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A CHARACTERIZATION OF DISTRIBUTIONS BY RANDOM SUMMATION

Chin-Yuan Hu and Tsung-Lin Cheng

Abstract. In this paper, we consider a problem of characterizing distribution through the constructive property of random sum pS_N, where $0 < p < 1$ and $N \geq 0$ is an integer-valued random variable. This problem will be solved under some regular conditions. We extend the characterization of exponential distribution to a general case. For example, the gamma distribution, the positive Linnik distribution and the scale mixture of stable distribution are characterized. Two new results in the vein are obtained. Finally, the problem of characterizing distribution by the property of the first order statistics is also investigated.

1. INTRODUCTION

The problem of characterizing distribution through a random sum has long been a subject for study; e.g., Feller (1971), Kakosyan et al. (1984), Kotz and Steutel (1988), Pakes (1994, 1995), Rao and Shanbhag (1994) and the references therein. Consider a sequence of independent and identically distributed (i.i.d.) nonnegative random variables $X, X_1, X_2, \ldots$ with common distribution $F(x) = P(X \leq x), x \geq 0$. Assume that $N \geq 0$, independent of $\{X_n\}_{n=1}^{\infty}$, is an integer-valued random variable with probabilities $p_n = P(N = n), n = 0, 1, 2, \ldots$. Set $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. Replacing n by the random variable N, S_N becomes a random sum. In this case, S_N is called compound and the distribution function (d.f.) of S_N is dubbed as a compound distribution.

In particular, if $N_1 \geq 1$ is a geometric random variable with parameter $0 < p < 1$, namely, $P(N_1 = n) = p(1-p)^{n-1}$ for $n \geq 1$, the random sum S_{N_1} is called a geometric compound of the sequence $\{X_n\}_{n=1}^{\infty}$. This geometric compounding model is useful in many fields, such as risk theory, queueing theory, reliability and distribution theory (see, for examples, Feller (1971); Rolski et al. (1999); Hu and Lin (2001) and the references therein). An important characterization result in this vein can be restated as follows (Arnold (1973); Azlarov et al. (1972)). Under the geometric compounding model, the

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distributional equation $X \stackrel{d}{=} pS_{N_1}$ has the exponential distribution solution, where $\stackrel{d}{=}$ stands for equality in distribution. Note that $E(N_1) = 1/p$ and $E(N_1^2) < \infty$. In this paper, we shall consider the characterization of distributions through the distributional equation $X \stackrel{d}{=} pS_N$ for general compound -N with $E(N) = 1/p$ and $E(N^2) < \infty$, where $0 < p < 1$. Besides, we also obtain some related results. Main results of this paper will be stated in Section 2 under an additional condition. Theorem 1 can be viewed as an extension of the characterization of exponential distribution (Arnold (1973) and Azlarov et al. (1972)) and it can be also regarded as a characterization of scale mixture of stable distribution. Theorem 3 is a uniqueness theorem. The applications of the main theorems are given in Section 3. Examples 1 and 3 are new results in this vein. Some related problems are also investigated there.

2. Main Results

In order to establish the main results, we need some notations and definitions in the sequel. Throughout this section, we assume that all random variables are nonnegative, that the distribution functions are right continuous and that the interval of integration is closed (and may be replaced by $[0, \infty)$). We also adopt the same notations as given in Section 1. Let X be a nonnegative random variable with distribution function F and its Laplace-Stieltjes transform be $\hat{F}$; that is,

$$\hat{F}(s) = E(e^{-sX}), \quad s \geq 0.$$

The probability generating function of a nonnegative integer-valued random variable N will be denoted by P_N; i.e.,

$$P_N(t) = E(t^N), \quad 0 \leq t \leq 1.$$

It is well-known that the Laplace-Stieltjes transform $\hat{F}_{S_N}$ of the random sum $S_N = X_1 + \ldots + X_N$, as given in Section 1, can be expressed in terms of the probability generating function P_N of N and the Laplace-Stieltjes transform $\hat{F}_X$ of X; namely,

$$\hat{F}_{S_N}(s) = P_N(\hat{F}_X(s)), \quad s \geq 0. \tag{1}$$

This relationship can be extended to arbitrary power series with positive coefficients (see Feller (1971), p. 437; Steutel and Van Harn (2004), p.10). Next, we are going to solve the following functional equation

$$\hat{F}(s) = P_N(\hat{F}(p^{1/\alpha}s)), \quad s \geq 0, \tag{2}$$

where $N \geq 0$ is a given integer-valued random variable, $0 < p < 1$ and $0 < \alpha \leq 1$ are fixed constants. Under some regularity conditions, the problems of existence and uniqueness are established in Theorem 1 below. For $\alpha = 1$, Theorem 1 can be read as an extension of the characterization of exponential distribution (see Arnold (1973) and Azlarov et al. (1972)); for $0 < \alpha < 1$, Theorem 1 can be viewed as a characterization of scale mixture of stable distribution. Let for each α, $0 < \alpha < 1$, T_α be a nonnegative random variable. T_α is called (strictly) stable with the exponent α, if the Laplace-Stieltjes transform of T_α has the form

$$\hat{F}_{T_\alpha}(s) = e^{-s^\alpha}, \quad s \geq 0$$

(see Feller (1971), p. 448; Steutel and Van Harn (2004), p. 241). We also note that some properties of two important types of mixtures can be found in Steutel and Van Harn (2004, p. 329). For example, the distribution of a random variable X is said to be a scale mixture if

$$X \stackrel{d}{=} ZY,$$

where $Z \geq 0$ and Y are independent random variables. In particular, let Y_1 and T_α be independent random variables with Laplace-Stieltjes transforms $\hat{F}_1$ and e^{-s^α}, respectively. Then $T_\alpha Y_1^{1/\alpha}$ has the Laplace-Stieltjes transform $\hat{F}_1(s^\alpha)$ (Feller (1971), p. 463). Let $Y_\alpha = T_\alpha Y_1^{1/\alpha}$, the distribution of Y_α is called a scale mixture of stable distribution with exponent α. To establish the main results, we need two auxiliary lemmas.

Lemma 1. *Let X be a nonnegative random variable with distribution F and finite non-zero variance. Then the following inequality holds*

$$\hat{F}(s) \leq 1 - \frac{\mu_1^2}{\mu_2} + \frac{\mu_1^2}{\mu_2} e^{-(\mu_2/\mu_1)s}, s \geq 0, \tag{3}$$

where μ_j, $j = 1, 2$, is the j-th moment of X.

Proof. The proof of this lemma can be found in Eckberg (1977), Guljas-Pearce-Pecaric (1998), or Hu and Lin (2008). Note that if $\mu_2 = \infty$ then the inequality is reduced to $\hat{F}(s) \leq 1$, and there is nothing to prove.

The following lemma is a characterization of the degenerate distribution with finite and non-zero mean.

Lemma 2. *Let $0 < \mu < \infty$ and $0 < p < 1$ be two constants. Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with common distribution F and $E(X) = \mu$. Assume that $N \geq 0$, independent of $\{X_n\}_{n=1}^{\infty}$, is an integer-valued random variable. Then, the distributional equation*

$$X \stackrel{d}{=} pS_N, \tag{4}$$

has exactly one degenerate distribution solution at $x = \mu$ if and only if there exists an integer $n_0 \geq 2$ such that

$$P(N = n_0) = 1 \quad \text{and} \quad n_0 p = 1.$$

Proof. Note that the condition $E(X) = \mu$ is equivalent to the following condition

(5) $$\lim_{s \to 0^+} \frac{1 - \hat{F}(s)}{s} = \mu.$$

The proof of this lemma is easy. Therefore, the detailed proof is omitted.

Theorem 1. *Let $0 < \alpha \leq 1$, $\lambda_\alpha > 0$ and $0 < p < 1$ be three constants. Suppose that $X, X_1, X_2, \ldots$, be a sequence of i.i.d. nonnegative random variables with common distribution F. Assume that $N \geq 0$, independent of $\{X_n\}_{n=1}^{\infty}$, is an integer-valued random variable with $E(N) = 1/p$ and $E(N^2) < \infty$, and that the following condition is fulfilled*

(6) $$\lim_{s \to 0^+} \frac{1 - \hat{F}(s)}{s^\alpha} = \lambda_\alpha,$$

where $\hat{F}(s) = E(e^{-sX})$, $s \geq 0$. Then the distributional equation

(7) $$X \stackrel{d}{=} p^{1/\alpha} S_N$$

has exactly one distributional solution F_α. In particular, if $\alpha = 1$ then the unique solution F_1 has mean λ_1 and finite variance. Furthermore the following relationship holds

(8) $$Y_\alpha \stackrel{d}{=} T_\alpha Y_1^{1/\alpha},$$

where Y_α has distribution function F_α and the random variable T_α, independent of Y_1, has a stable distribution with exponent α.

Proof. First, we prove the case when $\alpha = 1$. Note that condition (6) with $\lambda_1 = \mu$ is then reduced to the limiting condition (5), which is in turn tantamount to the condition $E(X) = \mu$. The distributional equation (7) with $\alpha = 1$ is equivalent to the functional equation

(9) $$\hat{F}(s) = P_N(\hat{F}(pS)), \quad s \geq 0.$$

Second, we turn to the proof of the existence of an distributional solution. For $n \geq 1$, define recursively

(10) $$\hat{F}_n(s) = P_N(\hat{F}_{n-1}(ps)), \quad s \geq 0,$$

in which $\hat{F}_0(s)$ is the Laplace-Stieltjes transform of an initial random variable Y_0. Clearly, $\hat{F}_1 = \hat{F}_{S_N}$ is the Laplace-Stieltjes transform of the random sum $S_N = Y_{01} + Y_{02} + \ldots + Y_{0N}$, where $Y_0 \stackrel{d}{=} Y_{01}$ and $Y_{01}, Y_{02}, \ldots$ are i.i.d. random variables. Hence, for $n \geq 1$, $\hat{F}_n$ is a well-defined Laplace-Stieltjes transform. Let Y_n be a random variable with the Laplace-Stieltjes transform $\hat{F}_n$. From assumptions $E(N) = 1/p$ and $E(N^2) < \infty$, it is easy to show that the initial random variable Y_0 has finite second moment, and so does Y_n for $n \geq 1$. In fact, we have the following more general recursive form

$$E(Y_n^2) = pE(Y_{n-1}^2) + p^2 E(N(N-1))[E(Y_0)]^2, \quad n \geq 1 \tag{11}$$

and $E(Y_n) = E(Y_0)$ for every $n \geq 1$. Now, we choose the initial random variable Y_0 having the Laplace-Stieltjes transform

$$\hat{F}_{Y_0}(s) = 1 - \frac{\mu_1^2}{\mu_2} + \frac{\mu_1^2}{\mu_2} e^{-(\mu_2/\mu_1)s}, \quad s \geq 0,$$

where $\mu_1 = \mu$, $\mu_2 = \frac{p^2 E(N(N-1))}{1-p}\mu^2$ and $\hat{F}_0 = \hat{F}_{Y_0}$.

The condition $E(N) = 1/p > 1$ and $E(N^2) < \infty$ implies that this Laplace-Stieltjes transform $\hat{F}_{Y_0}$ is well-defined, $E(Y_0) = \mu_1$ and $E(Y_0^2) = \mu_2$. By using (11) with $n = 1$, we obtain $E(Y_1) = \mu_1$ and $E(Y_1^2) = \mu_2$.

To exploit Lemma 1 with $X = Y_1$, the random variable Y_1 requires a finite non-zero variance. When the conditions $E(N) = 1/p > 1$ and $E(N^2) < \infty$ are fulfilled, $var(Y_1) = \mu_2 - \mu_1^2 < \infty$, which in turn implies that

$$var(Y_1) = 0 \quad \text{iff} \quad E(N^2) = (E(N))^2; \quad \text{namely,}$$

the integer-valued random variable $N \geq 0$ is degenerate. Remember that the degenerate case has been dealt with in Lemma 2. Therefore, we may assume that Y_1 has a finite non-zero variance. Applying Lemma 1, we obtain

$$\hat{F}_1(s) \leq 1 - \frac{\mu_1^2}{\mu_2} + \frac{\mu_1^2}{\mu_2} e^{-(\mu_2/\mu_1)s} = \hat{F}_0(s), \quad s \geq 0.$$

The defining relationship (10) implies that, for $n \geq 2$,

$$\begin{aligned} \hat{F}_n(s) - \hat{F}_{n-1}(s) &= P_N\Big(\hat{F}_{n-1}(ps)\Big) - P_N\Big(\hat{F}_{n-2}(ps)\Big) \\ &= \sum_{k=0}^{\infty} P(N = k)\Big[\hat{F}_{n-1}^k(ps) - \hat{F}_{n-2}^k(ps)\Big], \quad s \geq 0. \end{aligned}$$

Hence, we get

$$\hat{F}_n(s) \leq \hat{F}_{n-1}(s), \quad s \geq 0, n \geq 1.$$

Since $0 < E(Y_n) = \mu < \infty$, the Jensen's inequality gives

$$e^{-\mu s} \leq \hat{F}_n(s), \quad s \geq 0, n \geq 0.$$

Combining these inequalities, we have

$$e^{-\mu s} \leq \hat{F}_n(s) \leq \hat{F}_0(s), \quad s \geq 0, n \geq 0.$$

Thus, $\hat{F}_n$, $n \geq 0$, are monotone and bounded and have a unique limit which we will denote it by $\hat{F}_\infty$. Since $\lim_{s \to 0^+} \hat{F}_\infty(s) = 1$, the continuity theorem (Steutel and Van Harn (2004), p. 479) implies that this $\hat{F}_\infty$ is the Laplace-Stieltjes transform of a random variable Y_∞. Next, we will show that $\hat{F}_\infty$ satisfies the functional equation (9), and that $E(Y_\infty) = \mu$ and $E(Y_\infty^2) < \infty$. Let X be a nonnegative random variable with distribution F and $E(X) = \mu$. Define a distribution F_w by

$$F_w(x) = \frac{1}{\mu} \int_0^x (1 - F(t))dt, \quad x \geq 0. \tag{12}$$

Note that this distribution F_w is called the "equilibrium distribution of F". By Feller (1971,p.435), we have the following two identities

$$\frac{1 - \hat{F}(s)}{s} = \int_0^\infty e^{-sx}(1 - F(x))dx, \quad s > 0,$$

and

$$\frac{\hat{F}(s) - 1 + \mu s}{s^2} = \mu \int_0^\infty e^{-sx}(1 - F_w(x))dx, \quad s > 0.$$

Hence, the above functions are decreasing in $s > 0$. By the above inequalities, we have

$$\frac{1 - e^{-\mu s}}{s} \geq \frac{1 - \hat{F}_n(s)}{s} \geq \frac{1 - \hat{F}_0(s)}{s}, \quad s > 0, n \geq 0$$

and

$$\frac{e^{-\mu s} - 1 + \mu s}{s^2} \leq \frac{\hat{F}_n(s) - 1 + \mu s}{s^2} \leq \frac{\hat{F}_0(s) - 1 + \mu s}{s^2}, \quad s > 0, n \geq 0.$$

By monotonicity, letting firstly $n \to \infty$ and then $s \to 0^+$, we have

$$E(Y_\infty) = \mu \quad \text{and} \quad E(Y_\infty^2) \leq \mu_2 < \infty.$$

Finally, by the dominated convergence theorem, this $\hat{F}_\infty(s)$ satisfies the functional equation (9). Therefore, the existence is guaranteed.

To verify the uniqueness, we may assume that there are two distributions F and G with the same μ and both of their Laplace-Stieltjes transforms $\hat{F}$ and $\hat{G}$ satisfy the functional equation (9). Verifying $F = G$ is equivalent to verifying $\hat{F} = \hat{G}$. By definition,

$$
\begin{aligned}
|\hat{F}(s)-\hat{G}(s)| &\le \sum_{k=0}^{\infty} P(N=k)|\hat{F}^k(ps)-\hat{G}^k(ps)| \\
&\le \sum_{k=0}^{\infty} P(N=k)k|\hat{F}(ps)-\hat{G}(ps)| \\
&= E(N)|\hat{F}(ps)-\hat{G}(ps)| \\
&= \frac{1}{p}|\hat{F}(ps)-\hat{G}(ps)|, \quad s \ge 0.
\end{aligned} \tag{13}
$$

Now define,

$$g(s)=\Big|\frac{\hat{F}(s)-\hat{G}(s)}{s}\Big|, \quad s>0.$$

Since F and G have the same mean μ, we have

$$\lim_{s\to 0^+}\frac{1-\hat{F}}{s}=\lim_{s\to 0^+}\frac{1-\hat{G}}{s}=\mu<\infty;$$

therefore, we may define

$$g(0)=\lim_{s\to 0^+} g(s)=0.$$

Dividing both sides of inequality (13) by s and iterating, we have

$$0\le g(s)\le g(ps)\le\cdots\le g(p^n s), \quad n\ge 1 \text{ and } s>0,$$

where $0<p<1$. Let $n\to\infty$. We obtain $g(s)=0$, for all $s>0$; that is, $\hat{F}=\hat{G}$. Hence $F=G$ and this completes the proof of uniqueness.

Secondly, we shall prove Theorem 1 for $0<\alpha<1$. This part of existence can be obtained by using a special transformation as shown below. The distributional equation (7) is equivalent to the functional equation (2), that is

$$\hat{F}(s)=P_N(\hat{F}(p^{1/\alpha}s)), \quad s\ge 0.$$

Let F_1 be the unique solution of Theorem 1 for $\alpha=1$ (without ambiguity, for convenience we use the same notation F_1 here and after). Note that this distribution has a finite mean $\lambda_1=\mu$, and the following condition holds

$$\lim_{s\to 0^+}\frac{1-\hat{F}_1(s)}{s}=\mu.$$

By the mixture of F_1 and a stable distribution with exponent α (Feller (1971), p.463), we can define a Laplace-Stieltjes transform

$$\hat{F}_\alpha(s)=\hat{F}_1(s^\alpha), \quad s\ge 0. \tag{14}$$

Note that $\hat{F}_\alpha$ is called a scale mixture of stable distribution, as given in this section before Lemma 1. Next, we'll prove that $\hat{F}_\alpha$ is a solution of the functional equation

(2), and that it satisfies condition (6). These assertions follow immediately from the definition of $\hat{F}_\alpha$ and following equations,

$$\hat{F}_\alpha(s) = \hat{F}_1(s^\alpha) = P_N(\hat{F}_1(ps^\alpha)) = P_N(\hat{F}_\alpha(p^{1/\alpha}s)),\ s \geq 0$$

and

$$\lim_{s\to 0^+} \frac{1-\hat{F}_\alpha(s)}{s^\alpha} = \lim_{t\to 0^+} \frac{1-\hat{F}_1(t)}{t}, \text{ by setting } t = s^\alpha.$$

The latter limit exists and is finite. The proof of existence is complete. The proof of existence is done. The proof of uniqueness is essentially similar to the proof of Theorem 1 for $\alpha = 1$ case, and is omitted here. Therefore, we complete the proof of Theorem 1.

Corollary 1. *Let $0 < \mu < \infty$ and $0 < p < 1$ be two constants. Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with common distribution F and $E(X) = \mu$. Assume that $N \geq 0$, independent of $\{X_n\}_{n=1}^\infty$, is an integer-valued random variable with*

$$E(N) = \frac{1}{p} \quad and \quad E(N^2) < \infty. \tag{15}$$

Then the distributional equation

$$X \stackrel{d}{=} pS_N \tag{16}$$

has exactly one distributional solution F_1 with the mean μ and finite variance. Equivalently, the following functional equation

$$\hat{F}(s) = P_N(\hat{F}(ps)), \quad s \geq 0, \tag{17}$$

where $P_N(t) = E(t^N)$, $0 \leq t \leq 1$, has exactly one solution $\hat{F}_1$ with $\hat{F}_1'(0^+) = -\mu$ and $\hat{F}_1''(0^+) < \infty$. Furthermore, if N is not degenerate then the unique solution F_1 has a finite and nonzero variance.

Proof. The corollary follows immediately from Theorem 1. The last assertion of the corollary follows from Lemma 2, which means that the integer-valued random variable $N \geq 0$ is not degenerate, and so is the unique solution F_1.

Theorem 2. *Let $m > 1$ be a given integer, and $0 < \alpha \leq 1$, $0 < \lambda_\alpha < \infty$ be two constants. Let $X, X_1, \ldots, X_m$ be i.i.d. nonnegative random variables with common distribution F. Assume that $0 \leq B \leq 1$, independent of $\{X_n\}_{n=1}^m$, is any nonnegative random variable with*

$$E(B^2) < \frac{1}{m} = E(B) \tag{18}$$

and that the following limiting condition holds

(19) $$\lim_{s\to 0^+} \frac{1-\hat{F}(s)}{s^\alpha} = \lambda_\alpha.$$

Then the distributional equation

(20) $$X \stackrel{d}{=} B^{1/\alpha}(X_1 + \ldots + X_m)$$

has exactly one distributional solution F_α. In particular, if $\alpha = 1$ then the unique solution F_1 has the mean λ_1 and finite variance. Furthermore, the following relation holds

$$Y_\alpha \stackrel{d}{=} T_\alpha Y_1^{1/\alpha},$$

where Y_α has the distribution F_α, and the random variable T_α, independent of Y_1, has a stable distribution with exponent α.

Proof. A sketch of proof of Theorem 2 is given here, since it is essentially similar to that of Theorem1. It suffices to verify the case as $\alpha = 1$. In this case, condition (19) with $\alpha = 1$ is equivalent to the functional equation

(21) $$\hat{F}(s) = \int_0^1 [\hat{F}(us)]^m dF_B(u), \quad s \geq 0,$$

where F_B is the distribution function of B.

To prove the existence, let us define

(22) $$\hat{F}_n(s) = \int_0^1 [\hat{F}_{n-1}(us)]^m dF_B(u), \quad s \geq 0, n \geq 1,$$

and $\hat{F}_0$ is the Laplace-Stieltjes transform of an initial random variable Y_0. It is obvious that $\hat{F}_n$ is well-defined; that is, $\hat{F}_n$ is a Laplace-Stieltjes for $n \geq 1$, because the power $m > 1$ in the functional equation (22) is an integer (noting also that the function $\hat{F}_n$ may not be defined if $m > 1$ is a real number but not an integer, this is why we need the uniqueness in Theorem 3 below).

Let Y_n be a random variable with the Laplace-Stieltjes transform $\hat{F}_n$, $n \geq 1$. Under the condition (18), we have (here assume that $E(Y_0^2) < \infty$)

$$E(Y_n) = E(Y_0)$$

and

$$E(Y_n^2) = E(B^2)\Big[m(m-1)[E(Y_0)]^2 + mE(Y_{n-1}^2)\Big], \quad n \geq 1.$$

Using the same argument as that of Theorem 1, choose

$$\hat{F}_0(s) = 1 - \frac{\mu_1^2}{\mu_2} + \frac{\mu_1^2}{\mu_2} e^{-(\mu_2/\mu_1)s}, \quad s \geq 0$$

with $\mu_1 = \mu$ and

$$\mu_2 = \frac{m(m-1)E(B^2)}{1-mE(B^2)}\mu^2.$$

The condition $E(B^2) < 1/m = E(B)$ implies that the Laplace-Stieltjes transform $\hat{F}_0$ is well-defined. Note also that if the random variable B is degenerate, then $P(B = 1/m) = 1$ and there is nothing to prove. Thus, the same argument as that of Theorem ensures the existence. The proof of uniqueness is the same as that of Theorem 3 below, hence is omitted here. The proof is completed.

Theorem 3. (Uniqueness). *Let $r > 1$, $0 < \alpha \le 1$ and $\lambda_\alpha > 0$ be three constants, and $X \ge 0$ be a nonnegative random variable with distribution F. Assume that $0 \le B \le 1$ is a given nonnegative variable with distribution F_B and that*

$$0 < E(B) \le \frac{1}{r}. \tag{23}$$

Moreover, we assume that the following limit condition holds

$$\lim_{s\to 0^+} \frac{1-\hat{F}(s)}{s^\alpha} = \lambda_\alpha. \tag{24}$$

Then, the functional equation

$$\hat{F}(s) = \int_0^1 \left[\hat{F}(u^{1/\alpha}s)\right]^r dF_B(u), \quad s \ge 0 \tag{25}$$

has a solution only if the solution of this functional equation is unique. In this case, we denote the solution by $\hat{F}_\alpha$ and we have the following relation

$$Y_\alpha \overset{d}{=} T_\alpha Y_1^{1/\alpha},$$

where Y_α has the distribution F_α and the random variable T_α which is independent of Y_1 has a stable distribution with exponent α.

Proof. In order to prove the uniqueness, let us assume that there are two distributions F and G that satisfy (24), and that their Laplace-Stieltjes transforms $\hat{F}$ and $\hat{G}$ satisfy the functional equation (25). In the following, we want to prove $F = G$ or equivalently $\hat{F} = \hat{G}$. Note that for $r \ge 1$,

$$\begin{aligned} \left|\hat{F}(s) - \hat{G}(s)\right| &\le \int_0^1 \left|\hat{F}^r(u^{1/\alpha}s) - \hat{G}^r(u^{1/\alpha}s)\right| dF_B(u) \\ &\le r\int_0^1 \left|\hat{F}(u^{1/\alpha}s) - \hat{G}(u^{1/\alpha}s)\right| dF_B(u) \\ &\le \frac{1}{E(B)}\int_0^1 \left|\hat{F}(u^{1/\alpha}s) - \hat{G}(u^{1/\alpha}s)\right| dF_B(u), \quad s \ge 0. \end{aligned} \tag{26}$$

Since $\hat{F}$ and $\hat{G}$ satisfy (24), we can define a bounded continuous function g as below:

$$g(s) = \left| \frac{\hat{F}(s) - \hat{G}(s)}{s^{\alpha}} \right|, \quad s > 0 \text{ and } g(0) = \lim_{s \to 0^+} g(s).$$

As a result, the following function h is also well-defined

$$h(s) = \sup_{0 \le t \le s} g(t), \quad s \ge 0 \text{ with } h(0) = 0.$$

It is clear that $h(s)$ is increasing in $s \ge 0$. Dividing both sides of the inequality (26) by $s > 0$, we obtain

$$g(s) \le \int_0^1 g(u^{1/\alpha} s) dH(u), \quad s \ge 0,$$

where the distribution function H is defined by

$$H(x) = \frac{1}{E(B)} \int_0^x t dF_B(t), \quad 0 \le x \le 1.$$

Note that

$$\begin{aligned} \sup_{0 \le t \le s} g(t) &\le \sup_{0 \le t \le s} \int_0^1 g(u^{1/\alpha} t) dH(u) \\ &\le \int_0^1 \sup_{0 \le t \le s} g(u^{1/\alpha} t) dH(u) \qquad (27) \\ &\le h(s). \qquad (28) \end{aligned}$$

(27) is due to Jensen's inequality and (28) is due to the monotonicity of $h(s)$, $s \ge 0$. As a result, h must be the zero function and so is g. This leads to $\hat{F} = \hat{G}$, or equivalently, $F = G$. This completes the proof.

3. Applications and Examples

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with common distribution H. Let $X_{1,n} \le \ldots \le X_{n,n}$ be the corresponding order statistics of $\{X_k\}_{k=1}^n$ defined above. The following important property is well studied; that is, under the condition

$$\lim_{x \to 0^+} \frac{1 - \bar{H}(x)}{x} = \lambda, \quad \text{for some } \lambda > 0, \qquad (29)$$

where $\bar{H} = 1 - H$, the distributional equation $X \overset{d}{=} nX_{1,n}$ (for some $n \ge 2$) characterizes H to be exponential (Gupta (1973), note that without the limit condition, the conclusion fails). Various restatements and extensions of this result are also known (see Galambos and Kotz (1978); Shimizu(1978,1979); Hu and Lin (2003)).

In the following, we consider a problem of characterizing distributions by the property of the first order statistics $X_{1,N}$, where $N \geq 1$ is an integer-valued random variable independent of $\{X_n\}_{n\geq 1}$. This problem is in a sense dual to the main result in section 2. The extension of Theorem 1 to the real line can be found in Theorem 7 below. The symmetric stable random variable on $\Re$ will be denoted by T_α, that is, the characteristic function ϕ_{T_α} of T_α is defined by

$$\phi_{T_\alpha}(t) = e^{-|t|^\alpha}, \quad t \in \Re,$$

where $0 < \alpha \leq 2$.

Theorem 4. *Let $\alpha > 0$, $\lambda_\alpha > 0$ and $0 < p < 1$ be three constants. Assume that $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. nonnegative random variables with common distribution H, and that $N \geq 1$ is an integer-valued random variable independent of $\{X_n\}_{n\geq 1}$ with $E(N) = 1/p$ and $E(N^2) < \infty$. Suppose that the following condition holds*

$$\lim_{x\to 0^+} \frac{1-\bar{H}(x)}{x^\alpha} = \lambda_\alpha, \tag{30}$$

where $\bar{H} = 1 - H$. Then the following equation has exactly one solution (say H_α)

$$X \stackrel{d}{=} p^{-1/\alpha} X_{1,N}. \tag{31}$$

Proof. Under the conditions in the statement of Theorem 4, (31) is equivalent to the following functional equation

$$\bar{H}(x) = P_N(\bar{H}(p^{1/\alpha}x)), \quad x \geq 0.$$

The desired result follows immediately from Corollary 1 with $\mu = \lambda_1$. Precisely, the unique solution of the distributional equation (31) is given by

$$H_\alpha(x) = 1 - \hat{F}_1(x^\alpha), \quad x \geq 0,$$

where $\hat{F}_1$ is the unique solution of the functional equation (17). Since the Laplace-Stieltjes transform $\hat{F}_1$ is a non-increasing function with

$$\lim_{s\to\infty} \hat{F}_1(s) = F_1(0),$$

that is, a possible atom of F_1 at the origin has the effect that $\hat{F}_1(\infty) > 0$. Therefore, we have to show that $1 - \hat{F}_1(x^\alpha)$ is actually a non-defective probability distribution function. The condition $N \geq 1$ together with the fact that $\hat{F}_1$ satisfying the functional equation (17) (by letting $s \to \infty$) imply $\hat{F}_1(\infty) = 0$. Hence H_α is a proper probability distribution function.

Theorem 5. *Let $n > 1$ be a given integer, and $\alpha > 0, \lambda_\alpha > 0$ be two positive constants. Let $X_1, \ldots, X_n$ be i.i.d. nonnegative random variables with common distribution H, and $X_{1,n} = Min\{X_1, \ldots, X_n\}$ be the first order statistics. Assume that $0 \le B \le 1$, independent of $\{X_1, \ldots, X_n\}$, is a given nonnegative random variable with*

$$E(B^2) < \frac{1}{n} = E(B), \tag{32}$$

and that the limit condition holds

$$\lim_{x \to 0^+} \frac{1 - \bar{H}(x)}{x^\alpha} = \lambda_\alpha, \tag{33}$$

where $\bar{H} = 1 - H$. Then the distributional equation

$$X \stackrel{d}{=} B^{-1/\alpha} X_{1,n} \tag{34}$$

has exactly one distributional solution H_α.

Proof. Under the given conditions, the distributional equation (34) is equivalent to the following functional equation

$$\bar{H}(x) = \int_0^1 [\bar{H}(u^{1/\alpha} x)]^n dF_B(u), x \ge 0.$$

The desired result follows from Theorem 2 with $m = n$, $\alpha = 1$; and $H_\alpha(x) = 1 - \hat{F}_1(x^\alpha), \quad x \ge 0$. Note that $1 - \hat{F}_1$ is a proper probability distribution function. This completes the proof.

Theorem 6. (Uniqueness). *Let $r > 1$, $\alpha > 0$ and $\lambda_\alpha > 0$ be three constants. Let $X \ge 0$ be a nonnegative random variable with distribution H. Assume that $0 \le B \le 1$ is a nonnegative random variable with distribution H_B and*

$$0 < E(B) \le \frac{1}{r},$$

and that the limit condition holds

$$\lim_{x \to 0^+} \frac{1 - \bar{H}(x)}{x^\alpha} = \lambda_\alpha,$$

where $\bar{H} = 1 - H$. In addition, if the functional equation

$$\bar{H}(x) = \int_0^1 [\bar{H}(u^{1/\alpha} x)]^r dH_B(u), \quad x \ge 0,$$

has a solution, then the solution of this functional equation is unique, by which we denote it $\bar{H}_\alpha$.

Proof. The proof is similar to Theorem 3 and we omit it.

Theorem 7. *Let* $0 < \alpha \leq 2$, $\lambda_\alpha > 0$ *and* $0 < p < 1$ *be three given constants. Let* $X, X_1, X_2, \ldots$ *be a sequence of i.i.d. symmetric random variables on the real line with common distribution* G. *Assume that* $N \geq 0$, *independent of* $\{X_n\}_{n=1}^{\infty}$, *is an integer-valued random variable with*

$$E(N) = \frac{1}{p} \quad \textit{and} \quad E(N^2) < \infty, \tag{35}$$

and that the limit condition holds

$$\lim_{t\to 0} \frac{1-\phi(t)}{|t|^\alpha} = \lambda_\alpha, \tag{36}$$

where $\phi(t), t \in \Re$, *is the characteristic function of* X. *Then the distributional equation*

$$X \stackrel{d}{=} p^{1/\alpha} S_N \tag{37}$$

has exactly one solution G_α. *Furthermore, the relationship holds*

$$Z_\alpha \stackrel{d}{=} T_\alpha Z_1^{1/\alpha} \tag{38}$$

where Z_α *has distribution function* G_α *and the random variable* T_α, *independent of* Z_1, *has a symmetric stable distribution on* $\Re$ *with exponent* α.

Proof. First note that X is symmetric if and only if its characteristic function is real-valued, and that the distributional equation (37) is equivalent to the following functional equation

$$\phi(t) = P_N(\phi(p^{1/\alpha} t)), \quad t \in \Re.$$

The unique solution of this functional equation is given by

$$\phi_\alpha(t) = \hat{F}_1(|t|^\alpha), \quad t \in \Re,$$

where $\hat{F}_1$ is the unique solution of the functional equation (17) in Corollary 1 with $\mu = \lambda_1$. the detailed proof is omitted here.

Example 1. Let $0 < \theta < 1$ and $0 < \mu < \infty$ be two given constants. Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with common distribution F and $E(X) = \mu$. Assume that $N \geq 0$, independent of $\{X_n\}_{n=1}^{\infty}$, is an integer-valued random variable with probabilities

$$P(N = n) = \Big(\frac{\theta}{1+\theta}\Big)\Big(\frac{1}{1+\theta}\Big)^n, \quad n = 0, 1, 2, \ldots.$$

Note that $E(N) = \frac{1}{\theta}$ and $E(N^2) < \infty$. Now applying Corollary 1 with $p = \theta$, the distributional equation $X \stackrel{d}{=} \theta S_N$ has exactly one distributional solution F_1 and the Laplace-Stieltjes transform of F_1 is given by

$$\hat{F}_1(s) = \theta + (1-\theta)\frac{1}{1+\mu s}, \quad s \geq 0,$$

that is, the unique solution F_1 is a mixture of the exponential distribution. This result can be obtained by the following fact. The Laplace-Stieltjes transform $\hat{F}_1$ satisfies the functional equation

(39)
$$\hat{F}(s) = \frac{1}{1 + \frac{1}{\theta}(1 - \hat{F}(\theta s))}, \quad s \geq 0,$$

which is equivalent to the distributional equation $X \stackrel{d}{=} \theta S_N$. Similarly, under the limit condition of Theorem 1 and $0 < \alpha \leq 1$. Theorem 1 implies that the distributional equation $X \stackrel{d}{=} \theta^{1/\alpha} S_N$ has exactly one distributional solution F_α and the relationship holds

$$Y_\alpha \stackrel{d}{=} T_\alpha Y_1^{1/\alpha}$$

where Y_α has the distributional F_α and the random variable T_α, independent of Y_1, has a stable distribution with exponent α (Feller (1971), p.448 and p. 463). That is, the Laplace-Stieltjes transform of F_α is given by

$$\hat{F}_1(s) = \theta + (1-\theta)\frac{1}{1+\mu s^\alpha}, \quad s \geq 0.$$

This means that the unique solution F_α is a mixture of the Linnik distribution (Lukacs (1970), p.97). Note that all solutions of this example belong to the compound exponential type distribution and hence infinitely divisible (Steutel and Van Harn (2004), p. 99).

The unique solution F_1 above has the following remarkable property. Let X be a nonnegative random variable with distribution F and $E(X) = \mu$, $0 < \mu < \infty$. Then F_1 is the unique solution of the following distributional equation

(40)
$$W \stackrel{d}{=} X + \theta W,$$

where $0 < \theta < 1$, X and W are independent, and the random variable W has the so-called equilibrium distribution. That is, the distribution of W is defined by

$$F_w(x) = \frac{1}{\mu}\int_0^x (1 - F_X(t))dt, \quad x \geq 0.$$

Note that this distributional equation (40) is equivalent to the functional equation (39) as mentioned before. Some related results can be found in Steutel and Van Harn (2004, p.445).

Example 2. Set $\lambda = 1/p > 1$ and $0 < \mu < \infty$ to be constants. Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with common distribution F and $E(X) = \mu$. Assume that $N_\lambda \geq 0$, independent of $\{X_n\}_{n=1}^{\infty}$, is the Poisson random variable with the parameter λ, namely,

$$P(N_\lambda = n) = \frac{e^{-\lambda}\lambda^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Note that $E(N_\lambda) = \lambda$ and $E(N_\lambda^2) = \lambda(\lambda + 1) < \infty$. Applying Corollary 1 with $p = 1/\lambda$, the distributional equation

$$X \stackrel{d}{=} pS_{N_\lambda}$$

has exactly one distributional solution, say F_λ. Equivalently, the Laplace-Stieltjes transform $\hat{F}_\lambda$ of F_λ is the unique solution of the following functional equation

$$\hat{F}(s) = e^{-\frac{(1-\hat{F}(ps))}{p}}, \quad s \geq 0. \tag{41}$$

It is not easy to express the solution of this functional equation in a closed form. Note that in the degenerate case $X = 0$ (or $\hat{F}(s) = 1, s \geq 0$) is not a solution because $E(X) = \mu > 0$. The unique solution F_λ above possesses the following properties:

(a) F_λ is the unique solution of the distributional equation

$$Z \stackrel{d}{=} X + pZ, \tag{42}$$

where $p = 1/\lambda$, X and Z are independnet, and the random variable Z has the so-called length-biased distribution, that is, the distribution F_Z of Z is defined by

$$F_Z(x) = \frac{1}{\mu}\int_0^x t dF_X(t), \quad x \geq 0.$$

The distributional equation above is well-known and often mentioned in the literature on perpetuities. Some related results can be found in Iksanov and Kim (2004a, 2004b). Note that this distributional equation (42) is equivalent to the functional equation (41).

(b) F_λ has finite moments of all orders. Let μ_n be the n-th moment of F_λ, then the following moment recurrence holds,

$$\begin{cases} \mu_{n+1} = \frac{1}{1-p^n}\sum_{k=0}^{n-1}\binom{n}{k}\mu_{k+1}\mu_{n-k}p^k, \quad n \geq 1 \\ \qquad\qquad\qquad\qquad \mu_0 = 1 \text{ and } \mu_1 = \mu, \end{cases}$$

where $p = 1/\lambda$, $0 < p < 1$ and $E(X) = \mu$. Furthermore, the n-th moment has the form

$$\mu_n = g_n(p)\mu^n, \quad n \geq 1,$$

where $g_n(p)$, independent of μ, is a rational function of p. For example, $\mu_1 = \mu$, $\mu_2 = \frac{1}{1-p}\mu^2$, $\mu_3 = \frac{1+2p}{(1+p)(1-p)^2}\mu^3$, ..., etc.

This means that the unique distribution F_λ is completely determined by its first moment $\mu_1 = \mu$ (because $\mu_0 = 1$ is always true) and the moment recurrence before. Note that the cumulant K_n of order n of F_λ can be obtained by

$$K_n = \mu_n p^{n-1}, \quad n \geq 1.$$

(c) Clearly, F_λ is an infinitely divisible distribution and its Laplace-Stieltjes transform $\hat{F}_\lambda$ satisfies the following infinite product

$$\hat{F}_{Z_\lambda}(s) = \prod_{n=0}^{\infty} \hat{F}_\lambda(p^n s), \quad s \geq 0,$$

where $0 < p = 1/\lambda < 1$ and F_{Z_λ} is the length-biased distribution induced by F_λ, that is

$$F_{Z_\lambda}(x) = \frac{1}{\mu_1}\int_0^x t dF_\lambda(t), \quad x \geq 0,$$

where μ_1 is the mean of F_λ.

Example 3. Let $r = 1 + (\beta_2/\beta_1)$, where $\beta_1 > 0$ and $\beta_2 > 0$ are two constants. Let $X \geq 0$ be a nonnegative random variable with distribution F. Assume that $B \stackrel{d}{=} B(\beta_1, \beta_2)$ has a beta distribution F_B with parameters β_1 and β_2, and the following condition holds

$$\lim_{s \to 0^+} \frac{1 - \hat{F}(s)}{s^\alpha} = \lambda_\alpha,$$

where $0 < \alpha \leq 1$ and $\lambda_\alpha > 0$ are two constants. Then, the functional equation

$$\hat{F}(s) = \int_0^1 \left[\hat{F}(u^{1/\alpha}s)\right]^r dF_B(u), \quad s \geq 0,$$

has exactly one solution, say $\hat{F}_\alpha$. Precisely, this unique solution is given by

$$\hat{F}_\alpha(s) = \left(\frac{1}{1 + cs^\alpha}\right)^{\beta_1}, \quad s \geq 0,$$

where the constant $c = \lambda_1/\beta_1$ is uniquely determined by the limit condition above with $\alpha = 1$. For $0 < \alpha < 1$, this means that the unique solution F_α is a scale mixture of

F_1 and the stable distribution with exponent α. Note that this result follows from the uniqueness Theorem 3 in Section 2 and the well-known identity below,

$$\Big(\frac{1}{1+s}\Big)^{\beta_1} = \frac{\Gamma(\beta_1+\beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^1 \Big(\frac{1}{1+us}\Big)^{\beta_1+\beta_2} u^{\beta_1-1}(1-u)^{\beta_2-1} du, \quad s \geq 0,$$

(See Stuart (1962), Bondesson (1992), p. 14). Some related results in this vein can be found in Kotz and Steutel (1988), Milne and Yeo (1989), Pakes (1992, 1994, 1995) and Rao and Shanbhag (1994, p. 150).

Example 4. The following problem seems interesting enough. Let N have the uniform distribution with probabilities

$$P(N=k) = \frac{1}{n}, \quad k = 1, 2, \ldots, n,$$

where $n \geq 2$ is a given integer. Then Corollary 1 implies the existence of a nontrivial solution of the following functional equation

$$f\Big(\frac{(n+1)s}{2}\Big) = \frac{1}{n}\sum_{k=1}^{n} f^k(s), \quad s \geq 0.$$

What is the closed form of the nontrivial solution? The problem is still in question.

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REFERENCES

1. B. C. Arnold, Some Characterizations of the exponential distribution by geometric compounding, *SIAM J. Appl. Math.*, **24** (1973), 242-244.

2. T. A. Azlarov, A. A. Dzamirzaev and M. M. Sultanova, Characterizing properties of the exponential distribution and their stability, *Random Processes and Statistical Inference* (*Russian*), N. 11, **94** (1972), 10-19; *Izdat. Fan Uzbek, SSR, Tashkent.*, [MR **48** (1974), 3150].

3. L. Bondesson, *Generalized gamma convolutions and related classes of distributions and densities*, Lecture Notes in Statistics, Vol. 76, New York, Springer-Verlag, 1992.

4. A. E. Jr. Eckberg, Sharp bounds on Laplace-Stieltjes transforms, with applications to various queueing problems, *Math. Oper. Res.*, **2** (1977), 135-142.

5. W. Feller, *An introduction to probability theory and its applications*, V2., 2nd ed., Wiley, New York, 1971.

6. J. Galambos and S. Kotz, *Characterizations of probability distributions*, A unified approach with an emphasis on exponential and related models, Lecture Notes in Math., 675, Springer, New York, 1978.

7. B. Gukhas, C. E. M. Pearce and J. Pecarić, Jensen's inequality for distributions possessing higher moments, with application to sharp bounds for Laplace-Stieltjes transforms, *J. Austral. Math. Sco., Ser. B*, **40** (1998), 80-85.

8. R. C. Gupta, A characteristic property of the exponential distribution, *Sankhya, Ser. B.*, **35** (1973), 365-366.

9. C.-Y. Hu and G. D. Lin, On the geometric compounding model with applications, *Proba. Math. Statis.*, **21** (2001), 135-147.

10. C.-Y. Hu and G. D. Lin, Characterizations of the exponential distribution by stochastic ordering properites of the geometric compound, *Ann. Inst. Statis. Math.*, **55(3)** (2003), 499-506.

11. C.-Y. Hu and G. D. Lin, Some inequalities for Laplace transforms, *J. Math. Anal. Appl.*, **340** (2008), 675-686.

12. A. M. Iksanov and C. S. Kim, On a Pitman-Yor problem. Statis. & Probab, *Letters*, **68** (2004a), 61-72.

13. A. M. Iksanov and C. S. Kim, New explicit examples of fixed points of Poisson shot noise transforms, *Aust. N. Z. J. Statis.*, **46(2)** (2004b), 313-321.

14. A. V. Kakosyan, L. B. Klebanov and J. A. Melamed, *Characterization of distributions by the method of intensity monotone operators*, Lecture Notes in Math., 1088, Springer, New York, 1984.

15. S. Kotz and F. W. Steutel, Note on a characterization of the exponential distributions. Statis. & Probab, *Letters*, **6** (1988), 201-203.

16. E. Lukacs, *Characteristic functions*, 2nd ed., Griffin, London, 1970.

17. R. K. Milne and G. F. Yeo, Random sum characterization, *Math. Sci.*, **14** (1989), 120-126.

18. A. G. Pakes, A characterization of gamma mixtures of stable laws motivated by limit theorem, *Statistica Neerlandica*, **46** (1992), 209-218.

19. A. G. Pakes, Necessary conditions for characterization of laws via mixed sums, *Ann. Inst. Stat. Math.*, **63** (1994), 285-310.

20. A. G. Pakes, Characterization of discrete laws via mixed sums and Markov branching processes, *Stoch. Proc. Appl.*, **55** (1995), 285-300.

21. C. R. Rao and D. N. Shanbhag, *Choquet-Deny type functional equations with applications to stochastic models*, Wiley, New York, 1994.

22. T. Rolski, H. Schmidli, V. Schmidt and J. Teugels, *Stochastic processes for insurance and finance*, John Wiley, Chichester, 1999.

23. R. Shimizu, Solution to a functional equation and its applications to some characterization problems, *Sankhya., Ser. A.*, **40** (1978), 319-332.

24. R. Shimizu, On a lack of memory of the exponential distribution, *Ann. Inst. Stati Math.*, **31** (1979), 309-313.

25. F. W. Steutel and K. Van Harn, *Infinite divisibility of probability distributions on the real line*, Marcel Dekker, Inc., New York, 2004.

26. A. Stuart, Gamma distributed products of random variables, *Biometrika*, **49** (1962), 564-565.

Chin-Yuan Hu
Department of Business Education
National Changhua University of Education
Changhua 50058, Taiwan

Tsung-Lin Cheng
Graduate Institute of Statistics and Information
National Changhua University of Education
Changhua 50058, Taiwan
E-mail: tlcheng@cc.ncue.edu.tw.