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THE EXISTENCE RESULTS FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY QUASI-VARIATIONAL INEQUALITIES IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, some existence results for optimal control problems governed by abstract quasi-variational inequalities are proved in reflexive Banach spaces. As an application, an existence of the optimal control for the bilateral obstacle optimal control problem is also given under some suitable conditions, in which the state satisfies a quasilinear elliptic variational inequality with a source term.

1. INTRODUCTION

The obstacle problems and the optimal control of obstacle have attracted much attention in recent years (see, for example, [1-18]). Necessary and sufficient conditions for optimal control problems governed by variational inequalities have been investigated. Different methods have been used to consider this problem.

The optimal control problem for a abstract variational inequality proposed by Zhou et al. [10] is the following minimization problem:

$$\begin{aligned} &\text{min } J_1(w,u) \\ &\text{subject to } (w,u) \in U_{ad} \times K_1 \text{ and } u \in S_1(w), \end{aligned} \tag{1.1}$$

where, for each $w \in U_{ad}$, $S_1(w)$ is the solution set of the following variational inequality problem:

$$\langle A_1(u), v-u\rangle \geq \langle F_1(u) - B_1(w), v-u\rangle,\ \forall\ v \in K_1, \tag{1.2}$$

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and K_1 is a closed and convex cone of a reflexive Banach space V with a dual space V^*, U_{ad} is a nonempty closed set of a reflexive Banach space $\bar{U}$, $J : U_{ad} \times K_1 \to \mathbf{R}$ is a real-valued function and $A_1,\ F_1 : K_1 \to V^*,\ B_1 : U \to V^*$ are three given mappings. If there exist $(w_0, u_0) \in U_{ad} \times K_1$ and $u_0 \in S(w_0)$ such that

$$J_1(u_0, w_0) = \min_{(w,u)\in U_{ad}\times K_1,\ u\in S_1(w)} J_1(u, w),$$

then w_0 is called an optimal control for the minimization problem (1.1). The problem (1.1) extends the corresponding problem in [5, 6] in many aspects, for details, see [10]. The first work on the optimal obstacle control problem was that of Adams et al. [11] in 1998. Recently, Adams and Lenhart continued the work begun in [11] and a nonzero source term was added to the right hand side of the state equation. They soon found that even such a minor change is not a trivial alteration (see [12]). In addition, Lou [15] considered the existence and regularity of the control problem governed by quasilinear elliptic variational inequality. Ye and Chen [19] considered an obstacle control problem where the state satisfies a quasilinear elliptic variational inequality and given existence and necessary conditions for the obstacle control problem. Ye et al. [20] studied the existence of an optimal control problem for a quasi-linear elliptic obstacle variational inequality in which the obstacle is taken as the control. Ye et al. [21] obtained the existence and incomplete necessary condition of an obstacle control problem where the state satisfies a quasilinear elliptic variational inequality with a source term and the control functions are the upper and the lower obstacles. Chen and Ye [23] considered existence and necessary conditions for bilateral obstacle optimal control defined by a quasilinear elliptic variational inequalities.

Motivated and inspired by the work mentioned above, we establish the existence results for optimal control problems governed by abstract quasi-variational inequalities in reflexive Banach spaces. As an application, we consider a bilateral obstacle optimal control problem where the state satisfies a quasilinear elliptic variational inequality with a source term and give an existence of the optimal control for the bilateral obstacle optimal control problem under some suitable conditions. The results presented in this paper extend and improve some corresponding results in [10, 14, 21, 23].

2. Existence Results for Optimal Control Problems Governed by Abstract Quasi-variational Inequalities

Throughout this paper, unless otherwise stated, we assume that $\mathbf{R} = (-\infty, +\infty)$, W, X are two reflexive Banach spaces, X^* is the dual space of X, U is a nonempty closed convex set of W and $K : U \to 2^X$ is a set-valued mapping. We use $\to$ for convergence in strong sense and $\rightharpoonup$ for convergence in weak sense. Let $J : U \times K(U) \to \mathbf{R}$ be a real-valued function and $A : K(U) \to 2^{X^*}$, $B : U \to X^*$ be two given mappings. Consider the following optimal control problem governed by a generalized quasi-variational inequality:

$$\begin{aligned}&\min J(u,w)\\&\text{subject to } (w,u)\in U\times K(w) \text{ and } u\in S(w),\end{aligned} \tag{2.1}$$

where for each $w \in U$, $S(w) = \{u : (u, u^*) \in \hat{S}(w)\}$ and $\hat{S}(w)$ is the solution set of the following abstract generalized quasi-variational inequality problem: find $u \in K(w)$ and $u^* \in A(u)$ such that

$$\langle u^* + B(w), v - u\rangle \geq 0,\ \forall\ v \in K(w). \tag{2.2}$$

Some special cases are as follows:

(I) If $A_2, F_2 : K(U) \to X^*$ are single-valued mappings and $A = A_2 - F_2$, then the problem (2.1) becomes the following problem:

$$\begin{aligned}&\min J(u,w)\\&\text{subject to } (w,u)\in U\times K(w) \text{ and } u\in S_2(w),\end{aligned} \tag{2.3}$$

where for each $w \in U$, $S_2(w)$ is the solution set of the following abstract variational inequality problem: find $u \in K(w)$ such that

$$\langle A_2(u) - F_2(u) + B(w), v - u\rangle \geq 0,\ \forall\ v \in K(w). \tag{2.4}$$

(II) If K_2 is a nonempty closed and convex cone of X and for each $w \in U$, $K(w) = K_2$, then the problem (2.3) becomes the problem (1.1). Thus the problem (2.3) also contains the corresponding problem in [5, 6] as special cases, for details, see [10].

Definition 2.1. Let D be a nonempty subset of X. A mapping $T : D \to X^*$ is a single-valued mapping.

(1) T is said to be of class $(S)_+$ if for any sequence $\{y_j\} \subset D$, $y_j \rightharpoonup y_0 \in D$ satisfying
$\limsup_{j\to\infty}\langle T(y_j), y_j - y_0\rangle \leq 0$ implies that $y_j \to y_0$.

(2) T is said to be generalized pseudo-monotone if for each sequence $\{y_j\} \subset D$, $y_j \rightharpoonup y_0 \in D$, $T(y_j) \rightharpoonup w_0$ and $\limsup_{j\to\infty}\langle T(y_j), y_j - y_0\rangle \leq 0$, then we have $w_0 = T(y_0)$ and $\langle T(y_j), y_j\rangle \to \langle w_0, y_0\rangle$.

(3) T is said to be demicontinuous if for any sequence $\{y_j\} \subset D$, $y_j \to y_0 \in D$, we have $T(y_j) \rightharpoonup T(y_0)$.

(4) T is said to be continuous if for any sequence $\{y_j\} \subset D$, $y_j \to y_0 \in D$, we have $T(y_j) \to T(y_0)$.

(5) T is said to be pseudo-monotone if for each sequence $\{y_j\} \subset D$, $y_j \rightharpoonup y_0 \in D$ and
$\limsup_{j\to\infty}\langle T(y_j), y_j - y_0\rangle \le 0$ imply

$$\langle T(y_0), y_0 - w\rangle \le \liminf_{j\to\infty}\langle T(y_j), y_j - w\rangle$$

for all $w \in D$.

(6) T is said to be monotone if for any $\langle T(y) - T(x), y - x\rangle \ge 0,\ \forall\ x,\ y \in D.$

(7) T is said to be uniformly monotone if for any $\langle T(y) - T(x), y - x\rangle \ge a(\|x - y\|)\|x-y\|,\ \forall\ x,\ y \in D$, where the continuous function $a : [0, +\infty) \to [0, +\infty)$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \to +\infty$, as $t \to +\infty$.

(8) T is said to be strongly monotone if there is a constant $c > 0$ such that

$$\langle T(y) - T(x), y - x\rangle \ge c\|x - y\|^2,\ \forall\ x,\ y \in D.$$

(9) T is said to be coercive if

$$\lim_{\|u\|\to+\infty}\frac{\langle T(u), u\rangle}{\|u\|} = +\infty.$$

(10) T is said to be strongly continuous if $y_j \rightharpoonup y_0$ in D, then $T(y_j) \to T(y_0)$.

(11) T is said to be hemicontinuous if $t \mapsto \langle T(u + tv), w\rangle$ with $(u + tv) \in D$ is continuous on $[0, 1]$ for all $u,\ v \in D$, $w \in X$.

(12) T is said to be compact if T is continuous and $T(C)$ is relatively compact for any bounded subset C of D, i.e., $\overline{T(C)}$ is a compact set.

Remark 2.1. (See [24], pages 501 and 596).

(1) If T is demicontinuous and of class $(S)_+$, then T is pseudo-monotone.

(2) If D is a nonempty closed convex subset of X and $T : D \to X^*$ is a monotone and hemicontinuous, then T is pseudo-monotone.

(3) If T is uniformly monotone, then T is of class $(S)_+$.

(4) If $T : D \subset X \to X^*$ is demicontinuous if and only if T is continuous from the topology of X to the weakly star topology of X^*.

(5) Let $A,\ B : D \to X^*$ be operators. If A is of class $(S)_+$ and B is compact, then $A + B$ is of class $(S)_+$. In addition, If A is pseudo-monotone and B is strongly continuous, then $A + B$ is pseudo-monotone.

(6) We have the following implications:

$$T \text{ is uniformly monotone} \Rightarrow T \text{ is monotone},$$

$$T \text{ is continuous} \Rightarrow T \text{ is demicontinuous} \Rightarrow T \text{ is hemicontinuous},$$

$$\text{and } T \text{ is strongly monotone} \Rightarrow T \text{ is uniformly monotone} \Rightarrow T \text{ is coercive.}$$

Definition 2.2. Let D be a nonempty subset of X. A mapping $T_1 : D \to 2^{X^*}$ is a set-valued mapping. T_1 is said to be pseudo-monotone if for each sequence $\{y_j\} \subset D$, $y_j \rightharpoonup y_0 \in D$, $y_j^* \in T_1(y_j)$ and $\limsup_{j\to\infty}\langle y_j^*, y_j - y_0\rangle \leq 0$, then for each $v \in D$, there exists $y_v^* \in T(y_0)$ with the property that

$$\langle y_v^*, y_0 - v\rangle \leq \liminf_{j\to\infty}\langle y_j^*, y_j - v\rangle.$$

Definition 2.3. Let $G : X_1 \to 2^{X_2}$ be a multi-valued mapping from a topological space X_1 into a topological space X_2. G is said to be

(1) with convex (or closed, or bounded etc.) values if for each $x \in X_1$, $G(x)$ is a convex (or closed, or bounded etc.) subset of X_2;

(2) upper semicontinuous at $x_0 \in X_1$ if for every open set V_0 with $G(x_0) \subset V_0$ in X_2, there is an open neighborhood $U(x_0)$ of x_0 in X_1, such that $G(x) \subset V_0$ for all $x \in U(x_0)$;

(3) upper semicontinuous if G is upper semicontinuous for each $x \in X_1$.

(4) bounded if $G(\tilde{C})$ is a bounded set for any bounded subset $\tilde{C}$ of X_1.

Definition 2.4. Let O be a nonempty subset of X and $K' : O \to 2^X$ be a set-valued mapping. For any $\{w_n\} \subset O$ with $w_n \rightharpoonup w_0 \in O$, we say that the sequence of sets $K(w_n)$ Mosco-converges to $K(w)$ if the following two assumptions are satisfied:

(i) for every sequence $u_n \in K'(w_n)$ such that u_n weakly converges to u_0, then $u_0 \in K'(w_0)$;

(ii) for every $u_0 \in K'(w_0)$, there exists $u_n \in K'(w_n)$ (for n large enough) such that u_n strongly converges to u_0.

Definition 2.5. Let $\tilde{O}$ be a nonempty subset of X and $T_1' : \tilde{O} \to 2^{E^*}$ be a set-valued mapping. T_1' is said to be upper semicontinuous on finite-dimensional subspaces of X if for each finite dimensional subspace L' of X, $T_1' \mid_{L'}: L' \cap \tilde{O} \to 2^{X^*}$ is upper semicontinuous continuous, where X^* equipped with its weakly star topology.

Remark 2.2. If T_1 is a single-valued mapping, then Definition 2.2 becomes (5) in Definition 2.1. If $O = X$, then Definition 2.4 becomes the Definition 2 in [25]. In Definition 2.5, if T_1' is a single valued mapping, then T_1' is called continuous on finite-dimensional subspaces of X. It is clear that, if T_1' is demicontinuous, then it is continuous on finite-dimensional subspaces of X, for details, see [26].

Lemma 2.1. ([26]). *Suppose that C is a convex and compact set in R^n and $H : C \to 2^{R^n}$ is an upper semicontinuous set-valued mapping with compact convex values. Then there exist $u \in C$ and $u^* \in H(u)$ such that*

$$\langle u^*, v - u\rangle \geq 0, \ \forall\, v \in C.$$

Lemma 2.2. ([27]). *Let $\tilde{K}'$ be a nonempty convex set in a vector space X' and D' a nonempty compact convex subset of a Hausdorff topological vector space Y'. Suppose that $\bar{f}$ is a real-valued function on $\tilde{K}' \times D'$ such that*

(i) *for each fixed $x \in \tilde{K}'$, $\bar{f}(x, y)$ is lower semicontinuous and convex on D';*

(ii) *for each fixed $y \in D'$, $\bar{f}(x, y)$ is concave on $\tilde{K}'$. Then*

$$\sup_{x \in \tilde{K}'} \inf_{y \in D'} \bar{f}(x, y) = \inf_{y \in D'} \sup_{x \in \tilde{K}'} \bar{f}(x, y).$$

Lemma 2.3. *Let X be a reflexive Banach space and D be a nonempty subset of X. If $T : D \to X^*$ is a bounded generalized pseudo-monotone mapping from D into X^*, then T is pseudo-monotone.*

Proof. For any $\{y_j\} \subset D$ with $y_j \rightharpoonup y_0 \in D$ and $\limsup_{j\to\infty}\langle T(y_j), y_j - y_0\rangle \le 0$, we now show that

$$\langle T(y_0), y_0 - w\rangle \le \liminf_{j\to\infty}\langle T(y_j), y_j - w\rangle, \ \forall w \in D.$$

Suppose on the contrary that the assertion is false. Then there exists $v \in D$ such that

$$\langle T(y_0), y_0 - v\rangle > \liminf_{j\to\infty}\langle T(y_j), y_j - v\rangle.$$

Since T is bounded and $y_j \rightharpoonup y_0$, $\{T(y_j)\}$ is bounded. Without loss of generality, we can assume that $T(y_j) \rightharpoonup w_0 \in X^*$. The generalized pseudo-monotonicity of T implies that $w_0 = T(y_0)$ and $\langle T(y_j), y_j\rangle \to \langle w_0, y_0\rangle$. Hence

$$\langle T(y_0), y_0 - v\rangle = \liminf_{j\to\infty}\langle T(y_j), y_j - v\rangle,$$

which is a contradiction. Thus the assertion is true and T is pseudo-monotone. This completes the proof. ■

Theorem 2.1. *Assume that W, X are two reflexive Banach spaces, U is a nonempty closed convex set of W and $K : U \to 2^X$ is a mapping with nonempty closed and convex values such that, for each $w \in U$, $0 \in K(w)$. Suppose that for each $w \in U$, $A : K(w) \to 2^{X^*}$ is a pseudo-monotone mapping and the following conditions are satisfied:*

(i) *A is upper semicontinuous on finite dimensional subspaces of X;*

(ii) *for each $x \in K(U)$, $A(x)$ is a closed convex and bounded set;*

(iii) *for any $w \in U$,*

$$\lim_{u\in K(w),\ \|(w,u)\|\to+\infty} \inf_{u^*\in A(u)} \langle u^* + B(w), u\rangle = +\infty. \tag{2.5}$$

Then, for each $w \in U$, the abstract generalized quasi-variational inequality problem (2.2) has a solution and so $S(w) \neq \emptyset$.

Proof. For any given $w \in U$, letting $B_r = \{v \in X : \|v\| \leq r\}$ and $K^r = B_r \cap K(w)$, we get that K^r is a bounded, closed and convex subset of X. We claim that, for this $w \in U$, there exist $u_r \in K^r$ and $u_r^* \in A(u_r)$ such that

$$\langle u_r^* + B(w), v - u_r \rangle \geq 0, \ \forall\ v \in K^r. \tag{2.6}$$

Indeed, denote by $\mathcal{F}$ the set of all finite dimensional subspaces L of X such that $L \cap K^r \neq \emptyset$. Fix a subspace $L \in \mathcal{F}$ and consider a mapping $\alpha_L : X^* \to L$ defined by $\langle \alpha_L x^*, y \rangle = \langle x^*, y \rangle$ for all $y \in L$. Put $K_L^r = K^r \cap L$ and define the mapping $A_L : K_L^r \to 2^L$ by the formula

$$A_L(x) = \{\alpha_L x^* : x^* \in A(x) + B(w)\}.$$

By (i) and (ii), A_L is upper semicontinuous on K_L^r and has compact convex values. Since K_L^r is a compact convex set, by Lemma 2.1, we know that there exists $u_L \in K_L^r$ and $u^*{}_L \in A_L(u_L)$ such that

$$\langle u^*{}_L, v - u_L \rangle \geq 0, \ \forall\ v \in K_L^r.$$

Since $u^*{}_L = \alpha_L \tilde{u}^*$ for some $\tilde{u}^* \in A(u_L) + B(w)$, there exists $u^* \in A(u_L)$ such that

$$\langle u^* + B(w), v - u_L \rangle \geq 0, \ \forall\ v \in K_L^r$$

and so

$$\sup_{u^* \in A(u_L)} \langle u^* + B(w), v - u_L \rangle \geq 0, \ \forall\ v \in K_L^r. \tag{2.7}$$

For each $Y \in \mathcal{F}$, denote by S_Y the set of all $\hat{u} \in K^r$ such that there exists a subspace $L \supseteq Y$ with the property that $\hat{u} \in K_L^r$ and

$$\sup_{u^* \in A(\hat{u})} \langle u^* + B(w), v - \hat{u} \rangle \geq 0, \ \forall\ v \in K_L^r.$$

We show that the family $\{\bar{S}_Y\}$ has the finite intersection property, where $\bar{S}_Y$ is the weak closure of S_Y in X. Indeed, for each $Y \in \mathcal{F}$, by putting $L = Y$, we deduce from (2.7) that $u_Y \in S_Y$. Hence S_Y is nonempty. Take subspaces $L_1, L_2, \cdots, L_n \in \mathcal{F}$ and put $M = \text{span}\{L_1, L_2, \cdots, L_n\}$. Then $M \in \mathcal{F}$ and

$$S_M \subset \textstyle\bigcap_{i=1}^n S_{L_i}.$$

This implies that

$$\emptyset \neq S_M \subset \bar{S}_M \subset \overline{\textstyle\bigcap_{i=1}^n S_{L_i}} \subset \textstyle\bigcap_{i=1}^n \bar{S}_{L_i}$$

and so $\{\bar{S}_Y\}$ has the finite intersection property.

Since $\bar{S}_Y \subset K^r$ and K^r is weakly compact, we obtain

$$\bigcap_{Y\in\mathcal{F}} \bar{S}_Y \neq \emptyset.$$

This means that there exists a point $u_r \in K^r$ such that $u_r \in \bar{S}_Y$ for all $Y \in \mathcal{F}$. Fix any $v \in K^r$ and choose $Y \in \mathcal{F}$ such that Y contains v and u_r. Since $u_r \in \bar{S}_Y$, there exists a sequence $\{u_n\} \subset S_Y$ such that $u_n \rightharpoonup u_r$. By the definition of S_Y, we have

$$\sup_{u^*\in A(u_n)} \langle u^* + B(w), y - u_n\rangle \geq 0,\ \forall\ y \in K^r_Y$$

and so

$$\inf_{y\in K^r_Y} \sup_{u^*\in A(u_n)} \langle u^* + B(w), y - u_n\rangle \geq 0.$$

It follows from Lemma 2.2 that

$$\sup_{u^*\in A(u_n),} \inf_{y\in K^r_Y} \langle u^* + B(w), y - u_n\rangle = \inf_{y\in K^r_Y} \sup_{u^*\in A(u_n)} \langle u^* + B(w), y - u_n\rangle \geq 0.$$

Since the extended-real-valued function $u^* \mapsto \inf_{y\in K^r_Y} \langle u^*, y - u_n\rangle$ is upper semicontinuous, there exists $u^*{}_n \in A(u_n)$ such that

$$\inf_{y\in K^r_Y} \langle u^*{}_n + B(w), y - u_n\rangle \geq 0$$

and so

$$\langle u^*{}_n + B(w), y - u_n\rangle \geq 0,\ \forall\ y \in K^r_Y.$$

In particular,

$$\langle u^*{}_n + B(w), v - u_n\rangle \geq 0 \text{ and } \langle u^*{}_n + B(w), u_r - u_n\rangle \geq 0. \tag{2.8}$$

From (2.8), we get

$$\limsup_{n\to\infty} \langle u^*{}_n, u_n - u_r\rangle = \limsup_{n\to\infty} \langle u^*{}_n + B(w), u_n - u_r\rangle \leq 0.$$

By the pseudo-monotonicity of A, there exists $u'^*_r \in A(u_r)$ such that

$$\langle u'^*_r, u_r - v\rangle \leq \liminf_{n\to\infty} \langle u^*{}_n, u_n - v\rangle. \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\begin{aligned}\langle u'^*_r + B(w), u_r - v\rangle &\leq \liminf_{n\to\infty} \langle u^*{}_n, u_n - v\rangle + \liminf_{n\to\infty} \langle B(w), u_n - v\rangle. \\ &\leq \liminf_{n\to\infty} \langle u^*{}_n + B(w), u_n - v\rangle \leq 0.\end{aligned}$$

This implies that $\langle u'^*_r + B(w), v - u_r\rangle \geq 0$ and so

$$\inf_{v\in K^r} \sup_{u'^*_r\in A(u_r)} \langle u'^*_r + B(w), v - u_r\rangle \geq 0.$$

Using Lemma 2.2 again, we can prove that there exists $u_r^* \in A(u_r)$ such that

$$\langle u_r^* + B(w), v - u_r \rangle \geq 0, \ \forall \ v \in K^r.$$

Thus, the claim is proved and (2.6) is true.

In particular, taking $v = 0$ in (2.6), there are $u_r \in K^r$ and $u_r^* \in A(u_r)$ such that

$$\langle u_r^* + B(w), u_r \rangle \leq 0. \tag{2.10}$$

It follows from condition (2.5) that $\{u_r\}$ is bounded. Otherwise, if $\|u_r\| \to \infty$, then by (2.5), we get

$$\lim_{\|u_r\| \to +\infty} \inf_{u^* \in A(u_r)} \langle u^* + B(w), u_r \rangle = +\infty,$$

which contradicts (2.10). Therefore, $\{u_r\}$ is bounded and so $\|u_r\| \leq M$ for some real number $M > 0$. Let $r = M + 1$. For each $v \in K(w)$, we can choose $t \in (0, 1)$ small enough such that $z = u_r + t(v - u_r) \in K_r$. Substituting z into (2.6), we obtain that $u_r \in S(w)$ and the generalized variational inequality problem (2.2) has a solution. The proof is completed. ■

Corollary 2.1. *Assume that W, X are two reflexive Banach spaces, U is a nonempty closed convex set of W and $K : U \to 2^X$ is a mapping with nonempty bounded, closed and convex values. Suppose that for each $w \in U$, $A : K(w) \to 2^{X^*}$ is a pseudo-monotone mapping and the following conditions are satisfied:*

(i) *A is upper semicontinuous on finite dimensional subspaces of X;*

(ii) *for each $x \in K(U)$, $A(x)$ is a closed convex and bounded set.*

Then, for each $w \in U$, the abstract generalized quasi-variational inequality problem (2.2) has a solution and so $S(w) \neq \emptyset$.

Proof. Notice that for each $w \in U$, $K(w)$ is bounded. From (2.6), it follows that for each $w \in U$, $S(w) \neq \emptyset$. The proof is completed. ■

Corollary 2.2. *Let V, $\bar{U}$ be two reflexive Banach spaces, V^* be the dual space of V, U_{ad} be a nonempty closed convex set of W and K_1 be a nonempty closed and convex subset of V and $0 \in K_1$. Assume that $F_1 : K_1 \to V^*$ is a strongly continuous mapping and $A_1 : K_1 \to V^*$ is a demicontinuous pseudo-monotone mapping, or $A_1 - F_1$ is a demicontonuous, bounded and generalized pseudo-monotone mapping. Suppose that the following coercive condition is satisfied:*

$$\lim_{(w,u) \in U_{ad} \times K_1, \ \|(w,u)\| \to +\infty} \langle A_1(u) - F_1(u) + B_1(w), u \rangle = +\infty. \tag{2.11}$$

Then, for each $w \in U_{ad}$, $S_1(w) \neq \emptyset$, i.e., the abstract variational inequality problem (1.2) has a solution.

Proof. Since A_1 is a pseudo-monotone mapping and F_1 is strongly continuous, by Remark 2.1, we know $A_1 - F_1$ is a pseudo-monotone mapping. Since $A_1 - F_1$ is bounded and generalized pseudo-monotone, Lemma 2.3 implies that $A_1 - F_1$ is a pseudo-monotone mapping. Taking $A = A_1 - F_1$, by Theorem 2.1, we know for each $w \in U_{ad}$, $S_1(w) \neq \emptyset$. The proof is completed. ■

Corollary 2.3. *Assume that V, $\bar{U}$ are two reflexive Banach spaces, V^* is the dual space of V, U_{ad} is a nonempty closed convex set of W and K_1 is a closed and convex cone of V. Suppose that $F_1 : K_1 \to V^*$ is a compact mapping and $A_1 : K_1 \to V^*$ is a demicontinuous mapping of class $(S)_+$, or $A_1 - F_1$ is a demicontonuous, bounded and generalized pseudo-monotone mapping. Suppose that the following coercive condition is satisfied:*

$$\lim_{(w,u)\in U_{ad}\times K_1,\ \|(w,u)\|\to+\infty} \langle A_1(u) - F_1(u) + B_1(w), u\rangle = +\infty. \tag{2.12}$$

Then, for each $w \in U_{ad}$, $S_1(w) \neq \emptyset$, i.e., the abstract variational inequality problem (1.2) has a solution.

Proof. Since A_1 is a demicontinuous mapping of class $(S)_+$ and F_1 is a compact mapping, by (1) and (5) in Remark 2.1, we know $A_1 - F_1$ is a pseudo-monotone mapping. From Corollary 2.2, it follows that for each $w \in U$, $S_1(w) \neq \emptyset$. The proof is completed. ■

Remark 2.3. If A_1, $A_1 - F_1$ are continuous, then Corollary 2.4 reduces to Lemma 2.1 in [10].

Theorem 2.2. *Let W, X be two reflexive Banach spaces, U be a nonempty closed convex set of W and $K : U \to 2^X$ be a mapping with nonempty closed and convex values such that for each $w \in U$, $0 \in K(w)$. Assume that $A : K(U) \to 2^{X^*}$ is a bounded pseudo-monotone mapping, $J : U \times K(U) \to \mathbf{R}$ is a weakly lower semicontinuous function, $B : U \to X^*$ is strongly continuous from the weak topology of W to the topology of X^* and the following conditions are satisfied:*

(*i*) *A is upper semicontinuous on finite dimensional subspaces of X;*

(*ii*) *for each $x \in K(U)$, $A(x)$ is a closed convex and bounded set;*

(*iii*) *for any $w \in U$, $u \in K(w)$,*

$$\lim_{\|(w,u)\|\to+\infty} \inf_{u^*\in A(u)} \langle u^* + B(w), u\rangle = +\infty; \tag{2.13}$$

(*iv*) *for all $w_n \subset U$ with $w_n \rightharpoonup w$, $K(w_n)$ Mosco-converges to $K(w)$.*

Then there exists an optimal control $w_0 \in U$ for the problem (2.1).

Proof. From Theorem 2.1, it follows that for each $w \in U$, $S(w) \neq \emptyset$. Let $\{(w_n, u_n)\}_{n=1,2,\cdots} \subset U \times K(U)$ be a minimizing sequence for the problem (2.1) such that

$$\lim_{n\to\infty} J(w_n, u_n) = \min_{w\in U,\ u\in S(w)} J(w, u). \tag{2.14}$$

We claim that $\{(w_n, u_n)\}_{n=1,2,\cdots}$ is bounded. If not so, then there exists a subsequence $\{(w_{n_k}, u_{n_k})\}$ such that $\|(w_{n_k}, u_{n_k})\| \to +\infty$. It follows from the coercive condition (2.13) that

$$\lim_{k\to+\infty} \inf_{u^*_{n_k}\in A(u_{n_k})} \langle u^*_{n_k} + B(w_{n_k}), u_{n_k}\rangle = +\infty. \tag{2.15}$$

By $u_{n_k} \in S(w_{n_k})$, we know $u_{n_k} \in K(w_{n_k})$ and there exists $u^*_{n_k} \in A(u_{n_k})$ such that

$$\langle u^*_{n_k} + B(w_{n_k}), v - u_{n_k}\rangle \geq 0,\ \forall\ v \in K(w_{n_k}). \tag{2.16}$$

By taking $v = 0$ in (2.16), we get

$$\langle u^*_{n_k} + B(w_{n_k}), u_{n_k}\rangle \leq 0,$$

which contradicts (2.15). Hence, $\{(w_n, u_n)\}$ is bounded.

By the reflexivity of W and X, there exists a weakly convergent subsequence of $\{(w_n, u_n)\}$. Without lose of generality, we can assume that $w_n \rightharpoonup w_0 \in W$ and $u_n \in K(w_n) \rightharpoonup u_0 \in X$ as $n \to +\infty$.

Since U is closed and convex, U is a weakly closed set and $w_0 \in U$. By the assumptions, we know $K(w_n)$ Mosco-converges to $K(w_0)$ and so $u_0 \in K(w_0)$. According to Definition 2.4, there exists $\bar{u}_n \in K(w_n)$ such that $\bar{u}_n \to u_0$.

By $u_n \in S(w_n)$, we know that $u_n \in K(w_n)$ and there exists $u^*_n \in A(u_n)$ such that

$$\langle u^*_n + B(w_n), v - u_n\rangle \geq 0,\ \forall\ v \in K(w_n), \tag{2.17}$$

and so

$$\langle u^*_n + B(w_n), u_n - \bar{u}_n\rangle \leq 0. \tag{2.18}$$

Since B is strongly continuous from the weak topology of W to the topology of X^*, $B(w_n) \to B(w_0)$. Note that A is bounded. Without loss of generality, we can assume that $u^*_n \rightharpoonup \hat{u}^*$. (2.18) implies that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle u^*_n, u_n - u_0\rangle \\
&= \limsup_{n\to\infty}\langle u^*_n, u_n - u_0\rangle + \lim_{n\to\infty}\langle u^*_n, u_0 - \bar{u}_n\rangle + \lim_{n\to\infty}\langle B(w_n), u_n - \bar{u}_n\rangle \\
&= \limsup_{n\to\infty}\langle u^*_n + B(w_n), u_n - \bar{u}_n\rangle \leq 0.
\end{aligned}$$

Fixing any $v' \in K(w_0)$, from the pseudo-monotonicity of A, it follows that there exists $u^*_{v'} \in A(u_0)$ such that

$$\langle u^*_{v'}, u_0 - v' \rangle \le \liminf_{n\to\infty} \langle u^*_n, u_n - v' \rangle. \tag{2.19}$$

According to $w_n \rightharpoonup w_0$ and the condition (iv), there exists $v_n \in K(w_n)$ such that $v_n \to v'$. Since $u^*_n \in A(u_n)$ and $u^*_n \rightharpoonup \hat{u}^*$, we have

$$\begin{aligned} &\liminf_{n\to\infty} \langle u^*_n, u_n - v' \rangle \\ &= \liminf_{n\to\infty} \langle u^*_n, u_n \rangle + \lim_{n\to\infty} \langle u^*_n, -v' \rangle = \liminf_{n\to\infty} \langle u^*_n, u_n \rangle + \langle \hat{u}^*, -v' \rangle \\ &= \liminf_{n\to\infty} \langle u^*_n, u_n \rangle + \lim_{n\to\infty} \langle u^*_n, -v_n \rangle = \liminf_{n\to\infty} \langle u^*_n, u_n - v_n \rangle. \end{aligned} \tag{2.20}$$

(2.17), (2.19) and (2.20) imply that

$$\begin{aligned} &\langle u^*_{v'}, u_0 - v' \rangle + \langle B(w_0), u_0 - v' \rangle \\ &\le \liminf_{n\to\infty} \langle u^*_n, u_n - v' \rangle + \lim_{n\to\infty} \langle B(w_n), u_n - v_n \rangle \\ &= \liminf_{n\to\infty} \langle u^*_n, u_n - v_n \rangle + \lim_{n\to\infty} \langle B(w_n), u_n - v_n \rangle \\ &= \liminf_{n\to\infty} \langle u^*_n + B(w_n), u_n - v_n \rangle \le 0. \end{aligned}$$

This implies that

$$\langle u^*_{v'} + B(w_0), v' - u_0 \rangle \ge 0 \text{ and so } \sup_{\tilde{u}^* \in A(u_0)} \langle \tilde{u}^* + B(w_0), v' - u_0 \rangle \ge 0$$

and so

$$\inf_{v' \in K(w_0)} \sup_{\tilde{u}^* \in A(u_0)} \langle \tilde{u}^* + B(w_0), v' - u_0 \rangle \ge 0.$$

Using Lemma 2.2 again, we can prove that there exists $u^*_0 \in A(u_0)$ such that

$$\langle u^*_0 + B(w_0), v' - u_0 \rangle \ge 0, \ \forall \ v' \in K(w_0).$$

Therefore, $u_0 \in S(w_0)$.

Since $J(w, u)$ is a weakly lower semicontinuous function, it follows from (2.14) that

$$J(w_0, u_0) \le \lim_{n\to\infty} J(w_n, u_n) = \min_{w\in U,\ u\in S(w)} J(w, u)$$

and so

$$J(w_0, u_0) = \min_{w\in U, u\in S(w)} J(w, u).$$

Therefore, $w_0 \in U$ is an optimal control for the problem (2.1). This completes the proof. ■

Corollary 2.4. *Let W, X be two reflexive Banach spaces, U be a nonempty closed convex set of W and $K : U \to 2^X$ be a bounded mapping with nonempty closed and convex values. Assume that $A : K(U) \to 2^{X^*}$ is a bounded pseudo-monotone mapping, $J : U \times K(U) \to \mathbf{R}$ is a weakly lower semicontinuous function, $B : U \to X^*$ is strongly continuous from the weak topology of W to the topology of X^* and the following conditions are satisfied:*

(i) *A is upper semicontinuous on finite dimensional subspaces of X;*

(ii) *for each $x \in K(U)$, $A(x)$ is a closed convex and bounded set;*

(iii) $\lim_{w\in U,\ \|w\|\to+\infty} J(w,u) = +\infty,\ \forall\ u \in K(U)$*;*

(iv) *for all $w_n \subset U$ with $w_n \rightharpoonup w$, $K(w_n)$ Mosco-converges to $K(w)$.*

Then there exists an optimal control $w_0 \in U$ for the problem (2.1).

Proof. Since K is a bounded mapping, K is a mapping with bounded values. From Corollary 2.1, it follows that for each $w \in U$, $S(w) \neq \emptyset$. Let $\{(w_n, u_n)\}_{n=1,2,\cdots} \subset U \times K(U)$ be a minimizing sequence for the problem (2.1) such that

$$\lim_{n\to\infty} J(w_n, u_n) = \min_{w\in U,\ u\in S(w)} J(w,u). \tag{2.21}$$

The condition (iii) implies that $\{w_n\}_{n=1,2,\cdots}$ is bounded. Since K is a bounded mapping and $u_n \in S(w_n) \subset K(\{w_n\})$, $\{u_n\}$ is a bounded sequence.

By the reflexivity of W and X, there exists a weakly convergent subsequence of $\{(w_n, u_n)\}$. Without lose of generality, we can assume that $w_n \rightharpoonup w_0 \in W$ and $u_n \in K(w_n) \rightharpoonup u_0 \in X$ as $n \to +\infty$. By using similar arguments to the proof of Theorem 2.2, we can show that $w_0 \in U$ is an optimal control for the problem (2.1). The proof is completed. ■

Corollary 2.5. *Let W, X be two reflexive Banach spaces, U be a nonempty closed convex set of W and $K : U \to 2^X$ be a bounded mapping with nonempty closed and convex values. Assume that $A_2 : K(U) \to X^*$ is a bounded demicontinuous pseudo-monotone mapping and $F_2 : K(U) \to X^*$ is a strongly continuous mapping from the weak topology of X to the topology of X^* or $A_2 - F_2$ is a demicontonuous, bounded and generalized pseudo-monotone mapping. Suppose $J : U \times K(U) \to \mathbf{R}$ is a weakly lower semicontinuous function, $B : U \to X^*$ is strongly continuous from the weak topology of W to the topology of X^* and the following conditions are satisfied:*

(i) $\lim_{w\in U,\ \|w\|\to+\infty} J(w,u) = +\infty,\ \forall\ u \in K(U)$*;*

(ii) *for all $w_n \subset U$ with $w_n \rightharpoonup w$, $K(w_n)$ Mosco-converges to $K(w)$.*

Then there exists an optimal control $w_0 \in U$ *for the problem* (*2.3*).

Proof. Since K is a bounded mapping, K is a mapping with bounded values. Since A_2 is a pseudo-monotone mapping and F_2 is strongly continuous, by Remark 2.1, we know $A_2 - F_2$ is a pseudo-monotone mapping. Since $A_2 - F_2$ is bounded and generalized pseudo-monotone, Lemma 2.3 implies that $A_2 - F_2$ is a pseudo-monotone mapping. Taking $A = A_2 - F_2$, by Corollary 2.4, we know there exists an optimal control $w_0 \in U$ for the problem (2.3). The proof is completed. ■

Corollary 2.6. *Let* V, $\bar{U}$ *be two reflexive Banach spaces,* V^* *be the dual space of* V, U_{ad} *be a nonempty closed convex set of* W, K_1 *be a nonempty closed and convex subset of* V *and* $0 \in K_1$. *Assume that* $F_1 : K_1 \to V^*$ *is a strongly continuous mapping from the weak topology of* $\bar{U}$ *to the topology of* V^* *and* $A_1 : K_1 \to V^*$ *is a demicontinuous pseudo-monotone mapping, or* $A_1 - F_1$ *is a demicontinuous, bounded and generalized pseudo-monotone mapping. Suppose that* $B_1 : U_{ad} \to X^*$ *is a strongly continuous mapping from the weak topology of* $\bar{U}$ *to the topology of* V^* *and the following coercive condition is satisfied:*

$$\lim_{(w,u)\in U_{ad}\times K_1,\ \|(w,u)\|\to+\infty} \langle A_1(u) - F_1(u) + B_1(w), u\rangle = +\infty. \tag{2.22}$$

Then there exists an optimal control $w_0 \in U_{ad}$ *for the problem* (*1.1*).

Proof. From Corollary 2.2, it follows that for each $w \in U_{ad}$, $S_1(w) \neq \emptyset$. Let $\{(w_n, u_n)\}_{n=1,2,\cdots} \subset U \times K_1$ be a minimizing sequence for the problem (1.1) such that

$$\lim_{n\to\infty} J_1(w_n, u_n) = \min_{w\in U,\ u\in S_1(w)} J_1(w, u). \tag{2.23}$$

We claim that $\{(w_n, u_n)\}_{n=1,2,\cdots}$ is bounded. If not so, then there exists a subsequence $\{(w_{n_k}, u_{n_k})\}$ such that $\|(w_{n_k}, u_{n_k})\| \to +\infty$. It follows from the coercive condition (2.22) that

$$\lim_{(w,u)\in U_{ad}\times K_1,\ \|(w,u)\|\to+\infty} \langle A_1(u) - F_1(u) + B_1(w), u\rangle = +\infty. \tag{2.24}$$

By $u_{n_k} \in S_1(w_{n_k})$, there exists $u_{n_k} \in K_1$ such that

$$\langle A_1(u_{n_k}) - F_1(u_{n_k}) + B_1(w_{n_k}), v - u_{n_k}\rangle \geq 0,\ \forall\, v \in K_1. \tag{2.25}$$

By taking $v = 0$ in (2.25), we get

$$\langle A_1(u_{n_k}) - F_1(u_{n_k}) + B_1(w_{n_k}), u_{n_k}\rangle \leq 0,$$

which contradicts (2.24). Hence, $\{(w_n, u_n)\}$ is bounded.

By the reflexivity of W and X, there exists a weakly convergent subsequence of $\{(w_n, u_n)\}$. Without lose of generality, we can assume that $w_n \in U_{ad} \rightharpoonup w_0$ and $u_n \in K_1 \rightharpoonup u_0 \in X$ as $n \to +\infty$.

Since U_{ad} and K_1 are closed and convex, U_{ad} and K_1 are weakly closed sets and so $w_0 \in U_{ad}$, $u_0 \in K_1$.

By $u_n \in S_1(w_n)$, we know $u_n \in K_1$ such that

$$\langle A_1(u_n) - F_1(u_n) + B_1(w_n), v - u_n \rangle \geq 0, \ \forall \ v \in K_1. \tag{2.26}$$

Since B is strongly continuous, $B(w_n) \to B(w_0)$.

(i) If A_1 is a demicontinuous pseudo-monotone mapping and F_1 is strongly continuous, then $F_1(u_n) \to F_1(u_0)$. (2.26) implies that

$$\begin{aligned} &\limsup_{n\to\infty} \langle A_1(u_n), u_n - u_0 \rangle \\ &= \limsup_{n\to\infty} \langle A_1(u_n), u_n - u_0 \rangle + \lim_{n\to\infty} \langle B_1(w_n) - F_1(u_n), u_n - u_0 \rangle \\ &= \limsup_{n\to\infty} \langle A_1(u_n) + B_1(w_n) - F_1(u_n), u_n - u_0 \rangle \leq 0. \end{aligned}$$

From the pseudo-monotonicity of A_1, it follows that for any $v \in K_1$,

$$\langle A_1(u_0), u_0 - v \rangle \leq \liminf_{n\to\infty} \langle A_1(u_n), u_n - v \rangle.$$

(2.26) implies that

$$\begin{aligned} &\langle A_1(u_0), u_0 - v \rangle + \langle B_1(w_0) - F_1(u_0), u_0 - v \rangle \\ &\quad \leq \liminf_{n\to\infty} \langle A_1(u_n), u_n - v \rangle + \lim_{n\to\infty} \langle B_1(w_n) - F_1(u_n), u_n - v \rangle \\ &\quad = \limsup_{n\to\infty} \langle A_1(u_n) + B_1(w_n) - F_1(u_n), u_n - v \rangle \leq 0. \end{aligned}$$

This implies that

$$\langle A_1(u_0) + B_1(w_0) - F_1(u_0), u_0 - v \rangle \leq 0. \tag{2.27}$$

Thus, there exists $u_0 \in K_1$ such that

$$\langle A_1(u_0) + B(w_0) - F_1(u_0), u_0 - v \rangle \leq 0, \ \forall \ v \in K_1.$$

Therefore, $u_0 \in S_1(w_0)$.

(ii) If $A_1 - F_1$ is a demicontinuous, bounded and generalized pseudo-monotone mapping, then Lemma 2.3 implies that $A_1 - F_1$ is a demicontinuous and pseudo-monotone mapping. By using the same argument as the proof of (i), we know $u_0 \in S_1(w_0)$.

Since $J_1(w,u)$ is a weakly lower semicontinuous function, it follows from (2.23) that

$$J_1(w_0,u_0) \le \lim_{n\to\infty} J_1(w_n,u_n) = \min_{w\in U,\ u\in S_1(w)} J_1(w,u)$$

and so

$$J_1(w_0,u_0) = \min\nolimits_{w\in U, u\in S_1(w)} J_1(w,u).$$

Therefore, $w_0 \in U_{ad}$ is an optimal control for problem (1.1). This completes the proof. ■

According to Lemma 2.3, Remark 2.1 and Corollary 2.6, it is easy to obtain the following result.

Corollary 2.7. *Let V, $\bar{U}$ be two reflexive Banach spaces, V^* be the dual the space of V, U_{ad} be a nonempty closed convex subset of W and K_1 be a nonempty closed and convex cone of V. Assume that $F_1 : K_1 \to V^*$ is a compact mapping and $A_1 : K_1 \to V^*$ is a demicontinuous mapping of class $(S)_+$, or $A_1 - F_1$ is a demicontonuous, bounded and generalized pseudo-monotone mapping. Suppose that $B_1 : U \to X^*$ is strongly continuous from the weak topology of $\bar{U}$ to the topology of V^* and the following coercive condition is satisfied:*

$$\lim_{(w,u)\in U_{ad}\times K_1,\ \|(w,u)\|\to+\infty} \langle A_1(u) - F_1(u) + B_1(w), u\rangle = +\infty. \tag{2.28}$$

Then there exists an optimal control $w_0 \in U_{ad}$ for the problem (1.1).

Remark 2.4. If A_1, $A_1 - F_1$ are continuous, then Corollary 2.6 reduces to Theorem 2.1 in [10].

According to Lemma 2.3, Remark 2.1 and Theorem 2.2, it is easy to obtain the following result.

Corollary 2.8. *Let W, X be two reflexive Banach spaces, U be a nonempty closed convex set of W and $K : U \to 2^X$ be a mapping with nonempty closed and convex values such that for each $w \in U$, $0 \in K(w)$. Assume that $A_2 : K(U) \to X^*$ is a bounded demicontinuous pseudo-monotone mapping and $F_2 : K(U) \to X^*$ is strongly continuous, or $A_2 - F_2$ is a demicontinuous, bounded and generalized pseudo-monotone mapping. Suppose that $B : U \to X^*$ is strongly continuous, $J : U \times K(U) \to \mathbf{R}$ is a weakly lower semicontinuous function and the following conditions are satisfied:*

(i) *for any $w \in U$, $u \in K(w)$,*

$$\lim_{\|(w,u)\|\to+\infty} \langle A_2(u) - F_2(u) + B(w), u\rangle = +\infty; \tag{2.29}$$

(ii) *for all $w_n \subset U$ with $w_n \rightharpoonup w$, $K(w_n)$ Mosco-converges to $K(w)$.*

Then there exists an optimal control $w_0 \in U$ *for the problem* (*2.3*).

3. The Existence Result for a Bilateral Obstacle Optimal Control Problem

In recently years, optimal obstacle control problems for variational inequalities have been considered in many different venues aspects. One of the main features is the use of the obstacle as the control. For example, see [13, 14, 15, 19, 20, 21, 22, 23] and the references cited therein. The optimal control problem (1.1) may (or can) be called as indirect obstacle optimal control problem. In this section, we obtain existence results for a bilateral obstacle optimal control problem by applying Corollary 2.5. It is worth noting that the method used here is different from [13, 14, 15, 19, 20, 21, 22, 23].

Let Ω be a bounded domain of R^N with Lipschitz boundary and $1 < p < N$. Let $X = W_0^{1,p}(\Omega)$, $W_1 = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and $W = W_1 \times W_1$. Set

$$U = \{w = (\varphi, \psi) \in W_1 \times W_1 = W : \varphi \leq \psi \ a.e.\ \Omega\}.$$

Let $1 < q < p^* = \frac{Np}{N-p}$, $\bar{f} : \Omega \times \mathbf{R} \to \mathbf{R}$ be a function and $\bar{\tau} : U \to L^{q'}(\Omega)$ $(q' = \frac{q}{q-1})$ be a mapping. For any $w = (\varphi, \psi) \in W$, we define

$$K(w) = \{v \in W_0^{1,p}(\Omega),\ \varphi \leq v \leq \psi \ a.e.\ \Omega\}. \tag{3.1}$$

For each $w = (\varphi, \psi) \in U$, we define $u \in K(w)$ (the state of the system) as the solution of the following quasilinear elliptic variational inequality:

$$\int_\Omega \tilde{a}(\nabla u)\nabla(v-u)dx \geq \int_\Omega (\bar{f}(x,u) - \bar{\tau}(w))(v-u)dx,\ \forall\ v \in K(w), \tag{3.2}$$

where $\tilde{a}(u) = (\tilde{a}_1(x,u), \cdots, \tilde{a}_N(x,u))$. We denote the solution set of the variational inequality (3.2) by $S_2(w)$. Let $z \in L^2(\Omega)$ be a given target profile. We seek a pair of $(\varphi, \psi) = w \in W$ so that the corresponding state $u = u(\varphi, \psi)$ is close to a desired target profile z and the norm of w is not too large in W. For this purpose, we take our objective functional J as

$$J(w,u) = J((\varphi,\psi),u) = \int_\Omega \{\frac{1}{2}(u-z)^2 + \frac{1}{p}(|\Delta\varphi|^p + |\Delta\psi|^p)\}dx,$$

which we try to minimize. More precisely, we pose the following optimal control problem: find $w_0 = (\varphi_0, \psi_0) \in U$ (optimal control), $u_0 \in K(w_0)$ and $u_0 \in S(w_0)$ such that

$$J(w_0, u_0) = \min_{(w,u) \in U \times K(U),\ u \in S_2(w)} J(w,u). \tag{3.3}$$

In the sequel, we introduce the following assumptions:

$(\tilde{H}_1)$ For all $\eta = (\eta_1, \cdots, \eta_N),\ \eta' = (\eta'_1, \cdots, \eta'_N) \in R^N$,

$$\sum_{i=1}^{N} (\tilde{a}_i(x, \eta) - \tilde{a}_i(x, \eta'))(\eta_i - \eta'_i) \geq 0.$$

$(\tilde{H}_2)$ There is a constant $c_1 > 0$ such that

$$\sum_{i=1}^{N} \tilde{a}_i(x, \eta)\eta_i \geq c_1 \sum_{i=1}^{N} |\eta_i|^p.$$

$(\tilde{H}_3)$ For all i, the function $\tilde{a}_i : \Omega \times \mathbf{R} \to \mathbf{R}$ has the following properties:

(i) $x \to \tilde{a}_i(x, t)$ is measurable on Ω for all $t \in \mathbf{R}$;

(ii) $t \to \tilde{a}_i(x, t)$ is continuous on $\mathbf{R}$ for almost all $x \in \Omega$;

(iii) there exists a constant $c_2 > 0$ such that $|\tilde{a}_i(x, \eta)| \leq c_2(1 + |\eta|^{p-1})$.

Define $A_2 : K(U) \to X^*$, $F_2 : K(U) \to X^*$ and $B : U \to X^*$ as follows: for all $u, v \in K(U)$ and $w \in U$,

$$\begin{aligned} &\langle A_2(u), v\rangle = \int_\Omega \tilde{a}(\nabla u)\nabla v dx,\ \langle F_2(u), v\rangle \\ &= \int_\Omega \bar{f}(x, u) v dx,\ \langle B(w), v\rangle = \int_\Omega \bar{\tau}(w) v dx. \end{aligned} \tag{3.4}$$

In addition, we define $\hat{A}_2 : X \to X^*$ and $\hat{F}_2 : X \to X^*$ as follows: for all $u, v \in X$,

$$\langle \hat{A}_2(u), v\rangle = \int_\Omega \tilde{a}(\nabla u)\nabla v dx,\ \langle \hat{F}_2(u), v\rangle = \int_\Omega \bar{f}(x, u) v dx. \tag{3.5}$$

Remark 3.1. (i) If $p = 2$ and $\tau' = 0$, then the variational inequality (3.2) becomes the quasilinear elliptic variational inequality (1.1) in [21]. (ii) If $\bar{f} = \bar{\tau} = 0$, then the problem (4.3) reduces to the optimal control problem (1.4) in [14]. (iii) If $p = 2$, $\bar{f} = \bar{\tau} = 0$, then the problem (4.3) reduces to the optimal control problem (1.3) in [23].

Lemma 3.1. *If K is defined by (3.1), then for all $w_n = (\varphi_n, \psi_n) \in U$ with $w_n \rightharpoonup w = (\varphi, \psi)$, $K(w_n)$ Mosco-converges to $K(w)$.'*

Proof. Let $v_n \in K(w_n)$ such that $v_n \rightharpoonup v$. Since

$$\begin{aligned} K(w_n) &= \{\tilde{v} \in X : \varphi_n \leq \tilde{v} \leq \psi_n\ a.e.\ \Omega\} \\ &= \{\tilde{v} \in X : \tilde{v} \leq \psi_n\ a.e.\ \Omega\} \cap \{\tilde{v} \in X : \tilde{v} \geq \varphi_n\ a.e.\ \Omega\}, \end{aligned}$$

$\varphi_n \rightharpoonup \varphi$, $\psi_n \rightharpoonup \psi$ and $\{\tilde{v} \in X : \tilde{v} \leq 0, \text{or } \geq 0\ a.e.\ \Omega\}$ is weakly closed, $v - \varphi \in \{\tilde{v} \in X : \tilde{v} \geq 0\ a.e.\ \Omega\}$ and $v - \psi \in \{\tilde{v} \in X : \tilde{v} \leq 0\ a.e.\ \Omega\}$. Thus $v \in K(w)$.

On the other hand, for any $v' \in K(w) = \{\tilde{v} \in X : \varphi \leq \tilde{v} \leq \psi \ a.e.\ \Omega\}$, there exist $k_1 \in \{\tilde{v} \in X : \tilde{v} \geq 0 \ a.e.\ \Omega\}$ and $k_2 \in \{\tilde{v} \in X : \tilde{v} \leq 0 \ a.e.\ \Omega\}$ such that $v' = k_1 + \varphi = k_2 + \psi$. Since $w_n \in U$, $\varphi_n \rightharpoonup \varphi$ and $\psi_n \rightharpoonup \psi$ in $W_0^{1,p} \cap W^{2,p}$, by using Sobolev embedding theorem, we get $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $W_0^{1,p} = X$. Put $v_n' = k_1 + \varphi_n \in K(w_n)$, for n large enough. It is clear that $v_n' = k_1 + \varphi_n \to k_1 + \varphi = k_2 + \psi = v'$. Therefore, $K(w_n)$ Mosco-converges to $K(w)$. This proof is completed. ∎

Lemma 3.2. If $\bar{\tau} : U \subset W \to L^{q'}(\Omega)$ ($q' = \frac{q}{q-1}$) is a strongly continuous mapping from the weak topology of W to the topology of $L^{q'}$, then $B : U \to X^*$ is a strongly continuous mapping from the weak topology of W to the topology of X^*.

Proof. Let $w_n = (\varphi_n, \phi_n) \rightharpoonup w_0 = (\varphi_0, \phi_0)$ in U. Since $\bar{\tau} : U \to L^{q'}(\Omega)$ is a strongly continuous mapping, we have $\bar{\tau}(w_n) \to \tau(w_0)$ in $L^{q'}(\Omega)$. Since $1 < q < N$, $1 < \frac{q}{q-1} < q < p^*$ and so $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$. Hence

$$\begin{aligned}\|B(w_n) - B(w_0)\| &= \sup_{\|v\| \leq 1} |\langle B(w_n) - B(w_0), v\rangle| \\ &\leq \sup_{\|v\| \leq 1} \left(\left| \int_\Omega |\bar{\tau}(w_n) - \bar{\tau}(w_0)\right|^{q'} dx)^{\frac{1}{q'}} \int_\Omega |v|^q dx \right)^{\frac{1}{q}} \\ &\leq c_6 \left(\int_\Omega |\bar{\tau}(w_n) - \bar{\tau}(w_0)|^{q'} dx \right)^{\frac{1}{q'}} \to 0\end{aligned}$$

for some constant $c_6 > 0$. Therefore, $\lim_{n \to +\infty} B(w_n) = B(w_0)$ in X^* and so $B : U \to X^*$ is a strongly continuous mapping from the weak topology of W to the topology of X^*. This proof is completed. ∎

Lemma 3.3. *Let $\hat{A}_2$, $\hat{F}_2$ be defined by (3.5). Then the following conclusions hold.*

(i) *Under assumptions $(\tilde{H}_1) - (\tilde{H}_3)$, $\hat{A}_2 : V \to V^*$ is a continuous monotone bounded and coercive.*

(ii) *If $\bar{f} : \Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function and satisfies*

$$\lim_{|t| \to +\infty} \frac{\bar{f}(x,t)}{\tilde{b}(x) t^{(s-1)}} = \bar{\lambda}_0, \tag{3.6}$$

uniformly a.e. with respect to $x \in \Omega$, where $\bar{\lambda}_0 \geq 0$, $1 < s < p^*$, $0 \leq \tilde{b}(x) \in L^r(\Omega)$, $r = \frac{p^*}{p^*-s}$. Then, $\hat{F}_2 : X \to V^*$ is a strongly continuous mapping from the weak topology of X to the topology of X^*.

Proof. By using the similar arguments to the proof of Lemma 3.1 in [10], we can proof $\hat{F}_2$ is a strongly continuous mapping. We only show (1) is true. In fact, set

$\tilde{a}_i(u) = \tilde{a}_i(x, \nabla u(x))$ for all $x \in \Omega$. From $(\tilde{H}_3)$ and Proposition 26.6 in [24], it follows that the operator $\tilde{a}_i : L^p(\Omega) \to L^{p'}(\Omega)$ $(p' = \frac{p}{p-1})$ is continuous and $\|\tilde{a}_i(u)\|_{p'} \leq C_3(1 + \|u\|_p^{p-1})$ for some constant $C_3 > 0$. Sobolev embedding theorem implies that the embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ is compact. Thus $\tilde{a}_i : X \to L^{p'}(\Omega)$ is continuous and $\|\tilde{a}_i(u)\|_{p'} \leq C_3'(1+\|u\|^{p-1})$ for some constant $C_3' > 0$. $|\langle \hat{A}_2(u), v\rangle| = \int_\Omega |\tilde{a}(\nabla u)\nabla v| dx \leq C_4(1+\|u\|^{p-1})\|v\|$ for some constant $C_4 > 0$. Hence $\|\hat{A}_2(u)\| \leq C_4(1 + \|u\|^{p-1})$ and $\hat{A}_2$ is bounded. Let $u_n \to u$ in X. Since $\tilde{a}_i : X \to L^{p'}(\Omega)$ is continuous, $\tilde{a}_i(u_n) \to a_i(u)$ in $L^{p'}$ as $n \to +\infty$. By the Hölder inequality

$$|\langle \hat{A}_2(u_n) - \hat{A}_2(u), v\rangle| \leq \sum_{i=1}^{N} \|\tilde{a}_i(u_n) - \tilde{a}_i(u)\|_{p'}\|v\|,$$

for all $v \in X$. This implies

$$\|\hat{A}_2(u_n) - \hat{A}_2(u)\| \leq \sum_{i=1}^{N} \|\tilde{a}_i(u_n) - \tilde{a}_i(u)\|_{p'},$$

and so $\|\hat{A}_2(u_n) - \hat{A}_2(u)\| \to 0$ as $n \to +\infty$. Thus, the operator $\hat{A}_2 : V \to V^*$ is continuous. By $(\tilde{H}_1)$, for all $u,\ v \in X$

$$\langle \hat{A}_2(u) - \hat{A}_2(v), u - v\rangle \geq 0,$$

i.e., $\hat{A}_2$ is monotone. By $(\tilde{H}_2)$, for all $u \in X$,

$$\langle \hat{A}_2(u), u\rangle = \int_\Omega \tilde{a}(\nabla u)\nabla u dx \geq c_1\|u\|^p.$$

Since $p > 1$, $\frac{\langle \hat{A}_2(u),u\rangle}{\|u\|} \geq c_1\|u\|^{p-1} \to +\infty$, as $\|u\| \to +\infty$. Thus $\hat{A}_2$ is coercive. This proof is completed. ■

Theorem 3.1. *Let* $(\tilde{H}_1) - (\tilde{H}_3)$ *be satisfied,* $1 < p < N$ *and* $1 < q < p^*$ *. Suppose that* $\bar{f} : \Omega \times \mathbf{R} \to \mathbf{R}$ *is a continuous function and satisfies*

$$\lim_{|t|\to+\infty} \frac{\bar{f}(x,t)}{\bar{b}(x)t^{(s-1)}} = \bar{\lambda}_0, \tag{3.7}$$

uniformly a.e. with respect to $x \in \Omega$*, where* $\bar{\lambda}_0 \geq 0$, $1 < s < p^*$, $0 \leq \bar{b}(x) \in L^r(\Omega)$, $r = \frac{p^*}{p^*-s}$. *If* $\bar{\tau} : U \to L^{q'}(\Omega)$ $(q' = \frac{q}{q-1})$ *is a strongly continuous mapping. Then, there exists an optimal control* $w_0 \in U$ *for the problem* (3.3).

Proof. Notice that for each $w = (\varphi, \psi) \in U$, $K(w) = \{v \in W_0^{1,p} : \varphi(x) \leq v(x) \leq \psi(x)\ a.e.$ in $\Omega\}$ is a bounded, closed and convex subset of X and K is a bounded mapping. Since $\bar{f}$ satisfies (3.7), it also satisfies (3.6) in Lemma 3.3. By Lemma 3.3, we know $\hat{A}_2$ is a continuous monotone bounded and coercive mapping and $\hat{F}_2$ is a strongly continuous mapping. Thus we know $\hat{F}_2|_{K(U)} = F_2$ is a strongly continuous mapping. From Remark 2.1, it follows that $\hat{A}_2$ is a continuous pseudo-monotone mapping and so $\hat{A}_2|_{K(U)} = A_2$ is a continuous pseudo-monotone mapping. By Lemma 3.2 we know B is a strongly continuous mapping. From Lemma 3.1 it follows that for all $w_n = (\varphi_n, \psi_n) \in U$ with $w_n \rightharpoonup w = (\varphi, \psi)$, $K(w_n)$ Mosco-converges to $K(w)$. The weakly lower semi-continuity of the norm implies that J is weakly lower semicontinuous. In addition, due to the form of J, we know the condition (i) in Corollary 2.5 is satisfied. Therefore, the conclusion of Theorem 3.1 holds by virtue of Corollary 2.5. This proof is completed. ■

Remark 3.2. Theorem 3.1 improves and extends Theorem 2.3 in [21] in the following aspects: (i) if $p = 2$ and $\tau' = 0$, then problem (3.3) becomes the problem (1.6) in [21]; (ii) from the Lemma 2.2 in [21], it is easy to see that our assumptions $(\tilde{H}_1) - (\tilde{H}_3)$ are weaker than the assumptions (H_1) and (H_2) in [21]. (iii) Our proof method is quite different from the one of Theorem 2.3 in [21]. Similarly, Theorem 3.1 also improves and extends Theorem 2 in [14] and Theorem 3.1 in [23]. In addition, the proof method of Theorem 3.1 is different from the ones used in Theorem 2.3 of [13], Theorem 1.1 of [15], Theorem 3.1 of [19], Theorem 3.1 of [20] and Theorem 3.1 of [22].

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