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TWO NONTRIVIAL SOLUTIONS FOR A CLASS OF ANISOTROPIC VARIABLE EXPONENT PROBLEMS

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Abstract. We study an anisotropic problem involving variable exponent growth conditions on a bounded domain $\Omega \subset \mathbb{R}^N$. We prove the existence of at least two nontrivial weak solutions using as main tool a result due to Ricceri.

1. Introduction

Equations involving variable exponent growth conditions have been extensively studied in the last decade. The large number of papers studying problems involving variable exponent growth conditions is motivated by the fact that this type of equations can serve as models in the theory of electrorheological fluids (see, e.g. [29]), image processing (see, e.g. [4]), the theory of elasticity (see, e.g. [35]), biology (see, e.g. [12]), the study of dielectric breakdown, electrical resistivity, and polycrystal plasticity (see, e.g. [1, 3]) or in the study of some models for growth of heterogeneous sandpiles (see, [2]). In this context, we just refer to the survey paper [13] and the references therein.

In this paper we are concerned with the study of the following class of equations

$$(1)\quad \begin{cases} -\sum\limits_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\,\partial_{x_i}u\right) = \lambda f(x,u) + \mu g(x,u) & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $f, g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ are two given functions that satisfy certain properties, p_i are continuous functions on $\overline{\Omega}$ with $2 \leq p_i(x)$ for each $x \in \Omega$ and every $i \in \{1, 2, \ldots, N\}$, $\lambda > 0$, μ are real numbers.

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The differential operator involved in equation (1) is an anisotropic variable exponent operator which represents an extension of the operator $\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p(x)-2}\, \partial_{x_i} u\right)$ obtained in the case when for each $i \in \{1, ..., N\}$ we have $p_i(x) = p(x)$. Even though different from the $p(x)$-Laplace operator, i.e. $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, this last differential operator keeps some of its properties. Thus, the differential operator involved in this article can be regarded as an extension of the $p(x)$-Laplace operator to the anisotropic case. Such kind of operators can be seen as candidates for modeling phenomena which ask for distinct behavior of partial differential derivatives in various directions.

In this paper we will prove the existence of at least two nontrivial weak solutions for problem (1) by using as main tool a three critical point theorem due to Ricceri (see [28, Theorem 2]). Particularly, our result extends to the anisotropic case some earlier results obtained for the $p(x)$-Laplace operator by Mihăilescu [17], Fan and Deng [8], Liu [16] and Ji [14]. The results presented in the above quoted papers are also obtained by applying different critical points theorems due to Ricceri (from [26], [27] or [28]).

2. A Brief Overview on Variable Exponent Spaces

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x)\,.$$

For each $p \in C_+(\overline{\Omega})$, we recall the definition of the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}\,.$$

This space becomes a Banach space [15, Theorem 2.5] with respect to the *Luxemburg norm*, that is

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\, dx \le 1\right\}\,.$$

Moreover, $L^{p(\cdot)}(\Omega)$ is a reflexive space [15, Corollary 2.7] provided that $1 < p^- \le p^+ < \infty$. Furthermore, on such kind of spaces a Hölder type inequality is valid [15, Theorem 2.1]. More exactly, denoting by $L^{q(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for any $x \in \overline{\Omega}$, for each $u \in L^{p(\cdot)}(\Omega)$ and each $v \in L^{q(\cdot)}(\Omega)$ the Hölder type inequality reads as follows

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(\cdot)} |v|_{q(\cdot)}\,. \tag{2}$$

An immediate consequence of Hölder's inequality is connected with some inclusions between various Lebesgue spaces involving variable exponent growth [15, Theorem 2.8]: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents, such that $p_1(x) \le p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx\,,$$

provided that $p^+ < \infty$. Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [5].

We point out some relations which can be established between the Luxemburg norm and the modular. If (u_n), $u \in L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true

$$|u|_{p(\cdot)} < 1\ (> 1,\ = 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1\ (> 1\ = 1)$$

(3) $$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}$$

(4) $$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}$$

(5) $$|u_n - u|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u_n - u) \to 0$$

$$|u|_{p(\cdot)} \to \infty \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) \to \infty$$

$$|u|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) \to 0.$$

Next, we define the *variable exponent Sobolev space* $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(\cdot)}\,.$$

The space $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space, provided that $1 < p^- \le p^+ < \infty$. We recall that if Ω is a bounded, open domain in $\mathbb{R}^N$, $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact and continuous, where $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \ge N$. We refer to [22, 5, 6, 7, 9, 10, 15] for further properties of variable exponent Lebesgue-Sobolev spaces.

Finally, we recall the definition and properties of the anisotropic variable exponent Sobolev space as they were introduced in [21]. With that end in view, we assume in the sequel that Ω is a bounded open domain in $\mathbb{R}^N$ and we denote by $\overrightarrow{p}(\cdot):\overline{\Omega}\to\mathbb{R}^N$ the vectorial function $\overrightarrow{p}(\cdot)=(p_1(\cdot),...,p_N(\cdot))$. We define $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the *anisotropic variable exponent Sobolev space* as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)}=\sum_{i=1}^N|\partial_{x_i}u|_{p_i(\cdot)}\,.$$

In the case when $p_i(\cdot)\in C_+(\overline{\Omega})$ are constant functions for any $i\in\{1,...,N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1,...,p_N)$. The theory of this type of spaces was developed in [11, 23, 24, 25, 32, 33]. It was argued in [21] that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a reflexive Banach space. Also, $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a separable space.

On the other hand, in order to facilitate the manipulation of the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we introduce $\overrightarrow{P}_+$, $\overrightarrow{P}_-$ in $\mathbb{R}^N$ as

$$\overrightarrow{P}_+=(p_1^+,...,p_N^+),\quad \overrightarrow{P}_-=(p_1^-,...,p_N^-),$$

and $P_+^+,P_-^+,P_-^-\in\mathbb{R}^+$ as

$$P_+^+=\max\{p_1^+,...,p_N^+\},\quad P_-^+=\max\{p_1^-,...,p_N^-\},\quad P_-^-=\min\{p_1^-,...,p_N^-\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^N\frac{1}{p_i^-}>1 \tag{6}$$

and define $P_-^\star\in\mathbb{R}^+$ and $P_{-,\infty}\in\mathbb{R}^+$ by

$$P_-^\star=\frac{N}{\sum_{i=1}^N\frac{1}{p_i^-}-1},\quad P_{-,\infty}=\max\{P_-^+,P_-^\star\}.$$

Let us also recall a compactness result that will be essential in our approach (see, [21, Theorem 1] or [20, Proposition 2.1]):

Theorem 1. *Assume that $\Omega\subset\mathbb{R}^N$ ($N\geq 3$) is a bounded domain with smooth boundary. Assume relation (6) is fulfilled. For any $q\in C(\overline{\Omega})$ verifying*

$$1<q(x)<P_{-,\infty}\quad for\ all\ \ x\in\overline{\Omega}, \tag{7}$$

the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact.

For further results, properties and applications regarding anisotropic variable exponent spaces the reader can also consult [18, 19, 30, 31].

3. The Main Result

In this paper, we study problem (1) in the case when $f, g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ such that

$$|f(x,t)|,\ |g(x,t)| \le C(1 + |t|^{\rho(x)-1}) \quad \text{for all } \ (x,t) \in \overline{\Omega} \times \mathbb{R} \tag{8}$$

where C is a positive constant, $\rho : \overline{\Omega} \to (1, \infty)$ is a continuous function and

$$P_+^+ < \rho^- \le \rho^+ < P_{-,\infty}.$$

We seek solutions for problem (1) belonging to $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the sense given below.

Definition 1. We say that $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a *weak solution* of problem (1) if it satisfies

$$\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \, \partial_{x_i} v \, dx = \lambda \int_\Omega f(x,u) v \, dx + \mu \int_\Omega g(x,u) v \, dx \tag{9}$$

for all $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Theorem 2. *Assume that*

$$\max \left\{ \limsup_{t \to 0} \frac{\sup_{x \in \Omega} F(x,t)}{|t|^{P_+^+}},\ \limsup_{|t| \to +\infty} \frac{\sup_{x \in \Omega} F(x,t)}{|t|^{P_-^-}} \right\} \le 0 \tag{10}$$

and

$$\sup_{u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)} \int_\Omega F(x,u) \, dx > 0. \tag{11}$$

Set

$$\gamma = \inf \left\{ \frac{\displaystyle\sum_{i=1}^{N} \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} \, dx}{\displaystyle\int_\Omega F(x,u) \, dx} \ : \ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\ \int_\Omega F(x,u) \, dx > 0 \right\}.$$

Then, for each compact interval $[a,b] \subset (\gamma, +\infty)$, there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$ and every function g, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem (1) has at least three solutions whose norms are less than r.

4. Proof of Theorem 2

Let E denote the anisotropic variable exponent space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Our idea is to apply critical point theory in order to show that the energetic functional associated to problem (1) possesses at least three critical points corresponding to the three solutions of problem (1).

To be more specific, the corresponding energy functional of (1) is $I : E \to \mathbb{R}$ defined by

$$I(u) = \sum_{i=1}^{N} \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\, dx - \lambda \int_\Omega F(x,u)\, dx - \mu \int_\Omega G(x,u)\, dx$$

for all $u \in E$, where $F(x,u) = \displaystyle\int_0^u f(x,s)\, ds,\;\; G(x,u) = \int_0^u g(x,s)\, ds.$

Standard arguments assure that $I \in C^1(E, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle I'(u), v\rangle = \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\, \partial_{x_i} v\, dx - \lambda \int_\Omega f(x,u) v\, dx - \mu \int_\Omega g(x,u) v\, dx,$$

for all $u, v \in E$. Therefore, the weak solutions of problem (1) are exactly the critical points of I.

Further, our idea is to apply a variational principle due to B. Ricceri in order to show that I has three critical points and consequently problem (1) has three weak solutions that verify the properties described in Theorem 2. We recall Ricceri's result below. In order to do that we start by introducing a notation. If X is a real Banach space, we denote by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: if (u_n) is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then (u_n) has a subsequence converging strongly to u.

Theorem 3. [28, Theorem 2]. *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\Phi(x_0) = J(x_0) = 0$. Finally, setting*

$$\alpha = \max\left\{0,\ \limsup_{\|x\|\to\infty} \frac{J(x)}{\Phi(x)},\ \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\right\},$$

$$\beta = \sup_{x\in\Phi^{-1}(0,+\infty)} \frac{J(x)}{\Phi(x)},$$

assume that $\alpha < \beta$.

Then, for every compact interval $[a,b] \subset (\frac{1}{\beta}, \frac{1}{\alpha})$ *(with the conventions* $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$*) there exists* $r > 0$ *with the following property: for every* $\lambda \in [a,b]$ *and every* C^1 *functional* $\Psi : X \to \mathbb{R}$ *with compact derivative, there exits* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$*, the equation*

$$\Phi'(x) = \lambda J'(x) + \mu \Psi'(x)$$

has at least three solutions whose norms are less than r*.*

Define the functionals $\Phi,\ J,\ \Psi : E \to \mathbb{R}$ as

$$\Phi(u) = \sum_{i=1}^{N} \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\, dx,\ \ J(u) = \int_\Omega F(x,u)\, dx,\ \ \Psi(u) = \int_\Omega G(x,u)\, dx,$$

for any $u \in E$. Thus, $I(u) = \Phi(u) - \lambda J(u) - \mu \Psi(u)$ for all $u \in E$.

Standard arguments assure that functionals $\Phi, J, \Psi \in C^1(E, \mathbb{R})$ and their derivatives are

$$\langle \Phi'(u), v\rangle = \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\, \partial_{x_i} v\, dx,\ \ \forall\ u, v \in E,$$
$$\langle J'(u), v\rangle = \int_\Omega f(x,u)\, v\, dx,\ \ \langle \Psi'(u), v\rangle = \int_\Omega g(x,u)\, v\, dx,\ \ \forall\, u, v \in E.$$

Proposition 1. *Functional* Φ *is coercive, convex and bounded on each bounded subset of* E*. Mapping* $\Phi' : E \to E^*$ *is coercive, hemicontinuous, uniformly monotone and satisfies the property:* for any sequence $(u_n) \subset E$ and any $u \in E$ such that (u_n) converges weakly to u in E and $\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \le 0$, we have (u_n) converges strongly to u in E.

Proof. For every $u \in E$ with $\|u\|_{\overrightarrow{p}(\cdot)} > 1$, we define

$$\xi_i = \begin{cases} P_+^+, & \text{if } |\partial_{x_i} u|_{p_i(\cdot)} < 1, \\ P_-^-, & \text{if } |\partial_{x_i} u|_{p_i(\cdot)} > 1. \end{cases}$$

The following inequality

$$\sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{P_-^-} \ge N \left(\frac{\sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}}{N} \right)^{P_-^-} = \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-}}{N^{P_-^- - 1}}$$

holds, for all $u \in E$. Taking into account this inequality, for all $u \in E$ with $\|u\|_{\overrightarrow{p}(\cdot)} > 1$, we get to

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx &\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{\xi_i}\\ &\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{P_-^-} - \sum_{\{i;\,\xi_i=P_+^+\}} |\partial_{x_i} u|_{p_i(\cdot)}^{P_+^+}\\ &\geq \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}} - N\,,\end{aligned}$$

so

$$\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx \geq \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}} - N, \text{ for all } u \in E \text{ with } \|u\|_{\overrightarrow{p}(\cdot)} > 1. \tag{12}$$

Thus

$$\Phi(u) = \sum_{i=1}^{N} \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\,dx \geq \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-}}{P_+^+\, N^{P_-^- -1}} - N, \text{ for all } u \in E \text{ with } \|u\|_{\overrightarrow{p}(\cdot)} > 1,$$

that implies Φ is coercive.

It is obvious that Φ is convex since function $h : [0,\infty) \to \mathbb{R},\ h(t) = t^s$ with $s > 2$ is convex.

Using inequalities (3) and (4) it is easy to see that Φ is bounded on each bounded subset of E.

Next, we show what properties Φ' has. By relation (12) we have

$$\langle \Phi'(u), u\rangle = \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx \geq \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}}, \text{ for all } u \in E \text{ with } \|u\|_{\overrightarrow{p}(\cdot)} > 1,$$

that implies Φ' is coercive. The fact that Φ' is hemicontinuous can be verified using standard arguments.
Next, we show that Φ' is uniformly monotone. It is known that the following inequality

$$\left(|\eta|^{t-2}\eta - |\varrho|^{t-2}\varrho\right)(\eta - \varrho) \geq 2^{-t}|\eta - \varrho|^t, \text{ for all } \eta, \varrho \in \mathbb{R}^N, \tag{13}$$

is valid for all $t \geq 2$. Thus, we deduce that

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \geq \frac{1}{2^{P_+^+}} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i}(u - v)|^{p_i(x)}\,dx, \ \forall\ u, v \in E. \tag{14}$$

We define function $\Theta : [0,\infty) \to [0,\infty)$ by

$$\Theta(s) = \begin{cases} \dfrac{s^{P_+^+-1}}{2^{P_+^+}}, & \text{if } \ s \le 1, \\[2ex] \dfrac{s^{P_-^- -1}}{2^{P_+^+}}, & \text{if } \ s \ge 1. \end{cases}$$

It is easy to check that Θ is an increasing function with $\Theta(0) = 0$ and $\lim\limits_{t\to\infty} \Theta(t) = \infty$. Relation (3), (4) and (14) yield that

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \ge \Theta(\|u - v\|_{\overrightarrow{p}(\cdot)})\ \|u - v\|_{\overrightarrow{p}(\cdot)}, \ \ \forall \ \ u, v \in E,$$

that means Φ' is uniformly monotone.

Let $u \in E$ and $(u_n) \subset E$ such that (u_n) converges weakly to u in E and $\limsup\limits_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \le 0$. Since Φ' is continuous, bounded, strictly monotone, (u_n) weakly converges to u in E and $\limsup\limits_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \le 0$, it follows that

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle = 0,$$

thus

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2}\ \partial_{x_i} u_n\ \partial_{x_i}(u_n - u) = 0.$$

Using the fact that (u_n) weakly converges to u in E, by this equality we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega \Big(|\partial_{x_i} u_n|^{p_i(x)-2}\ \partial_{x_i} u_n - |\partial_{x_i} u|^{p_i(x)-2}\ \partial_{x_i} u\Big)\ \ (\partial_{x_i} u_n - \partial_{x_i} u) = 0.$$

Applying inequality (13) we deduce that

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\ dx = 0,$$

and, consequently, (u_n) strongly converges to u in E. ■

Proof of Theorem 2. From Proposition 1 we deduce that Φ is a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_E$ and bounded on each bounded subset of E. Since Φ' is coercive, hemicontinuous and uniformly monotone on E, using [34, Theorem 26.A (d)] we deduce that the inverse of Φ' is continuous. It is easy to see that J' and Ψ' are strongly continuous. Using Proposition 26.2(a) in [34] it follows that J' and Ψ' are compact. Thus, J, Ψ are C^1 functionals that admit compact derivative. Functional Φ has a strict local minimum at $u = 0$ with $\Phi(0) = J(0) = 0$.

We fix $\epsilon > 0$ arbitrary. By relation (10) we deduce that there exist $r_1, r_2,\ 0 < r_1 < 1 < r_2$ such that $F(x,t) \le \epsilon|t|^{P_+^+}$ for all $(x,t) \in \Omega \times [-r_1, r_1]$ and

$$F(x,t) \le \epsilon|t|^{P_-^-} \ \text{ for all } \ (x,t) \in \Omega \times (\mathbb{R} \setminus [-r_2, r_2]). \tag{15}$$

Thus, we have that $F(x,t) \le \epsilon |t|^{P_+^+}$ for all $(x,t) \in \Omega \times (\mathbb{R} \setminus ([-r_2,-r_1] \cup [r_1,r_2]))$.

Since F is bounded on each bounded subset of $\Omega \times \mathbb{R}$, we can choose a constant $C_\epsilon > 0$ and s with $P_+^+ < s < P_{-,\infty}$ such that

$$F(x,t) \le \epsilon |t|^{P_+^+} + C_\epsilon |t|^s \ \text{ for all } \ (x,t) \in \Omega \times \mathbb{R}. \tag{16}$$

By relation (4), for all $u \in E$ with $\|u\|_{\overrightarrow{p}(\cdot)} < 1$, we obtain

$$\begin{aligned} \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_+^+}}{N^{P_+^+-1}} &= N \left(\frac{\sum\limits_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}}{N} \right)^{P_+^+} \\ &\le \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{P_+^+} \le \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{p_i^+} \le \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, dx \end{aligned}$$

so

$$\Phi(u) \ge \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_+^+}}{P_+^+ \ N^{P_+^+-1}}, \ \text{ for all } \ u \in E \ \text{ with } \ \|u\|_{\overrightarrow{p}(\cdot)} < 1. \tag{17}$$

Using Theorem 1 we obtain by relation (16) that there exist two positive constants C_1, C_2 such that

$$J(u) \le C_1 \|u\|_{\overrightarrow{p}(\cdot)}^{P_+^+} \epsilon + C_2 \|u\|_{\overrightarrow{p}(\cdot)}^{s} \ C_\epsilon$$

and taking into account relation (17) this yields

$$\limsup_{u \to 0} \frac{J(u)}{\Phi(u)} \le C_1 \ P_+^+ \ N^{P_+^+-1} \epsilon. \tag{18}$$

By relations (15) and (12), for all $u \in E$ with $\|u\|_{\overrightarrow{p}(\cdot)} > N$ we have

$$\frac{J(u)}{\Phi(u)} \le P_+^+ \ N^{P_-^- -1} \left(\frac{\displaystyle\int_{\Omega \cap \{|u| \le r_2\}} F(x,u)\, dx}{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - N^{P_-^-}} + \frac{\displaystyle\int_{\Omega \cap \{|u| > r_2\}} F(x,u)\, dx}{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - N^{P_-^-}} \right)$$

so using Theorem 1 we get

$$\limsup_{\|u\|_{\overrightarrow{p}(\cdot)} \to \infty} \frac{J(u)}{\Phi(u)} \le C_3 \ P_+^+ \ N^{P_-^- -1} \epsilon, \tag{19}$$

where C_3 is a positive constant.

Since $\epsilon > 0$ was arbitrary fixed, taking into account relations (18) and (19) we conclude that

$$\max\left\{\limsup_{u\to 0} \frac{J(u)}{\Phi(u)}, \limsup_{\|u\|_{\overrightarrow{p}(\cdot)}\to\infty} \frac{J(u)}{\Phi(u)}\right\} \leq 0.$$

So, α, defined in Theorem 3, is equal to 0. By assumptions we deduce that $\beta > 0$. All the assumptions of Theorem 3 are satisfied, thus we can apply this theorem. Taking $\gamma = \frac{1}{\beta}$, the proof of Theorem 2 is complete. ∎

5. An Application of Theorem 2

Corollary 1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary, p_i continuous functions on $\overline{\Omega}$ and $2 \leq p_i(x)$ for each $x \in \Omega$ and every $i \in \{1, 2, \ldots, N\}$, $\lambda > 0$, μ real numbers, C_1, C_2 two positive constants, $a, b, c, d, h : \overline{\Omega} \to \mathbb{R}$ continuous functions such that*

$$a(x) > 0 \text{ for every } x \in \overline{\Omega}, \tag{20}$$

$$P_+^+ < b^- \leq b^+ < d^- \leq d^+ < P_{-,\infty} \tag{21}$$

and

$$P_+^+ < c(x) < h(x) < P_{-,\infty}$$

for every $x \in \overline{\Omega}$. Set

$$\gamma = \inf\left\{\frac{\sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\,dx}{\int_\Omega a(x)\left(\frac{C_1|u|^{b(x)}}{b(x)} - \frac{C_2|u|^{d(x)}}{d(x)}\right)dx} : u \in E,\ \int_\Omega a(x)\left(\frac{C_1|u|^{b(x)}}{b(x)} - \frac{C_2|u|^{d(x)}}{d(x)}\right)dx > 0\right\},$$

where $E = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Then, for each compact interval $[a, b] \subset (\gamma, +\infty)$, there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and for function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(x,t) = \begin{cases} |t|^{c(x)-2}t, & \text{for } |t| \leq 1, \\ |t|^{h(x)-2}t, & \text{for } |t| \geq 1, \end{cases} \tag{22}$$

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\, \partial_{x_i} u\right) \\ \quad = \lambda a(x)\left(C_1|u|^{b(x)-2}u - C_2|u|^{d(x)-2}u\right) + \mu g(x,u) & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega. \end{cases} \tag{23}$$

has at least three solutions whose norms are less than r.

Proof. We consider $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,t) = a(x)\left(C_1|t|^{b(x)-2}t - C_2|t|^{d(x)-2}t\right) \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

It is clear that f and g defined by relation (22) verify condition (8). We will show that function f satisfies conditions (10) and (11) by Theorem 2.

Function $F(x,t) = \displaystyle\int_0^t f(x,s)\, ds$ is

$$F(x,t) = a(x)\left(\frac{C_1|t|^{b(x)}}{b(x)} - \frac{C_2|t|^{d(x)}}{d(x)}\right).$$

For each $t > 0$ there exists $x_t \in \overline{\Omega}$ depending on t such that $\sup\limits_{x \in \overline{\Omega}} F(x,t) = F(x_t,t)$. We evaluate

$$\begin{aligned}
\frac{\sup\limits_{x\in\overline{\Omega}} F(x,t)}{|t|^{P_+^+}} &= \frac{F(x_t,t)}{|t|^{P_+^+}} \\
&= a(x_t)\left(\frac{C_1|t|^{b(x_t)-P_+^+}}{b(x_t)} - \frac{C_2|t|^{d(x_t)-P_+^+}}{d(x_t)}\right) \\
&\le a(x_t)|t|^{b(x_t)-P_+^+}\left(\frac{C_1}{b^-} - \frac{C_2|t|^{d(x_t)-b(x_t)}}{d^+}\right).
\end{aligned}$$

Since relations (20), (21) hold, for $|t|$ small enough, we have

$$\limsup_{t\to 0} \frac{\sup\limits_{x\in\overline{\Omega}} F(x,t)}{|t|^{P_+^+}} = 0. \tag{24}$$

Also, we evaluate

$$\begin{aligned}
\frac{\sup\limits_{x\in\overline{\Omega}} F(x,t)}{|t|^{P_-^-}} &= \frac{F(x_t,t)}{|t|^{P_-^-}} \\
&= a(x_t)\left(\frac{C_1|t|^{b(x_t)-P_-^-}}{b(x_t)} - \frac{C_2|t|^{d(x_t)-P_-^-}}{d(x_t)}\right) \\
&\le a(x_t)|t|^{b(x_t)-P_-^-}\left(\frac{C_1}{b^-} - \frac{C_2|t|^{d(x_t)-b(x_t)}}{d^+}\right).
\end{aligned}$$

Since relations (20), (21) hold, for $|t|$ large enough, we get

$$a(x_t)|t|^{b(x_t)-P_-^-}\left(\frac{C_1}{b^-}-\frac{C_2|t|^{d(x_t)-b(x_t)}}{d^+}\right)\leq 0,$$

thus

$$\limsup_{|t|\to+\infty}\frac{\sup_{x\in\overline{\Omega}}F(x,t)}{|t|^{P_-^-}}\leq 0. \tag{25}$$

By (24) and (25), we deduce that condition (10) is true.

Taking into account the assumptions of our corollary, for t small enough and constant, we obtain

$$\int_\Omega a(x)\left(\frac{C_1|t|^{b(x)}}{b(x)}-\frac{C_2|t|^{d(x)}}{d(x)}\right)\,dx>0.$$

Thus, function f satisfies the hypothesis in Theorem 2 and the conclusion holds. The proof of Corollary 1 is complete. ■

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