following system of partial differential equations*

$$\frac{\partial^2 \psi}{\partial s_i \partial s_j} + \frac{\partial \psi}{\partial s_i}\frac{\partial \psi}{\partial s_j} + \frac{\partial \psi}{\partial t_i}\frac{\partial \psi}{\partial t_j} = 0\,, \tag{5.1.a}$$

$$\frac{\partial^2 \psi}{\partial s_i \partial t_j} + \frac{\partial \psi}{\partial s_i}\frac{\partial \psi}{\partial t_j} + \frac{\partial \psi}{\partial t_i}\frac{\partial \psi}{\partial s_j} = 0\,, \quad i, j = 1, \ldots, h\,, \tag{5.1.b}$$

$$\frac{\partial^2 \psi}{\partial t_i \partial t_j} + \frac{\partial \psi}{\partial t_i}\frac{\partial \psi}{\partial t_j} + \frac{\partial \psi}{\partial s_i}\frac{\partial \psi}{\partial s_j} = 0 \tag{5.1.c}$$

are either given by

$$\psi = \frac{1}{2}\ln\left|\left[\left(\langle \mathbf{v}, z\rangle + c_1\right)^2 - \left(\langle \mathrm{j}\mathbf{v}, z\rangle + c_2\right)^2\right]\right|, \tag{5.2}$$

where $z = (s_1, s_2, \ldots, s_h, t_1, t_2, \ldots, t_h)$, $\mathbf{v} = (a_1, a_2, \ldots, a_h, 0, b_2, \ldots, b_h)$ is a constant vector in $\mathbb{R}^{2h}$ with $a_1 \neq 0$, $c_1, c_2 \in \mathbb{R}$ and $\mathrm{j}\mathbf{v} = (0, b_2, \ldots, b_h, a_1, a_2, \ldots, a_h)$;

or given by

$$\psi = \frac{1}{2}\ln\left|\left(\langle \mathbf{v_1}, z\rangle + c\right)\left(\langle \mathbf{v_2}, z\rangle + d\right)\right|, \tag{5.3}$$

where $\mathbf{v_1} = \big(0, a_2, \ldots, a_h, 0, \epsilon a_2, \ldots, \epsilon a_h\big)$, $\mathbf{v_2} = \big(b_1, \ldots, b_h, -\epsilon b_1, \ldots, -\epsilon b_h\big)$ *with* $b_1 \neq 0$*,* z *is as above and* $c, d \in \mathbb{R}$*.*

Here $\langle\ ,\ \rangle$ *denotes the Euclidean scalar product in* $\mathbb{R}^{2h}$*.*

Proof. Let us make some notations: $\psi_{s_i} := \frac{\partial\psi}{\partial s_i}$; $\psi_{s_i s_j} := \frac{\partial^2\psi}{\partial s_i \partial s_j}$, and similar for ψ_{t_i}, $\psi_{s_i t_j}$, respectively $\psi_{t_i t_j}$. The same notations for any other function.

If in (5.1.b) we take $i = j$ one gets $\psi_{s_i t_i} = -2\psi_{s_i}\psi_{t_i}$ for all $i = 1, \ldots, h$. Since ψ is non-constant, there exists i_0 such that at least one of $\psi_{s_{i_0}}$ or $\psi_{t_{i_0}}$ is different from 0. Without loss of the generality we suppose $i_0 = 1$. Both situations yield

$$e^{2\psi} = \zeta(t_1, s_2, t_2, \ldots, s_h, t_h) + \eta(s_1, s_2, t_2, \ldots, s_h, t_h)\,,$$

where ζ and η are functions of $2h-1$ variables such that $\zeta + \eta > 0$ on the domain of ψ. It follows that

$$\text{(5.4)} \qquad \begin{aligned} \psi_{s_1} &= \frac{\eta_{s_1}}{2(\zeta+\eta)}\ ,\ \psi_{s_1 s_1} = \frac{\eta_{s_1 s_1}(\zeta+\eta) - \eta_{s_1}^2}{2(\zeta+\eta)^2}\ ,\\ \psi_{t_1} &= \frac{\zeta_{t_1}}{2(\zeta+\eta)}\ ,\ \psi_{t_1 t_1} = \frac{\zeta_{t_1 t_1}(\zeta+\eta) - \eta_{t_1}^2}{2(\zeta+\eta)^2}\ . \end{aligned}$$

Using (5.1.a) and (5.1.c) we obtain

$$\text{(5.5)} \qquad 2\eta_{s_1 s_1}(\zeta+\eta) = \eta_{s_1}^2 - \zeta_{t_1}^2\ , \quad 2\zeta_{t_1 t_1}(\zeta+\eta) = \zeta_{t_1}^2 - \eta_{s_1}^2\ .$$

Since $\zeta + \eta \neq 0$, adding the previous relations, one gets

$$\eta_{s_1 s_1}(s_1, s_2, t_2, \ldots, s_h, t_h) + \zeta_{t_1 t_1}(t_1, s_2, t_2, \ldots, s_h, t_h) = 0$$

and hence, there exists a function F depending on $s_2, t_2, \ldots, s_h, t_h$ such that

$$\begin{aligned} \eta_{s_1 s_1}(s_1, s_2, t_2, \ldots, s_h, t_h) &= \ 2F(s_2, t_2, \ldots, s_h, t_h)\,,\\ \zeta_{t_1 t_1}(t_1, s_2, t_2, \ldots, s_h, t_h) &= -2F(s_2, t_2, \ldots, s_h, t_h)\ . \end{aligned}$$

At this point one integrates with respect to s_1 and t_1 respectively and one gets

$$\text{(5.6)} \qquad \begin{aligned} \eta(s_1, s_2, t_2, \ldots, s_h, t_h) &= \ Fs_1^2 + Gs_1 + H\,,\\ \zeta(t_1, s_2, t_2, \ldots, s_h, t_h) &= -Ft_1^2 - Kt_1 - L\,, \end{aligned}$$

where G, H, L and K are functions depending on $s_2, t_2, \ldots, s_h, t_h$ satisfying the following condition

$$\text{(5.7)} \qquad 4F(H - L) = G^2 - K^2.$$

It follows that $\eta + \zeta = \big(Fs_1^2 + Gs_1 + H\big) - \big(Ft_1^2 + Kt_1 + L\big)$.

Case 1. Suppose $F \neq 0$; being continuous, it preserves constant sign; denote it by ε. From (5.7) we have $H - L = \frac{G^2 - K^2}{4F}$ which combined with (5.6) yields

$$\eta + \zeta = \varepsilon \left[\left(\varepsilon \sqrt{\varepsilon F}\ s_1 + \frac{G}{2\sqrt{\varepsilon F}} \right)^2 - \left(\varepsilon \sqrt{\varepsilon F}\ t_1 + \frac{K}{2\sqrt{\varepsilon F}} \right)^2 \right].$$

We make some notations: $a = \varepsilon\sqrt{\varepsilon F}$, $\gamma = \frac{G}{2\sqrt{\varepsilon F}}$ and $\delta = \frac{K}{2\sqrt{\varepsilon F}}$, all of them being functions depending on $s_2, t_2, \ldots, s_h, t_h$. We are able to re-write the function ψ as

$$\psi = \frac{1}{2} \ln \varepsilon \left[(as_1 + \gamma)^2 - (at_1 + \delta)^2 \right]. \tag{5.8}$$

We compute now

$$\psi_{s_1} = \frac{a(as + \gamma)}{(as_1 + \gamma)^2 - (at_1 + \delta)^2}\ ,\ \psi_{t_1} = \frac{-a(at_1 + \delta)}{(as_1 + \gamma)^2 - (at_1 + \delta)^2} \tag{5.9}$$

and for $i \neq 1$

$$\begin{aligned} \psi_{s_i} &= \frac{(as_1 + \gamma)(a_{s_i} s_1 + \gamma_{s_i}) - (at_1 + \delta)(a_{s_i} t_1 + \delta_{s_i})}{(as_1 + \gamma)^2 - (at_1 + \delta)^2}, \\ \psi_{t_i} &= \frac{(as_1 + \gamma)(a_{t_i} s_1 + \gamma_{t_i}) - (at_1 + \delta)(a_{t_i} t_1 + \delta_{t_i})}{(as_1 + \gamma)^2 - (at_1 + \delta)^2}. \end{aligned} \tag{5.10}$$

Computing also $\psi_{s_1 s_i}$, we can use (5.1.a) for $j = 1$, $i = 2, \ldots, h$ and obtain

$$\begin{aligned} &[a(a_{s_i} s_1 + \gamma_{s_i}) + a_{s_i}(as_1 + \gamma)][(as_1 + \gamma)^2 - (at_1 + \delta)^2] \\ &-a(as_1 + \gamma)[(as_1 + \gamma)(a_{s_i} s_1 + \gamma_{s_i}) - (at_1 + \delta)(a_{s_i} t_1 + \delta_{s_i})] \\ &-a(at_1 + \delta)[(as_1 + \gamma)(a_{t_i} s_1 + \gamma_{t_i}) - (at_1 + \delta)(a_{t_i} t_1 + \delta_{t_i})] = 0. \end{aligned}$$

This represents a polynomial in s_1 and t_1, identically zero, and hence, all its coefficients must vanish. Analyzing the coefficients for s_1^3 and t_1^3 we obtain $a_{s_i} = 0$ and $a_{t_i} = 0$ for all $i = 2, \ldots, h$. Consequently a is a real constant.

Replacing in the previous equation we get

$$\delta_{s_i}(as_1 + \gamma) - \gamma_{s_i}(at_1 + \delta) - \gamma_{t_i}(as_1 + \gamma) + \delta_{t_i}(at_1 + \delta) = 0.$$

Looking at the coefficients of s_1 and t_1 we have

$$\delta_{s_i} = \gamma_{t_i} \text{ and } \delta_{t_i} = \gamma_{s_i},\ \forall i = 2, \ldots, h. \tag{5.11}$$

Therefore (5.10) gives

$$\psi_{s_i} = \frac{\gamma_{s_i}(as_1 + \gamma) - \delta_{s_i}(at_1 + \delta)}{(as_1 + \gamma)^2 - (at_1 + \delta)^2}\ ,\ \psi_{t_i} = \frac{\gamma_{t_i}(as_1 + \gamma) - \delta_{t_i}(at_1 + \delta)}{(as_1 + \gamma)^2 - (at_1 + \delta)^2}\ . \tag{5.12}$$

We may compute

$$
\begin{aligned}
\psi_{s_i t_j} &= \frac{\gamma_{s_i t_j}(as_1+\gamma)+\gamma_{s_i}\gamma_{t_j}-\delta_{s_i t_j}(at_1+\delta)-\delta_{s_i}\delta_{t_j}}{(as_1+\gamma)^2-(at_1+\delta)^2} \\
&\quad -2\,\frac{[\gamma_{t_j}(as_1+\gamma)-\delta_{t_j}(at_1+\delta)][\gamma_{s_i}(as_1+\gamma)-\delta_{s_i}(at_1+\delta)]}{[(as_1+\gamma)^2-(at_1+\delta)^2]^2}
\end{aligned} \tag{5.13}
$$

and using (5.1.b) with $i, j > 1$, we obtain again a polynomial in s_1 and t_1, identically zero. By comparing the coefficients of s_1^3 and t_1^3 we find $\gamma_{s_i t_j} = 0$ and $\delta_{s_i t_j} = 0$, for all $i, j = 2, \ldots, h$. It follows that γ_{s_i} depend only on $s_2, \ldots, s_h$ and δ_{t_i} depend only on $t_2, \ldots, t_h$, for all i. From (5.11) we know $\gamma_{s_i} = \delta_{t_i}$. Hence, there exist constants $a_i \in \mathbb{R}$ such that $\gamma_{s_i} = \delta_{t_i} = a_i$, $\forall i = 2, \ldots, h$. In the same way, there exist constants $b_i \in \mathbb{R}$ such that $\gamma_{t_i} = \delta_{s_i} = b_i$, $\forall i = 2, \ldots, h$. It follows that

$$
\begin{aligned}
\gamma(s_2, t_2 \ldots, s_h, t_h) &= \sum_{i=2}^{h} a_i s_i + \sum_{i=2}^{h} b_i t_i + c_1, \\
\delta(s_2, t_2 \ldots, s_h, t_h) &= \sum_{i=2}^{h} b_i s_i + \sum_{i=2}^{h} a_i t_i + c_2\ , \quad c_1, c_2 \in \mathbb{R}.
\end{aligned} \tag{5.14}
$$

We conclude with

$$
\begin{aligned}
\psi &= \frac{1}{2}\ \ln \varepsilon \big[(as_1 + a_2 s_2 + b_2 t_2 + \ldots + a_h s_h + b_h t_h + c_1)^2 \\
&\quad -(at_1 + b_2 s_2 + a_2 t_2 + \ldots + b_h s_h + a_h t_h + c_2)^2 \big].
\end{aligned}
$$

Hence the solution (5.2) is obtained with $a_1 = a \neq 0$.

Case 2. Let us come back to the case $F = 0$ (on a certain open set). From (5.7) we immediately find $\eta + \zeta = Gs_1 - Kt_1 + H$, where G, H, K are functions depending on $(s_2, \ldots, s_h, t_2, \ldots, t_h)$, and $K = \epsilon G$, $\epsilon = \pm 1$. Thus

$$
\psi = \frac{1}{2} \ln |(s_1 - \epsilon t_1)G + H|.
$$

We have

$$
\begin{aligned}
\psi_{s_1} &= \frac{G}{2[(s_1-\epsilon t_1)G+H]}\ ,\ \psi_{t_1} = -\frac{\epsilon G}{2[(s_1-\epsilon t_1)G+H]}, \\
\psi_{s_i} &= \frac{(s_1-\epsilon t_1)G_{s_i}+H_{s_i}}{2[(s_1-\epsilon t_1)G+H]}\ ,\ \psi_{t_i} = \frac{(s_1-\epsilon t_1)G_{t_i}+H_{t_i}}{2[(s_1-\epsilon t_1)G+H]}\ ,\ i = 2, \ldots, h, \\
\psi_{s_i s_1} &= \frac{G_{s_i}[(s_1-\epsilon t_1)G+H] - G[(s_1-\epsilon t_1)G_{s_i}+H_{s_i})}{2[(s_1-\epsilon t_1)G+H]^2}\ ,\ i = 2, \ldots, h.
\end{aligned}
$$

By applying (5.1.a) for $j = 1$ and $i = 2, \ldots, h$ we obtain

$$
2G_{s_i}[(s_1-\epsilon t_1)G+H] - G[(s_1-\epsilon t_1)G_{s_i}+H_{s_i}] - \epsilon G[(s_1-\epsilon t_1)G_{t_i}+H_{t_i}] = 0.
$$

By comparing the coefficients of s_1 and t_1 we find

$$G(G_{s_i} - \epsilon G_{t_i}) = 0, \quad 2G_{s_i}H - G(H_{s_i} + \epsilon H_{t_i}) = 0. \tag{5.15}$$

Since $G \neq 0$ we have $G_{t_i} = \epsilon G_{s_i}$. In the sequel, computing

$$\psi_{s_i s_j} = \frac{(s_1 - \epsilon t_1)G_{s_i s_j} + H_{s_i s_j}}{2[(s_1 - \epsilon t_1)G + H]} - \frac{[(s_1 - \epsilon t_1)G_{s_i} + H_{s_i}][(s_1 - \epsilon t_1)G_{s_j} + H_{s_j}]}{2[(s_1 - \epsilon t_1)G + H]^2}$$

for $i, j \geq 2$, replacing in (5.1.a) and comparing the coefficients of s_1^2 we find $G_{s_i s_j} = 0$. It follows also $G_{s_i t_j} = 0$ and $G_{t_i t_j} = 0$. Hence

$$G(s_2, t_2, \ldots, s_h, t_h) = \sum_{i=2}^{h} a_i(s_i + \epsilon t_i) + c, \quad a_i, c \in \mathbb{R}.$$

Moreover, H should satisfy

$$2GH_{s_i s_j} - G_{s_i}(H_{s_j} - \epsilon H_{t_j}) - G_{s_j}(H_{s_i} - \epsilon H_{t_i}) = 0\,, \tag{5.16}$$

$$2HH_{s_i s_j} - H_{s_i}H_{s_j} + H_{t_i}H_{t_j} = 0. \tag{5.17}$$

Case 2a. If G is a non-zero constant c (and this happens when all a_i vanish), then from the second equation in (5.15) we find $H_{s_i} + \epsilon H_{t_i} = 0$ for all $i \geq 2$. Therefore, H has the form

$$H(s_2, t_2, \ldots, s_h, t_h) = Q(s_2 - \epsilon t_2, \ldots, s_h - \epsilon t_h),$$

where Q is a function depending only on h variables. From (5.16) we get $H_{s_i s_j} = 0$ and then Q is an affine function. Thus $H = \sum_{i=2}^{h} b_i(s_i - \epsilon t_i) + d$, with $b_2, \ldots, b_h, d \in \mathbb{R}$. Consequently,

$$\psi = \frac{1}{2}\ln\left[\sum_{i=1}^{h} b_i(s_i - \epsilon t_i) + d\right], \quad b_1 = c \neq 0\,.$$

Case 2b. If there exists at least one $a_i \neq 0$, from the second equation in (5.15) we can express H in the form $H = QG$, where Q is a function on $s_2, t_2, \ldots, s_h, t_h$. Then, for every $i \geq 2$,

$$H_{s_i} + \epsilon H_{t_i} = 2a_i Q + G(Q_{s_i} + Q_{t_i})\,,$$

which combined with (5.15) gives $Q_{s_i} + \epsilon Q_{t_i} = 0$. Thus, $Q = Q(s_2 - \epsilon t_2, \ldots, s_h - \epsilon t_h)$. Using (5.16), it follows that Q is an affine function and hence $H = \sum_{i=2}^{h} b_i(s_i - \epsilon t_i) + d$, with $b_2, \ldots, b_h, d \in \mathbb{R}$. Consequently,

$$\psi = \frac{1}{2}\ln\left\{\Big[\sum_{i=1}^{h} b_i(s_i - \epsilon t_i) + d\Big]\Big[\sum_{j=2}^{h} a_i(s_i + \epsilon t_i) + c\Big]\right\}$$

with $b_1 = 1$. This completes the proof. ■

6. $\mathcal{P}R$-Warped Products in $\mathcal{P}^{h+p}$ Satisfying $S_\sigma = 2p||\nabla \ln f||_2$

In the following, we use letters i, j, k for indices running from 1 to h; a, b, c for indices from 1 to p; and A, B for indices between 1 and m with $m = h + p$.

On $\mathbb{E}^{2(h+p)}_{h+p}$ we consider the global coordinates $(x_i, x_{h+a}, y_i, y_{h+a})$ and the canonical flat para-Kähler structure defined as above.

Proposition 6.1. *Let $M = N_\top \times_f N_\perp$ be a space-like $\mathcal{P}R$-warped product in the para-Kähler $(h+p)$-plane $\mathcal{P}^{h+p}$ with $h = \frac{1}{2}\dim N_\top$ and $p = \dim N_\perp$. If M satisfies the equality case of* (4.14) *identically, then*

- *$N_\top$ is a totally geodesic submanifold in $\mathcal{P}^{h+p}$, and hence it is congruent to an open part of $\mathcal{P}^h$;*
- *$N_\perp$ is a totally umbilical submanifold in $\mathcal{P}^{h+p}$.*

Moreover, if $N_\perp$ is a real space form of constant curvature k, then the warping function f satisfies $||\nabla f||_2 = k$.

Proof. Under the hypothesis, we know from the proof of Theorem 4.1 that the second fundamental form satisfies

$$\sigma(\mathcal{D}, \mathcal{D}) = \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = \{0\}.$$

On the other hand, since $M = N_\top \times_f N_\perp$ is a warped product, $N_\top$ is totally geodesic and $N_\perp$ is totally umbilical in M. Thus we have the first two statements.

The last statement of the proposition can be proved as follows. If $R^\perp$ denotes the Riemann curvature tensor of $N_\perp$, then we have

$$R_{ZV}W = R^\perp_{ZV}W - ||\nabla \ln f||_2\big(g(V,W)Z - g(Z,W)V\big)$$

for any Z, V, W tangent to $N_\perp$. See for details [26, page 210] (pay attention to the sign; see also page 74). If $N_\perp$ is a space form of constant curvature k, then R takes the form

$$R_{ZV}W = \left(\frac{k}{f^2} - ||\nabla \ln f||_2\right)\big(g(V,W)Z - g(Z,W)V\big). \tag{6.1}$$

The equation of Gauss may be written, for vectors tangent to $N_\perp$, as

$$g\big(R_{ZV}W, U\big) = \langle \widetilde{R}_{ZV}W, U\rangle + \langle \sigma(V,W), \sigma(Z,U)\rangle - \langle \sigma(Z,W), \sigma(V,U)\rangle\ .$$

Since the ambient space is flat and $\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0$ due to the equality in (4.14), it follows that $g(R_{ZV}W, U) = 0$. Combining this with (6.1) gives $||\nabla \ln f||_2 = \frac{k}{f^2}$. This gives the statement. ■

Para-complex numbers were introduced by Graves in 1845 [20] as a generalization of complex numbers. Such numbers have the expression $v = x + \mathrm{j}y$, where x, y are real numbers and j satisfies $\mathrm{j}^2 = 1, \mathrm{j} \neq \pm 1$. The conjugate of v is $\bar{v} = x - \mathrm{j}y$. The multiplication of two para-complex numbers is defined by

$$(a + \mathrm{j}b)(s + \mathrm{j}t) = (as + bt) + \mathrm{j}(at + bs).$$

For each natural number m, we put $\mathbb{D}^m = \{(x_1 + \mathrm{j}y_1, \ldots, x_m + \mathrm{j}y_m) : x_i, y_i \in \mathbb{R}\}$. With respect to the multiplication of para-complex numbers and the canonical flat metric, $\mathbb{D}^m$ is a flat para-Kähler manifold of dimension $2m$. Once we identify $(x_1+\mathrm{j}y_1, \ldots, x_m+\mathrm{j}y_m) \in \mathbb{D}^m$ with $(x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{E}^{2m}_m$, we may identify $\mathbb{D}^m$ with the para-Kähler m-plane $\mathcal{P}^m$ in a natural way.

In the following we denote by $\mathbb{S}^p, \mathbb{E}^p$ and $\mathbb{H}^p$ the unit p-sphere, the Euclidean p-space and the unit hyperbolic p-space, respectively.

Theorem 6.2. *Let $N_\top \times_f N_\perp$ be a space-like $\mathcal{P}R$-warped product in the para-Kähler $(h + p)$-plane $\mathcal{P}^{h+p}$ with $h = \frac{1}{2}\dim N_\top$ and $p = \dim N_\perp$. Then we have*

$$S_\sigma \leq 2p||\nabla \ln f||_2. \tag{6.2}$$

The equality sign of (6.2) *holds identically if and only if $N_\top$ is an open part of a para-Kähler h-plane, $N_\perp$ is an open part of $\mathbb{S}^p$, $\mathbb{E}^p$ or $\mathbb{H}^p$, and the immersion is given by one of the following:*

1. $\Phi : D_1 \times_f \mathbb{S}^p \longrightarrow \mathcal{P}^{h+p}$;

$$\begin{aligned}\Phi(z, w) = \Bigg(& z_1 + \bar{v}_1(w_0 - 1)\sum_{j=1}^{h} v_j z_j, \ldots, z_h + \bar{v}_h(w_0 - 1)\sum_{j=1}^{h} v_j z_j, \\ & w_1 \sum_{j=1}^{h} \mathrm{j}v_j z_j, \ldots, w_p \sum_{j=1}^{h} \mathrm{j}v_j z_j\Bigg), \quad h \geq 2,\end{aligned} \tag{6.3}$$

with warping function

$$f = \sqrt{\langle \bar{v}, z\rangle^2 - \langle \mathrm{j}\bar{v}, z\rangle^2},$$

where $v = (v_1, \ldots, v_h) \in \mathbb{S}^{2h-1} \subset \mathbb{D}^h$, $w = (w_0, w_1, \ldots, w_p) \in \mathbb{S}^p$, $z = (z_1, \ldots, z_h) \in D_1$ and $D_1 = \left\{z \in \mathbb{D}^h : \langle \bar{v}, z\rangle^2 > \langle \mathrm{j}\bar{v}, z\rangle^2\right\}$.

2. $\Phi : D_1 \times_f \mathbb{H}^p \longrightarrow \mathcal{P}^{h+p}$;

$$\begin{aligned}\Phi(z, w) = \Bigg(& z_1 + \bar{v}_1(w_0 - 1)\sum_{j=1}^{h} v_j z_j, \ldots, z_h + \bar{v}_h(w_0 - 1)\sum_{j=1}^{h} v_j z_j, \\ & w_1 \sum_{j=1}^{h} \mathrm{j}v_j z_j, \ldots, w_p \sum_{j=1}^{h} \mathrm{j}v_j z_j\Bigg), \quad h \geq 1,\end{aligned} \tag{6.4}$$

with the warping function $f = \sqrt{\langle \bar{v}, z\rangle^2 - \langle \mathrm{j}\bar{v}, z\rangle^2}$, *where* $v = (v_1, \ldots, v_h) \in \mathbb{H}^{2h-1} \subset \mathbb{D}^h$, $w = (w_0, w_1, \ldots, w_p) \in \mathbb{H}^p$ *and* $z = (z_1, \ldots, z_h) \in D_1$.

3. $\Phi(z,u) : D_1 \times_f \mathbb{E}^p \longrightarrow \mathcal{P}^{h+p}$;

$$\Phi(z,u) = \Big(z_1 + \frac{\bar{v}_1}{2}\Big(\sum_{a=1}^{p} u_a^2\Big)\sum_{j=1}^{h} v_j z_j, \ldots, z_h + \frac{\bar{v}_h}{2}\Big(\sum_{a=1}^{p} u_a^2\Big)\sum_{j=1}^{h} v_j z_j, \quad u_1 \sum_{j=1}^{h} \mathrm{j} v_j z_j, \ldots, u_p \sum_{j=1}^{h} \mathrm{j} v_j z_j\Big), \quad h \geq 2, \tag{6.5}$$

with the warping function $f = \sqrt{\langle \bar{v}, z\rangle^2 - \langle \mathrm{j}\bar{v}, z\rangle^2}$, *where* $v = (v_1, \ldots, v_h)$ *is a light-like vector in* $\mathbb{D}^h$, $z = (z_1, \ldots, z_h) \in D_1$ *and* $u = (u_1, \ldots, u_p) \in \mathbb{E}^p$,

Moreover, in this case, each leaf $\mathbb{E}^p$ *is quasi-minimal in* $\mathcal{P}^{h+p}$.

4. $\Phi(z,u) : D_2 \times_f \mathbb{E}^p \longrightarrow \mathcal{P}^{h+p}$;

$$\Phi(z,u) = \Big(z_1 + \frac{v_1}{2}\sum_{a=1}^{p} u_a^2, \ldots, z_h + \frac{v_h}{2}\sum_{a=1}^{p} u_a^2, \frac{v_0}{2}u_1, \ldots, \frac{v_0}{2}u_p\Big), \ h \geq 1, \tag{6.6}$$

with the warping function $f = \sqrt{-\langle v, z\rangle}$, *where* $v_0 = \sqrt{b_1} + \epsilon \mathrm{j}\sqrt{b_1}$ *with* $b_1 > 0$, $D_2 = \{z \in \mathbb{D}^h : \langle v, z\rangle < 0\}$, $v = (v_1, \ldots, v_h) = (b_1 + \epsilon \mathrm{j} b_1, \ldots, b_h + \epsilon \mathrm{j} b_h)$, $\epsilon = \pm 1$, $z = (z_1, \ldots, z_h) \in D_2$ *and* $u = (u_1, \ldots, u_p) \in \mathbb{E}^p$.

In each of the four cases the warped product is minimal in $\mathbb{E}^{2(h+p)}_{h+p}$.

Proof. Inequality (6.2) is already given in Theorem 4.4. From now on, let us assume that $\Phi : N_\top \times_f N_\perp \longrightarrow \mathcal{P}^m$ is a space-like $\mathcal{P}R$-warped product satisfying the equality in (6.2) with $m = h + p$. Then it follows that $\nu = 0$ and hence

$$\sigma(X,Y) = 0, \ \sigma(Z,W) = 0, \ \sigma(X,Z) = \big(PX(\ln f)\big)FZ, \tag{6.7}$$

for all X, Y tangent to $N_\top$ and Z, W tangent to $N_\perp$. Thus, $N_\top$ is totally geodesic in $\mathcal{P}^m$ and $N_\perp$ is totally umbilical $\mathcal{P}^m$.

As $N_\top$ is invariant and totally geodesic in $\mathcal{P}^m$, it is congruent with $\mathcal{P}^h$ with the canonical (induced) para-Kähler structure [15]. On $\mathbb{E}^{2h}_h$ we may choose global coordinates $s = (s_1, \ldots, s_h)$ and $t = (t_1, \ldots, t_h)$ such that

$$g_\top = -\sum_{i=1}^{h} ds_i^2 + \sum_{i=1}^{h} dt_i^2, \ \ \mathcal{P}\partial_{s_i} = \partial_{t_i}, \ \ \mathcal{P}\partial_{t_i} = \partial_{s_i}, \tag{6.8}$$

for $i = 1, \ldots, h$.

Let us put $\partial_{s_i} = \frac{\partial}{\partial s_i}$, $\partial_{t_i} = \frac{\partial}{\partial t_i}$ and so on.

Now, we study the case $p > 1$.

Since $N_\perp$ is a space-like totally umbilical, non-totally geodesic submanifold in $\mathcal{P}^m$, it is congruent (cf. [1], [15, Proposition 3.6])

- either to the Euclidean p-sphere $\mathbb{S}^p$,
- or to the hyperbolic p-plane $\mathbb{H}^p$,
- or to a flat quasi-minimal submanifold $\mathbb{E}^p$.

In what follows we discuss successively, all the three situations.

On $\mathbb{S}^p$ we consider spherical coordinates $u = (u_1, \ldots, u_p)$ such that the metric $g_\perp$ is expressed by

$$g_\perp = du_1^2 + \cos^2 u_1 du_2^2 + \ldots + \cos^2 u_1 \ldots \cos^2 u_{p-1} du_p^2. \tag{6.9}$$

Thus, the warped metric on M is given by

$$g = g_\top(s,t) + f^2(s,t) g_\perp(u).$$

Then the Levi Civita connection ∇ of g satisfies

$$\nabla_{\partial_{s_i}} \partial_{s_j} = 0\ ,\ \nabla_{\partial_{s_i}} \partial_{t_j} = 0\ ,\ \nabla_{\partial_{t_i}} \partial_{t_j} = 0, \tag{6.10.a}$$

$$\nabla_{\partial_{s_i}} \partial_{u_a} = \frac{f_{s_i}}{f}\ \partial_{u_a}\ ,\ \nabla_{\partial_{t_i}} \partial_{u_a} = \frac{f_{t_i}}{f}\ \partial_{u_a}, \tag{6.10.b}$$

$$\nabla_{\partial_{u_a}} \partial_{u_b} = -\tan u_a \partial_{u_b} \quad (a < b), \tag{6.10.c}$$

$$\begin{aligned} \nabla_{\partial_{u_a}} \partial_{u_a} = {} & \prod_{b=1}^{a-1} \cos^2 u_b \sum_{i=1}^{h} \big(f f_{s_i} \partial_{s_i} - f f_{t_i} \partial_{t_i} \big) \\ & + \sum_{b=1}^{a-1} \big(\sin u_b \cos u_b \cos^2 u_{b+1} \ldots \cos^2 u_{a-1} \big) \partial_{u_b}, \end{aligned} \tag{6.10.d}$$

for $i, j = 1, \ldots, h$ and $a, b = 1, \ldots, p$.

From now on we put $\psi = \ln f$. Using the relations above, we find that the Riemann curvature tensor R satisfies

$$\begin{aligned} R(\partial_{s_i}, \partial_{u_a})\ \partial_{s_j} &= \left(\frac{\partial^2 \psi}{\partial s_i \partial s_j} + \frac{\partial \psi}{\partial s_i} \frac{\partial \psi}{\partial s_j} \right) \partial_{u_a} \\ R(\partial_{s_i}, \partial_{u_a})\ \partial_{t_j} &= \left(\frac{\partial^2 \psi}{\partial s_i \partial t_j} + \frac{\partial \psi}{\partial s_i} \frac{\partial \psi}{\partial t_j} \right) \partial_{u_a} \\ R(\partial_{t_i}, \partial_{u_a})\ \partial_{t_j} &= \left(\frac{\partial^2 \psi}{\partial t_i \partial t_j} + \frac{\partial \psi}{\partial t_i} \frac{\partial \psi}{\partial t_j} \right) \partial_{u_a}\ . \end{aligned} \tag{6.11}$$

Moreover we have

$$\sigma(\partial_{s_i}, \partial_{u_a}) = \frac{\partial \psi}{\partial t_i}\ F \partial_{u_a},\ \sigma(\partial_{t_i}, \partial_{u_a}) = \frac{\partial \psi}{\partial s_i}\ F \partial_{u_a}.$$

Applying Gauss' equation we find

$$\widetilde{g}\big(\widetilde{R}_{XZ}Y, W\big) = g\big(R_{XZ}Y, W\big) + \widetilde{g}\big(\sigma(X,Y), \sigma(Z,W)\big) - \widetilde{g}\big(\sigma(X,W), \sigma(Y,Z)\big),$$

for X, Y tangent to $N_\top$ and Z, W tangent to $N_\perp$. Using (6.7) and (6.11) we get

$$\begin{aligned}
&\frac{\partial^2\psi}{\partial s_i \partial s_j} + \frac{\partial\psi}{\partial s_i}\frac{\partial\psi}{\partial s_j} + \frac{\partial\psi}{\partial t_i}\frac{\partial\psi}{\partial t_j} = 0\\
&\frac{\partial^2\psi}{\partial s_i \partial t_j} + \frac{\partial\psi}{\partial s_i}\frac{\partial\psi}{\partial t_j} + \frac{\partial\psi}{\partial t_i}\frac{\partial\psi}{\partial s_j} = 0\\
&\frac{\partial^2\psi}{\partial t_i \partial t_j} + \frac{\partial\psi}{\partial s_i}\frac{\partial\psi}{\partial s_j} + \frac{\partial\psi}{\partial t_i}\frac{\partial\psi}{\partial t_j} = 0, \ i = 1, \ldots, h.
\end{aligned} \tag{6.12}$$

Let us first consider the case $h \geq 2$.

By applying Proposition 5.1 (case 1, in the proof), we know that there exists a constant vector $v = (a_1, a_2, \ldots, a_h, 0, b_2, \ldots, b_h)$, with $a_1 > 0$, such that

$$\psi = \frac{1}{2} \ln \left[\langle \bar{v}, z\rangle^2 - \langle \mathrm{j}\bar{v}, z\rangle^2\right],$$

where $z = (s_1, \ldots, s_h, t_1, \ldots, t_h)$ and $\langle\ ,\ \rangle$ denotes the pseudo-Euclidean product in $\mathbb{E}^{2h}_h$. If $a_1 < 0$ we are allowed to make the isometric transformation in $\mathbb{E}^{2h}_h$: $s_1 \mapsto -s_1$ and $t_1 \mapsto -t_1$. In the sequel, we apply Gauss' formula

$$\widetilde{\nabla}_{\Phi_* U}\Phi_* V = \Phi_* \nabla_U V + \sigma(U, V), \ \forall U, V \in \chi(M),$$

where Φ_* denotes the differential of the map Φ. Taking $U, V \in \mathcal{D}$ we obtain

$$\begin{aligned}
&\frac{\partial^2 x_A}{\partial s_i \partial s_j} = \frac{\partial^2 x_A}{\partial s_i \partial t_j} = \frac{\partial^2 x_A}{\partial t_i \partial t_j} = 0\\
&\frac{\partial^2 y_A}{\partial s_i \partial s_j} = \frac{\partial^2 y_A}{\partial s_i \partial t_j} = \frac{\partial^2 y_A}{\partial t_i \partial t_j} = 0.
\end{aligned} \tag{6.13}$$

For $U \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$ we have

$$\begin{aligned}
&\frac{\partial^2 x_A}{\partial s_i \partial u_a} = \psi_{s_i}\frac{\partial x_A}{\partial u_a} + \psi_{t_i}\frac{\partial y_A}{\partial u_a}\ , \quad \frac{\partial^2 x_A}{\partial t_i \partial u_a} = \psi_{t_i}\frac{\partial x_A}{\partial u_a} + \psi_{s_i}\frac{\partial y_A}{\partial u_a}\\
&\frac{\partial^2 y_A}{\partial s_i \partial u_a} = \psi_{s_i}\frac{\partial y_A}{\partial u_a} + \psi_{t_i}\frac{\partial x_A}{\partial u_a}\ , \quad \frac{\partial^2 y_A}{\partial t_i \partial u_a} = \psi_{t_i}\frac{\partial y_A}{\partial u_a} + \psi_{s_i}\frac{\partial x_A}{\partial u_a}.
\end{aligned} \tag{6.14}$$

Finally, taking $U, V \in \mathcal{D}^{\perp}$ we obtain

$$
\begin{aligned}
\frac{\partial^2 x_A}{\partial u_a \partial u_b} &= -\tan u_a \frac{\partial x_A}{\partial u_b}\,, \quad \frac{\partial^2 y_A}{\partial u_a \partial u_b} = -\tan u_a \frac{\partial y_A}{\partial u_b}\,,\ a<b\,,\\
\frac{\partial^2 x_A}{\partial u_a^2} &= \prod_{b=1}^{a-1}\cos^2 u_b \sum_{j=1}^{h}\Big(ff_{s_j}\frac{\partial x_A}{\partial s_j} - ff_{t_j}\frac{\partial x_A}{\partial t_j}\Big)\\
&\quad + \sum_{b=1}^{a-1}\big(\sin u_b \cos u_b \cos^2 u_{b+1}\ldots\cos^2 u_{a-1}\big)\frac{\partial x_A}{\partial u_b}\,,\\
\frac{\partial^2 y_A}{\partial u_a^2} &= \prod_{b=1}^{a-1}\cos^2 u_b \sum_{j=1}^{h}\Big(ff_{s_j}\frac{\partial y_A}{\partial s_j} - ff_{t_j}\frac{\partial y_A}{\partial t_j}\Big)\\
&\quad + \sum_{b=1}^{a-1}\big(\sin u_b \cos u_b \cos^2 u_{b+1}\ldots\cos^2 u_{a-1}\big)\frac{\partial y_A}{\partial u_b}\,.
\end{aligned}
\tag{6.15}
$$

From (6.13) we get

$$
\begin{aligned}
x_A(s,t,u) &= \sum_1^h \lambda_A^j(u)s_j + \sum_1^h \rho_A^j(u)t_j + C_A(u)\,,\\
y_A(s,t,u) &= \sum_1^h \tilde\rho_A^j(u)s_j + \sum_1^h \tilde\lambda_A^j(u)t_j + \tilde C_A(u)\,.
\end{aligned}
\tag{6.16}
$$

By combining (6.14) with (6.16) we obtain

$$
\begin{aligned}
\frac{\partial\tilde\lambda_A^i}{\partial u_a} = \frac{\partial\lambda_A^i}{\partial u_a} &= \psi_{s_i}\left[\frac{\partial\lambda_A^j}{\partial u_a}(u)s_j + \frac{\partial\rho_A^j}{\partial u_a}(u)t_j + \frac{\partial C_A}{\partial u_a}\right]\\
&\quad + \psi_{t_i}\left[\frac{\partial\rho_A^j}{\partial u_a}(u)s_j + \frac{\partial\lambda_A^j}{\partial u_a}(u)t_j + \frac{\partial \tilde C_A}{\partial u_a}\right]\,,\\
\frac{\partial\tilde\rho_A^i}{\partial u_a} = \frac{\partial\rho_A^i}{\partial u_a} &= \psi_{t_i}\left[\frac{\partial\lambda_A^j}{\partial u_a}(u)s_j + \frac{\partial\rho_A^j}{\partial u_a}(u)t_j + \frac{\partial C_A}{\partial u_a}\right]\\
&\quad + \psi_{s_i}\left[\frac{\partial\rho_A^j}{\partial u_a}(u)s_j + \frac{\partial\lambda_A^j}{\partial u_a}(u)t_j + \frac{\partial \tilde C_A}{\partial u_a}\right]\,.
\end{aligned}
\tag{6.17}
$$

For $i=1$ we have

$$
\psi_{s_1} = \frac{a_1\big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big)}{\big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big)^2 - \big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big)^2}\,,
$$

$$\psi_{t_1} = \frac{-a_1\big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big)}{\big(a_1s_1 + \sum_2^h a_js_j + \sum_2^h b_jt_j\big)^2 - \big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big)^2}\ .$$

Substituting in (6.17) we find polynomials in s and t. Comparing the coefficients corresponding to s_1s_i and s_1t_i, $i > 1$, we find

$$\lambda^i_A(u) = \tfrac{a_i}{a_1}\lambda_A(u) + \tfrac{b_i}{a_1}\rho_A(u) + \tfrac{c^i_A}{a_1}\ ,\ \rho^i_A(u) = \tfrac{b_i}{a_1}\lambda_A(u) + \tfrac{a_i}{a_1}\rho_A(u) + \tfrac{d^i_A}{a_1} \tag{6.18}$$

for $i = 2, \dots, h$, and $\lambda^1_A(u) = \lambda_A(u)$, $\rho^1_A(u) = \rho_A(u)$, where $c^i_A, d^i_A \in \mathbb{R}$.

Comparing the coefficients of s_1 and t_1 we find that C_A and $\tilde{C}_A$ are constants, and applying a suitable translation in $\mathbb{E}^{2m}_m$ if necessary, one may suppose $C_A = 0$ and $\tilde{C}_A = 0$, $A = 1, \dots, m$. Replacing in (6.16) and taking into account (6.17) we get

$$\begin{aligned} x_A(s,t,u) &= \frac{1}{a_1}\lambda_A(u)\big(a_1s_1 + \sum_2^h a_js_j + \sum_2^h b_jt_j\big)\\ &\quad + \frac{1}{a_1}\rho_A(u)\big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big)\\ &\quad + \frac{1}{a_1}\big(\sum_2^h c^j_As_j + \sum_2^h d^j_At_j\big),\\ y_A(s,t,u) &= \frac{1}{a_1}\lambda_A(u)\big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big)\\ &\quad + \frac{1}{a_1}\rho_A(u)\big(a_1s_1 + \sum_2^h a_js_j + \sum_2^h b_jt_j\big)\\ &\quad + \frac{1}{a_1}\big(\tilde{d}_As_1 + \tilde{c}_At_1 + \sum_2^h \tilde{d}^j_As_j + \sum_2^h \tilde{c}^j_At_j\big), \end{aligned} \tag{6.19}$$

where $\tilde{c}_A$, $\tilde{d}_A$, $\tilde{c}^i_A$ and $\tilde{d}^i_A$ are real numbers. The third equation in (6.15) for $a = 1$ gives

$$\begin{aligned} \frac{\partial^2 x_A}{\partial u_1^2} &= \big(a_1s_1 + \sum_2^h a_js_j + \sum_2^h b_jt_j\big)\left[a_1\frac{\partial x_A}{\partial s_1} + \sum_2^h a_j\frac{\partial x_A}{\partial s_j} - \sum_2^h b_j\frac{\partial x_A}{\partial t_j}\right]\\ &\quad + \big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big)\left[a_1\frac{\partial x_A}{\partial t_1} + \sum_2^h a_j\frac{\partial x_A}{\partial t_j} - \sum_2^h b_j\frac{\partial x_A}{\partial s_j}\right] \end{aligned}$$

which combined with the first equation in (6.19) yields

$$\begin{aligned}&\big(a_1s_1+\sum_2^h a_js_j+\sum_2^h b_jt_j\big)\left[\frac{\partial^2\lambda_A}{\partial u_1^2}(u)+\langle v,v\rangle\lambda_A(u)+D_A\right]\\&+\big(a_1t_1+\sum_2^h a_jt_j+\sum_2^h b_js_j\big)\left[\frac{\partial^2\rho_A}{\partial u_1^2}(u)+\langle v,v\rangle\rho_A(u)+\tilde{D}_A\right]=0,\end{aligned}\tag{6.20}$$

where $D_A=\sum_2^h(a_jc_A^j-b_jd_A^j)$ and $\tilde{D}_A=\sum_2^h(a_jd_A^j-b_jc_A^j)$.

Since $||\nabla f||_2=-a_1^2-\sum_2^h a_j^2+\sum_2^h b_j^2$, Proposition 6.1 implies $\langle v,v\rangle=1$. Hence, considering in (6.20) the coefficients of s_1 and t_1 one obtains the following PDEs:

$$\frac{\partial^2\lambda_A}{\partial u_1^2}(u)+\lambda_A(u)-D_A=0,\quad \frac{\partial^2\rho_A}{\partial u_1^2}(u)+\rho_A(u)-\tilde{D}_A=0.\tag{6.21}$$

We immediately get

$$\begin{aligned}&\lambda_A(u)=\cos u_1\Theta_A^{(1)}(u_2,\ldots,u_p)+\sin u_1 D_A^{(1)}(u_2,\ldots,u_p)+D_A,\\&\rho_A(u)=\cos u_1\tilde{\Theta}_A^{(1)}(u_2,\ldots,u_p)+\sin u_1\tilde{D}_A^{(1)}(u_2,\ldots,u_p)+\tilde{D}_A\end{aligned}\tag{6.22}$$

where $\Theta_A^{(1)}$, $D_A^{(1)}$, $\tilde{\Theta}_A^{(1)}$ and $\tilde{D}_A^{(1)}$ are functions depending on $u_2,\ldots,u_p$. The first equation in (6.15) for $a=1$ gives $\frac{\partial^2 x_A}{\partial u_1\partial u_b}=-\tan u_1\frac{\partial x_A}{\partial u_b}$, $b>1$ which combined with (6.19) yields

$$\frac{\partial^2\lambda_A}{\partial u_1\partial u_b}=-\tan u_1\frac{\partial\lambda_A}{\partial u_b}\ ,\ \frac{\partial^2\rho_A}{\partial u_1\partial u_b}=-\tan u_1\frac{\partial\rho_A}{\partial u_b}\ .$$

Using (6.22), we get $\frac{\partial D_A^{(1)}}{\partial u_b}=0$, and $\frac{\partial\tilde{D}_A^{(1)}}{\partial u_b}=0$, $\forall b>1$, hence $D_A^{(1)}$ and $\tilde{D}_A^{(1)}$ are real constants.

Returning to the third equation in (6.15) with $a=2$ we get

$$\begin{aligned}\frac{\partial^2x_A}{\partial u_2^2}&=\cos^2u_1\big(a_1s_1+\sum_2^h a_js_j+\sum_2^h b_jt_j\big)\left[a_1\frac{\partial x_A}{\partial s_1}+\sum_2^h a_j\frac{\partial x_A}{\partial s_j}-\sum_2^h b_j\frac{\partial x_A}{\partial t_j}\right]\\&+\cos^2u_2\big(a_1t_1+\sum_2^h a_jt_j+\sum_2^h b_js_j\big)\left[a_1\frac{\partial x_A}{\partial t_1}+!\sum_2^h a_j\frac{\partial x_A}{\partial t_j}-\sum_2^h b_j\frac{\partial x_A}{\partial s_j}\right]\\&+\sin u_1\cos u_1\frac{\partial x_A}{\partial u_1}\ .\end{aligned}$$

This relation together with (6.19) yield a polynomial in s and t, and considering the coefficients of s_1 and t_1 respectively, we obtain

$$\frac{\partial^2\lambda_A}{\partial u_2^2} - \sin u_1 \cos u_1 \tfrac{\partial \lambda_A}{\partial u_1} + (\cos^2 u_1)\lambda_A - D_A \cos^2 u_1 = 0\,,$$
$$\frac{\partial^2\rho_A}{\partial u_2^2} - \sin u_1 \cos u_1 \tfrac{\partial \rho_A}{\partial u_1} + (\cos^2 u_1)\rho_A - \tilde{D}_A \cos^2 u_1 = 0\,.$$

Using (6.22) one gets

$$\frac{\partial^2\Theta_A^{(1)}}{\partial u_2^2} + \Theta_A^{(1)} = 0\ ,\ \ \frac{\partial^2\tilde{\Theta}_A^{(1)}}{\partial u_2^2} + \tilde{\Theta}_A^{(1)} = 0$$

with the solutions

$$\Theta_A^{(1)} = \cos u_2 \Theta_A^{(2)}(u_3,\dots,u_p) + \sin u_2 D_A^{(2)}(u_3,\dots,u_p)\,,$$
$$\tilde{\Theta}_A^{(1)} = \cos u_2 \tilde{\Theta}_A^{(2)}(u_3,\dots,u_p) + \sin u_2 \tilde{D}_A^{(2)}(u_3,\dots,u_p),$$

where $\Theta_A^{(2)}$, $D_A^{(2)}$, $\tilde{\Theta}_A^{(2)}$ and $\tilde{D}_A^{(2)}$ are functions depending on $u_3,\dots,u_p$. Continuing such procedure sufficiently many times, we find

(6.23)
$$\begin{aligned}
\lambda_A(u) &= D_A^{(0)} \cos u_1 \dots \cos_{p-1} \cos u_p + D_A^{(p)} \cos u_1 \dots \cos_{p-1} \sin u_p \\
&\quad + D_A^{(p-1)} \cos u_1 \dots \sin_{p-1} + \dots \\
&\quad + D_A^{(2)} \cos u_1 \sin u_1 + D_A^{(1)} \sin u_1 + D_A\,, \\
\rho_A(u) &= \tilde{D}_A^{(0)} \cos u_1 \dots \cos_{p-1} \cos u_p + \tilde{D}_A^{(p)} \cos u_1 \dots \cos_{p-1} \sin u_p \\
&\quad + \tilde{D}_A^{(p-1)} \cos u_1 \dots \sin_{p-1} + \dots \\
&\quad + \tilde{D}_A^{(2)} \cos u_1 \sin u_1 + \tilde{D}_A^{(1)} \sin u_1 + \tilde{D}_A\,,
\end{aligned}$$

where $D_A^{(p)},\dots,D_A^{(0)}$, D_A, $\tilde{D}_A^{(p)},\dots,\tilde{D}_A^{(0)}$ and $\tilde{D}_A$ are real constants. At this point let us make the following notations

$$\begin{aligned}
w_0 &= \cos u_1 \dots \cos u_{p-1} \cos u_p \\
w_p &= \cos u_1 \dots \cos u_{p-1} \sin u_p \\
w_{p-1} &= \cos u_1 \dots \sin u_{p-1} \\
\dots &\ \ \dots\dots\dots\dots \\
w_2 &= \cos u_1 \sin u_2 \\
w_1 &= \sin u_1.
\end{aligned}$$

It follows that λ_A and ρ_A may be rewritten as

$$\lambda_A(w) = D_A + \sum_{a=0}^{p} D_A^{(a)} w_a\,, \quad \rho_A(w) = \tilde{D}_A + \sum_{a=0}^{p} \tilde{D}_A^{(a)} w_a. \tag{6.24}$$

Going back to (6.19) we get, after a re-scaling with $a_1 \neq 0$

$$\begin{aligned} x_A(s,t,w) &= \big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big) \sum_{a=0}^{p} D_A^{(a)} w_a \\ &\quad + \big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big) \sum_{a=0}^{p} \tilde{D}_A^{(a)} w_a + \sum_{j=1}^{h} (\alpha_A^j s_j + \beta_A^j t_j)\,, \\ y_A(s,t,w) &= \big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big) \sum_{a=0}^{p} \tilde{D}_A^{(a)} w_a \\ &\quad + \big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big) \sum_{a=0}^{p} D_A^{(a)} w_a + \sum_{j=1}^{h} (\tilde{\alpha}_A^j s_j + \tilde{\beta}_A^j t_j). \end{aligned} \tag{6.25}$$

Let us choose the initial conditions

(6.26.a) $\quad \Phi_* \partial_{s_i}(1,0,\ldots,0) = (0,\ldots,0,\overset{(i)}{1},0,\ldots,0,0,\ldots,0)\,,$

(6.26.b) $\quad \Phi_* \partial_{t_i}(1,0,\ldots,0) = (0,\ldots,0,0,\ldots,\overset{(m+i)}{1},0,\ldots,0)\,,\ i = 1,\ldots,h\,,$

(6.26.c) $\quad \Phi_* \partial_{u_b}(1,0,\ldots,0) = (0,\ldots,0,0,\ldots,\overset{(m+h+b)}{a_1,}\ 0,\ldots,0)\,,\ b = 1,\ldots,p\,.$

From (6.25) and (6.26.c) and taking into account that

$$\left.\frac{\partial w_a}{\partial u_b}\right|_{u=0} = \begin{cases} 0, \text{ if } a = 0 \\ 0, \text{ if } b \neq a,\ a \geq 1 \\ 1, \text{ if } b = a\,, \end{cases}$$

we obtain that

$$\begin{aligned} &D_i^{(b)} = 0,\ D_{h+a}^{(b)} = 0,\ \tilde{D}_i^{(b)} = 0,\ \tilde{D}_{h+a}^{(b)} = 0, (a \neq b),\ \tilde{D}_{h+b}^{(b)} = 1, \\ &i = 1,\ldots,h;\ a,b = 1,\ldots,p. \end{aligned} \tag{6.27}$$

From (6.25) and (6.26.a) we find

$$\begin{aligned} &a_i D_j^{(0)} + b_i \tilde{D}_j^{(0)} + \alpha_j^i = \delta_{ij}, \\ &a_i D_{h+a}^{(0)} + b_i \tilde{D}_{h+a}^{(0)} + \alpha_{h+a}^i = 0\,, \\ &a_i \tilde{D}_j^{(0)} + b_i D_j^{(0)} + \tilde{\alpha}_j^i = 0, \quad a_i \tilde{D}_{h+a}^{(0)} + b_i D_{h+a}^{(0)} + \tilde{\alpha}_{h+a}^i = 0\,, \\ &i,j = 1,\ldots,h,\ a = 1,\ldots,p,\ b_1 = 0. \end{aligned} \tag{6.28}$$

Finally, from (6.25) and (6.26.b) we get

$$\begin{gathered} b_i D_j^{(0)} + a_i \tilde{D}_j^{(0)} + \beta_j^i = 0, \quad b_i D_{h+a}^{(0)} + a_i \tilde{D}_{h+a}^{(0)} + \beta_{h+a}^i = 0\,, \\ b_i \tilde{D}_j^{(0)} + a_i D_j^{(0)} + \tilde{\beta}_j^i = \delta_{ij},\ b_i \tilde{D}_{h+a}^{(0)} + a_i D_{h+a}^{(0)} + \tilde{\beta}_{h+a}^i = 0\,, \\ i, j = 1, \ldots, h,\ a = 1, \ldots, p,\ b_1 = 0\,. \end{gathered} \tag{6.29}$$

Now, plugging (6.27), (6.28) and (6.29) in (6.25) we obtain

$$\begin{aligned} x_i(s,t,w) = s_i &+ D_i^{(0)}(w_0-1)\big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big) \\ &+ \tilde{D}_i^{(0)}(w_0-1)\big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big)\,, \end{aligned} \tag{6.30.a}$$

$$\begin{aligned} x_{h+a}(s,t,w) = &D_{h+a}^{(0)}(w_0-1)\big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big) \\ &+ \big[w_a + \tilde{D}_{h+a}^{(0)}(w_0-1)\big]\big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big)\,, \end{aligned} \tag{6.30.b}$$

$$\begin{aligned} y_i(s,t,w) = t_i &+ D_i^{(0)}(w_0-1)\big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big) \\ &+ \tilde{D}_i^{(0)}(w_0-1)\big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big)\,, \end{aligned} \tag{6.30.c}$$

$$\begin{aligned} y_{h+a}(s,t,w) = &D_{h+a}^{(0)}(w_0-1)\big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big) \\ &+ \big[w_a + \tilde{D}_{h+a}^{(0)}(w_0-1)\big]\big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big). \end{aligned} \tag{6.30.d}$$

Since Φ is an isometric immersion we have $\tilde{g}(\Phi_* U, \Phi_* V) = g(U,V)$ for every U and V tangent to M. From $\tilde{g}(\Phi_*\partial_{s_1}, \Phi_*\partial_{s_1}) = -1$ and (6.30) we get

$$(w_0-1)\langle D^{(0)}, D^{(0)}\rangle + 2\sum_{a=1}^p w_a \tilde{D}_{h+a}^{(0)} - \frac{2}{a_1}\ D_1^{(0)} - (w_0+1) = 0$$

for all $w \in \mathbb{S}^p$, where

$$D^{(0)} = \big(D_1^{(0)}, \ldots, D_h^{(0)}, D_{h+1}^{(0)}, \ldots, D_{2h}^{(0)}, \tilde{D}_1^{(0)}, \ldots, \tilde{D}_h^{(0)}, \tilde{D}_{h+1}^{(0)}, \ldots, \tilde{D}_{2h}^{(0)}\big).$$

Therefore

$$D_1^{(0)} = -a_1\,,\ \tilde{D}_{h+a}^{(0)} = 0, \forall a = 1,\dots,p,\ \langle D^{(0)}, D^{(0)}\rangle = 1\,. \tag{6.31}$$

From $\tilde{g}(\Phi_*\partial_{s_1}, \Phi_*\partial_{s_j}) = 0$ and $\tilde{g}(\Phi_*\partial_{s_1}, \Phi_*\partial_{t_j}) = 0$, $(j \geq 2)$, together with (6.30) and (6.31) it follows

$$D_j^{(0)} = -a_j - \frac{b_j}{a_1}\,\tilde{D}_1^{(0)}, \quad \tilde{D}_j^{(0)} = b_j + \frac{a_j}{a_1}\,\tilde{D}_1^{(0)},\ \forall j \geq 2. \tag{6.32}$$

Finally, from $\tilde{g}(\Phi_*\partial_{s_1}, \Phi_*\partial_{u_b}) = 0$, (6.30) and (6.31) we get $\tilde{D}_1^{(0)} = 0$. Hence from (6.32) one obtains $D_j^{(0)} = -a_j$ and $\tilde{D}_j^{(0)} = b_j$, for all $j = 1,\dots,h$ (recall $b_1 = 0$), which combined with $\langle D^{(0)}, D^{(0)}\rangle = 1$ yield $D_{h+a}^{(0)} = 0$.

We conclude from (6.30) the following

$$\begin{aligned}
x_i(s,t,w) &= s_i - a_i(w_0-1)\big(a_1s_1 + \sum_2^h a_js_j + \sum_2^h b_jt_j\big)\\
&\quad + b_i(w_0-1)\big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big),\\
x_{h+a}(s,t,w) &= w_a\big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big),\\
y_i(s,t,w) &= t_i - a_i(w_0-1)\big(a_1t_1 + \sum_2^h a_jt_j + \sum_2^h b_js_j\big)\\
&\quad + b_i(w_0-1)\big(a_1s_1 + \sum_2^h a_js_j + \sum_2^h b_jt_j\big),\\
y_{h+a}(s,t,w) &= w_a\big(a_1s_1 + \sum_2^h a_js_j + \sum_2^h b_jt_j\big).
\end{aligned} \tag{6.33}$$

Computing now $x_i + \mathrm{j}y_i$ and $x_{h+a} + \mathrm{j}y_{h+a}$ one gets (6.3).

Let consider the second situation when $N_\perp$ is the hyperbolic space $\mathbb{H}^p$. On $\mathbb{H}^p$ consider coordinates $u = (u_1, u_2, \dots, u_p)$ such that the metric $g_\perp$ is expressed by

$$g_\perp = du_1^2 + \sinh^2 u_1\left(du_2^2 + \cos^2 u_2 du_3^2 + \ldots + \cos^2 u_2 \ldots \cos^2 u_{p-1} du_p^2\right), \tag{6.34}$$

and the warped metric on M is given by $g = g_\top(s,t) + f^2(s,t)g_\perp(u)$. Then the Levi

Civita connection ∇ of g satisfies

(6.35.a) $\nabla_{\partial_{s_i}}\partial_{s_j} = 0\,,\ \nabla_{\partial_{s_i}}\partial_{t_j} = 0\,,\ \nabla_{\partial_{t_i}}\partial_{t_j} = 0,$

(6.35.b) $\nabla_{\partial_{s_i}}\partial_{u_a} = \dfrac{f_{s_i}}{f}\,\partial_{u_a}\,,\ \nabla_{\partial_{t_i}}\partial_{u_a} = \dfrac{f_{t_i}}{f}\,\partial_{u_a},$

(6.35.c) $\nabla_{\partial u_1}\partial_{u_b} = \coth u_1 \partial_{u_b} \quad (1 < b),$

(6.35.d) $\nabla_{\partial_{u_a}}\partial_{u_b} = -\tan u_a \partial_{u_b} \quad (1 < a < b),$

(6.35.e) $\nabla_{\partial_{u_1}}\partial_{u_1} = \sum_{i=1}^{h}\big(ff_{s_i}\partial_{s_i} - ff_{t_i}\partial_{t_i}\big),$

(6.35.f)
$$\begin{aligned}\nabla_{\partial_{u_a}}\partial_{u_a} = {}& \sinh^2 u_1 \prod_{b=2}^{a-1}\cos^2 u_b \sum_{i=1}^{h}\big(ff_{s_i}\partial_{s_i} - ff_{t_i}\partial_{t_i}\big)\\ & - \sinh u_1 \cosh u_1 \prod_{b=2}^{a-1}\cos^2 u_b\ \partial_{u_1}\\ & + \sum_{b=1}^{a-1}\big(\sin u_b \cos u_b \cos^2 u_{b+1}\ldots\cos^2 u_{a-1}\big)\partial_{u_b},\ \ (1 < a)\end{aligned}$$

for any $i, j = 1, \ldots, h$ and $a, b = 1, \ldots, p$.

In the following we proceed in the same way as in previous case. Since some computations are very similar we skip them, and we will focus only on the major differences between the two cases.

The function ψ is obtained from Proposition 5.1 (case 1 in the proof):

$$\psi = \frac{1}{2}\ln\left[\langle\bar{v}, z\rangle^2 - \langle \mathrm{j}\bar{v}, z\rangle^2\right],$$

where $v = (a_1, a_2, \ldots, a_h, 0, b_2, \ldots, b_h)$, with $a_1 > 0$ is a constant vector.

Applying Gauss' formula $\tilde{\nabla}_{\Phi_* U}\Phi_* V = \Phi_*\nabla_U V + \sigma(U, V)$ for $U, V \in \mathcal{D}$, respectively for $U \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$ we may write (6.19). Using Gauss' formula for $U = V = \partial_{u_1}$, we find

$$\frac{\partial\lambda_A}{\partial u_1^2} + \langle v, v\rangle\lambda_A - D_A = 0 \quad : \quad D_A = \sum a_j c_A^j - \sum b_j \tilde{c}_A^j$$

$$\frac{\partial\rho_A}{\partial u_1^2} + \langle v, v\rangle\rho_A - \tilde{D}_A = 0 \quad : \quad D_A = \sum b_j c_A^j - \sum a_j \tilde{c}_A^j.$$

Here $\langle v, v\rangle = ||\nabla f||_2 = -1$ and consequently

(6.36)
$$\begin{aligned}\lambda_A(u) &= \cosh u_1 D_A^{(0)}(u_2, \ldots, u_p) + \sinh u_1 \Theta_A^{(0)}(u_2, \ldots, u_p) - D_A\,,\\ \rho_A(u) &= \cosh u_1 \tilde{D}_A^{(0)}(u_2, \ldots, u_p) + \sinh u_1 \tilde{\Theta}_A^{(0)}(u_2, \ldots, u_p) - \tilde{D}_A.\end{aligned}$$

Taking $U = \partial_{u_1}$ and $V = \partial_{u_b}$, ($b > 1$) we find that $D_A^{(0)}$ and $\tilde{D}_A^{(0)}$ are constants. Next, applying the Gauss formula for $U = V = \partial_{u_2}$ and respectively for $U = \partial_{u_2}$ and $V = \partial_{u_b}$, ($b > 2$) we get

$$\Theta_A^{(0)} = \cos u_2 \Theta_A^{(1)}(u_3, \ldots, u_p) + D_A^{(1)} \sin u_2 \,,$$
$$\tilde{\Theta}_A^{(0)} = \cos u_2 \tilde{\Theta}_A^{(1)}(u_3, \ldots, u_p) + \tilde{D}_A^{(1)} \sin u_2, \quad D_A^{(1)}, \tilde{D}_A^{(1)} \in \mathbb{R}.$$

Continuing the procedure sufficiently many times we finally get

$$\lambda_A = -D_A + D_A^{(0)} \cosh u_1 + D_A^{(1)} \sinh u_1 \cos u_2 + D_A^{(2)} \sinh u_1 \cos u_2 \sin u_3 + \cdots$$
$$+ D_A^{p-1)} \sinh u_1 \cos u_2 \cdots \cos u_{p-1} \sin u_p + D_A^{(p)} \sinh u_1 \cos u_2 \cdots \cos u_p \,,$$
$$\rho_A = -\tilde{D}_A + \tilde{D}_A^{(0)} \cosh u_1 + \tilde{D}_A^{(1)} \sinh u_1 \cos u_2 + \tilde{D}_A^{(2)} \sinh u_1 \cos u_2 \sin u_3 + \cdots$$
$$+ \tilde{D}_A^{p-1)} \sinh u_1 \cos u_2 \cdots \cos u_{p-1} \sin u_p + \tilde{D}_A^{(p)} \sinh u_1 \cos u_2 \cdots \cos u_p.$$

Considering the hyperbolic space $\mathbb{H}^p$ embedded in $\mathbb{R}_1^{p+1}$ with coordinates

$$\begin{aligned} w_0 &= \cosh u_1 \\ w_1 &= \sinh u_1 \sin u_2 \\ w_2 &= \sinh u_1 \cos u_2 \sin u_3 \\ &\cdots \quad \cdots \quad \cdots \\ w_{p-1} &= \sinh u_1 \cos u_2 \ldots \cos u_{p-1} \sin u_p \\ w_p &= \sinh u_1 \cos u_2 \ldots \cos u_{p-1} \cos u_p \,, \end{aligned} \tag{6.37}$$

we may express λ_A and ρ_A in terms of $w = (w_0, w_1, \ldots, w_p)$:

$$\begin{aligned} \lambda_A &= -D_A + D_A^{(0)} w_0 + D_A^{(1)} w_1 + \ldots + D_A^{(p)} w_p \,, \\ \rho_A &= -\tilde{D}_A + \tilde{D}_A^{(0)} w_0 + \tilde{D}_A^{(1)} w_1 + \ldots + \tilde{D}_A^{(p)} w_p \,. \end{aligned} \tag{6.38}$$

After a rescaling with the factor $a_1 \neq 0$ we may write

$$x_A(s,t,w) = \big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big) \sum_{a=0}^p D_A^{(a)} w_a$$
$$+ \big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big) \sum_{a=0}^p \tilde{D}_A^{(a)} w_a + \sum_{j=1}^h (\alpha_A^j s_j + \beta_A^j t_j) \,,$$
$$y_A(s,t,w) = \big(a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j\big) \sum_{a=0}^p \tilde{D}_A^{(a)} w_a$$
$$+ \big(a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j\big) \sum_{a=0}^p D_A^{(a)} w_a + \sum_{j=1}^h (\tilde{\alpha}_A^j s_j + \tilde{\beta}_A^j t_j)$$

which is similar to (6.25). From now on we will put

$$S = a_1 s_1 + \sum_2^h a_j s_j + \sum_2^h b_j t_j \quad \text{and} \quad T = a_1 t_1 + \sum_2^h a_j t_j + \sum_2^h b_j s_j. \tag{6.39}$$

Choose the initial point $s_{\rm init}(1,0,\ldots,0)$, $t_{\rm init}=(0,0,\ldots,0)$, $u_{\rm init}=(\omega,0,\ldots,0)$ with $\omega\neq 0$ and the initial conditions

$$\begin{aligned}
&\Phi_*\partial_{s_i}(1,0,\cdots,0,\omega,0,\cdots,0) = (0,\cdots,0,\overset{(i)}{1},0,\cdots,0,0,\cdots,0)\,,\\
&\Phi_*\partial_{t_i}(1,0,\cdots,0,\omega,0,\cdots,0) = (0,\cdots,0,0,\cdots,\overset{(m+i)}{1},0,\cdots,0)\,,\ i=1,\cdots,h\,,\\
&\Phi_*\partial_{u_1}(1,0,\cdots,0,\omega,0,\cdots,0) = (0,\cdots,0,0,\cdots,\overset{(m+h+1)}{a_1,}\ 0,\cdots,0)\,,\\
&\Phi_*\partial_{u_b}(1,0,\cdots,0,\omega,0,\cdots,0) = (0,\cdots,0,0,\cdots,\overset{(m+h+b)}{a_1\sinh\omega},\,0,\cdots,0),\ b=2,\cdots,p\,.
\end{aligned}$$

A straightforward computations, similar to previous case, yield

$$\begin{aligned}
x_i(s,t,w) &= s_i + a_i\big(W_0-1\big)S - b_i\big(W_0-1\big)T\,,\\
x_{h+1}(s,t,w) &= W_pT,\ \ x_{h+a}(s,t,w) = w_{a-1}T\ ,\ \ a=2,\ldots,p\,,\\
y_i(s,t,w) &= t_i + a_i\big(W_0-1\big)T - b_i\big(W_0-1\big)S\,,\\
y_{h+1}(s,t,w) &= W_pS\,,\ \ y_{h+a}(s,t,w) = w_{a-1}S\ ,\ \ a=2,\ldots,p\,,
\end{aligned}$$

where $W_0 = w_0\cosh\omega - w_p\sinh\omega$ and $W_p = -w_0\sinh\omega + w_p\cosh\omega$. Moreover, since $W_0^2 - W_p^2 = w_0^2 - w_p^2$, it follows $(W_0,w_1,\ldots,w_{p-1},W_p)\in\mathbb{H}^p$ and after a re-notation we write

$$\begin{aligned}
x_i(s,t,w) &= s_i + a_i\big(w_0-1\big)S - b_i\big(w_0-1\big)T\,,\\
x_{h+a}(s,t,w) &= w_aT\,,\quad a=1,\ldots,p\,,\\
y_i(s,t,w) &= t_i + a_i\big(w_0-1\big)T - b_i\big(w_0-1\big)S\,,\\
y_{h+a}(s,t,w) &= w_aS\,,\quad a=1,\ldots,p\,,
\end{aligned}$$

where $(w_0,w_1,\ldots,w_p)\in\mathbb{H}^p$. Computing $x_i+{\rm j}y_i$ and $x_{h+a}+{\rm j}y_{h+a}$ we get (6.4).

Let consider the third situation when $N_\perp$ is the flat space $\mathbb{E}^p$, on which we take coordinates $u=(u_1,u_2,\ldots,u_p)$ such that the metric $g_\perp$ is expressed by

$$g_\perp = du_1^2 + \ldots + du_p^2. \tag{6.40}$$

Then the warped metric on M is given by $g = g_\top(s,t) + f^2(s,t)g_\perp(u)$. The Levi

Civita connection ∇ of g satisfies

(6.41.a) $$\nabla_{\partial_{s_i}}\partial_{s_j} = 0\ ,\ \nabla_{\partial_{s_i}}\partial_{t_j} = 0\ ,\ \nabla_{\partial_{t_i}}\partial_{t_j} = 0\,,$$

(6.41.b) $$\nabla_{\partial_{s_i}}\partial_{u_a} = \frac{f_{s_i}}{f}\ \partial_{u_a}\ ,\ \nabla_{\partial_{t_i}}\partial_{u_a} = \frac{f_{t_i}}{f}\ \partial_{u_a}\,,$$

(6.41.c) $$\nabla_{\partial_{u_a}}\partial_{u_b} = 0\,,\ (a \neq b)\,,$$

(6.41.d) $$\nabla_{\partial_{u_a}}\partial_{u_a} = \sum_{i=1}^{h}\left(ff_{s_i}\partial_{s_i} - ff_{t_i}\partial_{t_i}\right)\,,$$

for any $i, j = 1, \ldots, h$ and $a, b = 1, \ldots, p$.

In the following we will proceed in the same way as in previous cases. Again, we skip most computations, emphasizing only the major differences appearing in this situation. The function ψ is obtained from Proposition 5.1 (case 1 in the proof):

$$\psi = \frac{1}{2}\ln\left[\langle\bar{v}, z\rangle^2 - \langle \mathrm{j}\bar{v}, z\rangle^2\right]\,,$$

where $v = (a_1, \ldots, a_h, 0, t_2, \ldots, t_h)$, $a_1 > 0$, is a constant vector. Applying Gauss' formula $\widetilde{\nabla}_{\Phi_* U}\Phi_* V = \Phi_*\nabla_U V + \sigma(U, V)$ for $U, V \in \mathcal{D}$, respectively for $U \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$ we may write (6.19). Using Gauss' formula for $U = V = \partial_{u_1}$, we find

$$\frac{\partial\lambda_A}{\partial u_1^2} + \langle v, v\rangle\lambda_A - D_A = 0 \quad : \quad D_A = \sum a_j c_A^j - \sum b_j \tilde{c}_A^j$$

$$\frac{\partial\rho_A}{\partial u_1^2} + \langle v, v\rangle\rho_A - \tilde{D}_A = 0 \quad : \quad D_A = \sum b_j c_A^j - \sum a_j \tilde{c}_A^j\,.$$

Here $\langle v, v\rangle = ||\nabla f||_2 = 0$. Taking $U = \partial_{u_1}$ and $V = \partial_{u_b}$ ($b > 1$) we find that $\frac{\partial^2\lambda_A}{\partial u_1\partial u_b} = 0$ and $\frac{\partial^2\rho_A}{\partial u_1\partial u_b} = 0$. As consequence,

$$\lambda_A(u) = \frac{D_A}{2}\ u_1^2 + D_A^{(1)}u_1 + \Theta_A^{(1)}(u_2, \ldots, u_p)\,,$$

$$\rho_A(u) = \frac{\tilde{D}_A}{2}\ u_1^2 + \tilde{D}_A^{(1)}u_1 + \tilde{\Theta}_A^{(1)}(u_2, \ldots, u_p)\,,$$

where $D_A^{(1)}, \tilde{D}_A^{(1)}$ are constants. Continuing the computations in the same manner it turns that

(6.42) $$\begin{aligned}\lambda_A(u) &= \frac{D_A}{2}\ \sum_{a=1}^{p} u_a^2 + \sum_{a=1}^{p} D_A^{(a)}u_a + D_A^{(0)},\\ \rho_A(u) &= \frac{\tilde{D}_A}{2}\ \sum_{a=1}^{p} u_a^2 + \sum_{a=1}^{p} \tilde{D}_A^{(a)}u_a + \tilde{D}_A^{(0)},\end{aligned}$$

where $D_A^{(0)}, \tilde{D}_A^{(0)}$ and $D_A^{(a)}, \tilde{D}_A^{(a)}$, $a = 1, \ldots, p$ are constants. Choosing suitable initial conditions and taking into account the property of Φ to be isometric immersion, straightforward computations yield

$$x_i = s_i + \frac{1}{2}\big(a_i S - b_i T\big)\sum_1^p u_a^2\,, \quad x_{h+b} = u_b T\,,$$
(6.43)
$$y_i = t_i + \frac{1}{2}\big(a_i T - b_i S\big)\sum_1^p u_a^2\,, \quad y_{h+b} = u_b S\,,$$

where S and T are as in (6.39). Computing now $x_i + \mathrm{j} y_i$ and $x_{h+b} + \mathrm{j} y_{h+b}$ one gets (6.5). In the end, consider $N_\perp^0 = \{(s_0, t_0)\} \times \mathbb{E}^p$, where (s_0, t_0) is a fixed point in $\mathbb{E}_h^{2h}$. If $\sigma_\perp^0$ is the second fundamental form of $N_\perp^0$ in $\mathbb{E}_m^{2m}$, we find $||\sigma_\perp^0(\partial_{u_a}, \partial_{u_a})||_2 = 0$. So, the mean curvature vector of $N_\perp^0$ is a light-like vector, so it is nowhere zero.

If $h = 1$, then $v = (a_1, 0)$. Thus $||v||_2 < 0$. Hence, $N_\perp$ is an open part of the hyperbolic space $\mathbb{H}^p$. So, we obtain item **2**.

Let us now consider the case $p = 1$. In this case $N_\perp$ is a curve, which can be supposed to be parameterized by the arc-length u. Hence its metric is $g_\perp = du^2$. We can make the same computations as in previous case such that (6.19) holds. Yet, a first difference appear: we are not able to say something about the value of $||\nabla f||_2 = -\sum_{i=1}^h a_i^2 + \sum_{i=1}^h b_i^2$.

Using as usual Gauss' formula (for $U = V = \partial_{u_a}$) one gets

$$\frac{\partial^2 \lambda_A}{\partial u^2} = \langle v, v\rangle \lambda_A + D_A\,, \quad \frac{\partial^2 \rho_A}{\partial u^2} = \langle v, v\rangle \rho_A + \tilde{D}_A\,,$$

where $D_A, \tilde{D}_A \in \mathbb{R}$. Since $\langle v, v\rangle = -\sum_{i=1}^h a_i^2 + \sum_{i=1}^h b_i^2$ is an arbitrary constant, we have to distinguish three different cases: **Case (i)** $\langle v, v\rangle = -r^2$, **Case (ii)** $\langle v, v\rangle = r^2$ and **Case (iii)** $\langle v, v\rangle = 0$ ($r > 0$).

Solving the ordinary differential equations and doing the computations in the same manner as in the case when $p > 1$, and after a re-scaling of the vector v, we obtain the first three cases stated in the theorem.

At this point we recall that the PDE system in Proposition 5.1 has also other solutions. When Case 2a from the proof is considered, doing similar computations we easily get item **4** of the theorem.

Much more interesting is to consider Case 2b in the proof of Proposition 5.1. We have to examine again the three situations, namely when $N_\perp$ is $\mathbb{S}^p$, $\mathbb{H}^p$ or $\mathbb{E}^p$. In the following we give only few details for the case $M = \mathbb{E}_h^{2h} \times_f \mathbb{S}^p$, the other two being very similar. Here the warping function is $f = \sqrt{AB}$, where

$$A = \sum_{k=1}^{h} a_k(s_k + \epsilon t_k)\ , \quad B = \sum_{k=1}^{h} b_k(s_k - \epsilon t_k)\ ,$$

$\epsilon = \pm 1$, $a_1 = 0$, $b_1 = 1$, $a_2 \neq 0$. Moreover, by Proposition 6.1 we get $\sum\limits_{k=1}^{h} a_k b_k = -1$.

Direct computations, analogue to those done in the first part of the proof, yield

$$\begin{aligned} x_i &= s_i + \frac{w_0 - 1}{2}\big(b_i A + a_i B\big)\ , \quad x_{h+b} = \frac{u_b}{2}(A - B)\ , \\ y_i &= t_i + \epsilon\frac{w_0 - 1}{2}\big(b_i A - a_i B\big)\ , \quad x_{h+b} = \epsilon\frac{u_b}{2}(A + B)\ , \end{aligned} \tag{6.44}$$

where $(w_0, w_1, \ldots, w_p) \in \mathbb{S}^p$. Put $v_k = \frac{\epsilon}{2}(a_k + b_k) + \frac{1}{2}\mathrm{j}(a_k - b_k)$. We have $\langle v, v\rangle = 1$, where $v = (v_1, \ldots, v_p)$. Computing $x_i + \mathrm{j}y_i$ and $x_{h+b} + \mathrm{j}y_{h+b}$ we obtain (6.3). Moreover, the warping function could be written as $f = \sqrt{\langle \bar{v}, z\rangle^2 - \langle \mathrm{j}\bar{v}, z\rangle^2}$. So, we obtain again item **1** of the theorem.

The converse follows from direct computations. ■

Remark 6.3. In the case 3 of previous proof, if we choose $(s_0, t_0) = (1, 0, \ldots, 0)$, and $v = (1, 0, \ldots, 0, \sqrt{3} + 2\mathrm{j})$, we obtain the "initial" leaf $N_\perp^0$ given by

$$\Phi(1, 0, u) = \big(1 + \frac{1}{2}\sum u_a^2, 0, \ldots, 0, \frac{\sqrt{3}}{2}\sum^{(h)} u_a^2, 0, \ldots, 0, -\sum^{(m+h)} u_a^2, u_1, \ldots, u_p\big)\ ,$$

which represents the submanifold given in [15, Proposition 3.6] up to rigid motions.

Remark 6.4. By applying the same method we may also classify all time-like $\mathcal{P}R$-warped products $N_\top \times_f N_\perp$ in the para-Kähler $(h + p)$-plane $\mathcal{P}^{h+p}$ satisfying $h = \frac{1}{2}\dim N_\top$, $p = \dim N_\perp$ and $S_\sigma = 2p||\nabla \ln f||_2$.

Acknowledgment

The second author is supported by Fulbright Grant no. 498/2010 as a Fulbright Senior Researcher at Michigan State University, U.S.A.

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