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RELAXED PROJECTION-VISCOSITY APPROXIMATION METHOD

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Abstract. The purpose of this paper is to investigate the problem of finding a common element of the solution set of a general system of variational inequalities, the solution set of a variational inequality problem and the fixed point set of a strict pseudocontraction in a real Hilbert space. Based on the well-known viscosity approximation method, extragradient method and Mann's iteration method, we propose and analyze a relaxed projection-viscosity approximation method for computing a common element. Under very mild assumptions, we obtain a strong convergence theorem for three sequences generated by the proposed method. Our proposed method is quite general and flexible and develops some iterative methods considered in the earlier and recent literature.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and $\mathbf{R}$ be the set of all real numbers. For a given nonlinear mapping $A : C \to H$, consider the following classical variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by $VI(A, C)$. It is now well known that a variational inequality problem is equivalent to a fixed-point problem, the origin of which can be traced back to Lions and Stampacchia (see, e.g., [8]). This alternative formulation has been used to suggest and analyze Picard successive iterative method

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for solving variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous.

Let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by P_C the metric projection of H onto C.

A mapping $Q : C \to C$ is said to be a ρ-contraction if $\rho \in [0, 1)$ and

$$\|Qx - Qy\| \leq \rho\|x - y\|, \quad \forall x, y \in C.$$

A mapping $S : C \to C$ is called k-strictly pseudo-contractive if $0 \leq k < 1$ and

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

In this case, we also say that S is a k-strict pseudocontraction. In particular, whenever $k = 0,\ S$ becomes a nonexpansive self-mapping on C.

A mapping $A : C \to H$ is called α-inverse strongly monotone if $\alpha > 0$ and

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.3}$$

It is easy to see that every inverse strongly monotone mapping is a monotone and Lipschitz continuous mapping; see, e.g., [8].

Recently, Nadezhkina and Takahashi [9] and Zeng and Yao [10], motivated by the idea of Korpelevich [11] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality, proposed the so-called extragradient method. Further, these iterative methods were extended in [12] to develop a general iterative method for finding an element of $\mathrm{Fix}(S) \cap VI(A, C)$.

Let $B_1, B_2 : C \to H$ be two mappings. We consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.4}$$

which is called a general system of variational inequalities, where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. It was introduced and considered by Ceng, Wang and Yao [15]. Moreover, it was transformed into a fixed point problem in [15] in the following way.

Lemma 1.1. (see [15]). *For given $\bar{x}, \bar{y} \in C,\ (\bar{x}, \bar{y})$ is a solution of problem (1.4) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C, \tag{1.5}$$

and $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

In particular, if the mapping $B_i : C \to H$ is $\widehat{\beta}_i$-inverse strongly monotone for $i \in \{1, 2\}$, then the mapping G is nonexpansive, provided $\mu_i \in (0, 2\widehat{\beta}_i)$ for $i \in \{1, 2\}$.

Utilizing Lemma 1.1, Ceng, Wang and Yao proposed and analyzed in [15] a relaxed extragradient method for solving problem (1.4). Throughout this paper, the set of fixed points of the mapping G is denoted by Γ. Based on the extragradient method [11] and viscosity approximation method [14], Yao, Liou and Kang [1] introduced and studied (see Theorem 3.2 in [1]) a relaxed extragradient iterative algorithm for finding a common solution of problem (1.4) and the fixed point problem of a strict pseudocontraction in a real Hilbert space H.

Subsequently, Ceng, Ansari and Yao [8] also introduced and considered a new relaxed extragradient iterative algorithm (Theorem 3.1 in [8]) for finding a common solution of problem (1.1), problem (1.4), and the fixed point problem of a strict pseudocontraction in a real Hilbert space H.

Assume that $A : C \to H$ is α-inverse strongly monotone and $B_i : C \to H$ is $\widehat{\beta}_i$-inverse strongly monotone for $i \in \{1, 2\}$. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \cap VI(A, C) \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. Motivated and inspired by the research work going on in this area, we propose and analyze the following relaxed projection-viscosity approximation method for finding an element in Ω:

Given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{u_n\}, \{\tilde{u}_n\}$ be generated iteratively by

$$(1.6)\quad \begin{cases} u_n = \alpha_n Q x_n + (1-\alpha_n) P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ \tilde{u}_n = P_C(u_n - \lambda_n A u_n), \\ y_n = P_C(u_n - \lambda_n A \tilde{u}_n), \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n + (1 - \beta_n - \gamma_n - \delta_n) u_n, \quad \forall n \geq 0, \end{cases}$$

where $\mu_i \in (0, 2\widehat{\beta}_i)$ for $i \in \{1, 2\}$, $\{\lambda_n\} \subset (0, \alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\beta_n + \gamma_n + \delta_n \leq 1$ for all $n \geq 0$.

In this paper, it is proven that the sequences $\{x_n\}, \{u_n\}, \{\tilde{u}_n\}$ converge strongly to the same point $\bar{x} = P_\Omega Q \bar{x}$ under very mild conditions. Furthermore, $(\bar{x}, \bar{y})$ is a solution of the general system (1.4) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Our result supplements, extends and improves the corresponding theorems in the earlier and recent literature, see, for instance, Yao, Liou and Kang [1, Theorem 3.2] and Ceng, Ansari and Yao [8, Theorem 3.1].

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. We write $\to$ to indicate that the sequence $\{x_n\}$ converges strongly to x and $\rightharpoonup$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}. \tag{2.1}$$

It is obvious that, inequality (1.2) is equivalent to the following inequality:

$$\langle Sx - Sy, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

It is easy to see that if S is a k-strictly pseudo-contractive mapping, then the mapping $I - S$ is $\frac{1-k}{2}$-inverse strongly monotone and hence $\frac{2}{1-k}$-Lipschitz continuous. For further details, we refer to [6] and the references therein.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall x \in C.$$

The mapping P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C, that is, there holds the following relation

$$\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: for each $x \in H$ we have $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \ge 0, \tag{2.3}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.4}$$

for all $x \in H, y \in C$, see [4,5,7] for more details.

In order to prove our main result of this paper, we need the following lemmas. The following lemma is an immediate consequence of the definition of an inner product.

Lemma 2.1. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.5}$$

The following lemma was proved by Suzuki in [13].

Lemma 2.2. (see [13]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Recall that $S : C \to C$ is called a quasi-strict pseudocontraction if $\mathrm{Fix}(S)$ is nonempty and there exists a constant $0 \le k < 1$ such that

$$\|Sx - p\|^2 \le \|x - p\|^2 + k\|x - Sx\|^2 \quad \text{for all } x \in C \text{ and } p \in \mathrm{Fix}(S). \tag{2.6}$$

We also say that S is a k-quasi-strict pseudocontraction if condition (2.6) holds.

Lemma 2.3. (see [16, Proposition 2.1]). *Assume C is a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a self-mapping on C.*

(i) *If S is a k-strict pseudo-contractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C. \tag{2.7}$$

(ii) *If S is a k-strict pseudo-contractive mapping, then the mapping $I - S$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, then $(I - S)\tilde{x} = 0$, i.e., $\tilde{x} \in \mathrm{Fix}(S)$.*

(iii) *If S is a k-quasi-strict pseudocontraction, then the fixed-point set $\mathrm{Fix}(S)$ of S is closed and convex so that the projection $P_{\mathrm{Fix}(S)}$ is well defined.*

Some other auxiliary results are the following.

Lemma 2.4. (see [3]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq (1 - \delta_n)a_n + \delta_n\sigma_n, \quad \forall n \geq 0,$$

where $\{\delta_n\}, \{\sigma_n\}$ are sequences of real numbers such that

(i) $\{\delta_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \delta_n = \infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1 - \delta_n) := \lim_{n\to\infty} \prod_{j=0}^{n}(1 - \delta_j) = 0;$$

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$, *or*

(*ii'*) *′* $\sum_{n=0}^{\infty} \delta_n\sigma_n$ *is convergent.*

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. (see [1, Lemma 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a k-strictly pseudo-contractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)k \leq \gamma$. Then*

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C. \tag{2.8}$$

3. Main Results

We are now in a position to state and prove our main result.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone and $B_i : C \to H$ be $\widehat{\beta}_i$-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a k-strictly pseudo-contractive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \cap VI(A, C) \neq \emptyset$. Let $Q : C \to C$*

be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{u_n\}, \{\tilde{u}_n\}$ be generated iteratively by

$$\begin{cases} u_n = \alpha_n Q x_n + (1-\alpha_n) P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ \tilde{u}_n = P_C(u_n - \lambda_n A u_n), \\ y_n = P_C(u_n - \lambda_n A \tilde{u}_n), \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n + (1-\beta_n-\gamma_n-\delta_n) u_n, \quad \forall n \geq 0, \end{cases}$$

where $\mu_i \in (0, 2\widehat{\beta}_i)$ for $i \in \{1,2\}$, $\{\lambda_n\} \subset (0, \alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ such that

(*i*) $\beta_n + \gamma_n + \delta_n \leq 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$*;*

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(*iv*) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$ *and* $\lim_{n\to\infty}(\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}) = 0$*;*

(*v*) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequences $\{x_n\}, \{u_n\}, \{\tilde{u}_n\}$ converge strongly to the same point $\bar{x} = P_\Omega Q \bar{x}$ if and only if $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Furthermore, $(\bar{x}, \bar{y})$ is a solution of the general system (1.4) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, let $x^* \in \Omega := \mathrm{Fix}(S) \cap \Gamma \cap VI(A, C)$. Then $Sx^* = x^*,\ x^* = P_C(x^* - \lambda_n A x^*)$ and

$$x^* = P_C[P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)].$$

Since $A : C \to H$ is α-inverse strongly monotone and $0 < \lambda_n \leq \alpha$, we have for all $n \geq 0$,

$$\begin{aligned} \|\tilde{u}_n - x^*\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(x^* - \lambda_n A x^*)\|^2 \\ &\leq \|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^2 \\ &= \|(u_n - x^*) - \lambda_n (A u_n - A x^*)\|^2 \\ &\leq \|u_n - x^*\|^2 - \lambda_n (2\alpha - \lambda_n) \|A u_n - A x^*\|^2 \\ &\leq \|u_n - x^*\|^2. \end{aligned} \tag{3.2}$$

For simplicity, we write

$$\begin{aligned} y^* &= P_C(x^* - \mu_2 B_2 x^*),\ \tilde{x}_n = P_C(x_n - \mu_2 B_2 x_n),\ \bar{x}_n \\ &= P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \end{aligned}$$

for each $n \geq 0$. Then $u_n = \alpha_n Q x_n + (1-\alpha_n)\bar{x}_n$ for each $n \geq 0$. Since $B_i : C \to H$ is $\widehat{\beta}_i$-inverse strongly monotone and $0 < \mu_i < 2\widehat{\beta}_i$ for $i \in \{1, 2\}$, we know that for all $n \geq 0$,

$$
\begin{aligned}
&\|\bar{x}_n - x^*\|^2 = \|P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] - x^*\|^2 \\
&= \|P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\
&\quad - P_C[P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&\leq \|[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\
&\quad - [P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&= \|[P_C(x_n - \mu_2 B_2 x_n) - P_C(x^* - \mu_2 B_2 x^*)] \\
&\quad - \mu_1 [B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&\leq \|P_C(x_n - \mu_2 B_2 x_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad - \mu_1(2\widehat{\beta}_1 - \mu_1)\|B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\leq \|(x_n - \mu_2 B_2 x_n) - (x^* - \mu_2 B_2 x^*)\|^2 - \mu_1(2\widehat{\beta}_1 - \mu_1)\|B_1 \tilde{x}_n - B_1 y^*\|^2 \\
&= \|(x_n - x^*) - \mu_2(B_2 x_n - B_2 x^*)\|^2 - \mu_1(2\widehat{\beta}_1 - \mu_1)\|B_1 \tilde{x}_n - B_1 y^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \mu_2(2\widehat{\beta}_2 - \mu_2)\|B_2 x_n - B_2 x^*\|^2 \\
&\quad - \mu_1(2\widehat{\beta}_1 - \mu_1)\|B_1 \tilde{x}_n - B_1 y^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}
\tag{3.3}
$$

Furthermore, it follows from (2.4) that

$$
\begin{aligned}
&\|y_n - x^*\|^2 \\
&\leq \|u_n - \lambda_n A\tilde{u}_n - x^*\|^2 - \|u_n - \lambda_n A\tilde{u}_n - y_n\|^2 \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n \langle A\tilde{u}_n, x^* - y_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n \langle A\tilde{u}_n - Ax^*, x^* - \tilde{u}_n\rangle \\
&\quad + 2\lambda_n \langle Ax^*, x^* - \tilde{u}_n\rangle + 2\lambda_n \langle A\tilde{u}_n, \tilde{u}_n - y_n\rangle \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\lambda_n \alpha \|A\tilde{u}_n - Ax^*\|^2 + 2\lambda_n \langle A\tilde{u}_n, \tilde{u}_n - y_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - \tilde{u}_n\|^2 - 2\langle u_n - \tilde{u}_n, \tilde{u}_n - y_n\rangle - \|\tilde{u}_n - y_n\|^2 \\
&\quad + 2\lambda_n \alpha \|A\tilde{u}_n - Ax^*\|^2 + 2\lambda_n \langle A\tilde{u}_n, \tilde{u}_n - y_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - \tilde{u}_n\|^2 - \|\tilde{u}_n - y_n\|^2 - 2\lambda_n \alpha \|A\tilde{u}_n - Ax^*\|^2 \\
&\quad + 2\langle u_n - \lambda_n A\tilde{u}_n - \tilde{u}_n, y_n - \tilde{u}_n\rangle,
\end{aligned}
$$

where the second inequality follows from the assumption that A is α-inverse strongly monotone and x^* is a solution of problem (1.1). By using (2.3), we have

$$\begin{aligned}
&\langle u_n - \lambda_n A\tilde{u}_n - \tilde{u}_n, y_n - \tilde{u}_n\rangle \\
&= \langle u_n - \lambda_n A u_n - \tilde{u}_n, y_n - \tilde{u}_n\rangle + \langle \lambda_n A u_n - \lambda_n A\tilde{u}_n, y_n - \tilde{u}_n\rangle \\
&\le \langle \lambda_n A u_n - \lambda_n A\tilde{u}_n, y_n - \tilde{u}_n\rangle \\
&\le \frac{\lambda_n}{\alpha}\|u_n - \tilde{u}_n\|\|y_n - \tilde{u}_n\|.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|u_n - \tilde{u}_n\|^2 - \|\tilde{u}_n - y_n\|^2 - 2\lambda_n\alpha\|A\tilde{u}_n - Ax^*\|^2 \\
&\quad + 2\frac{\lambda_n}{\alpha}\|u_n - \tilde{u}_n\|\|y_n - \tilde{u}_n\| \\
&\le \|u_n - x^*\|^2 - \|u_n - \tilde{u}_n\|^2 - \|\tilde{u}_n - y_n\|^2 - 2\lambda_n\alpha\|A\tilde{u}_n - Ax^*\|^2 \\
&\quad + \frac{\lambda_n^2}{\alpha^2}\|u_n - \tilde{u}_n\|^2 + \|y_n - \tilde{u}_n\|^2 \\
&= \|u_n - x^*\|^2 - (1 - \frac{\lambda_n^2}{\alpha^2})\|u_n - \tilde{u}_n\|^2 - 2\lambda_n\alpha\|A\tilde{u}_n - Ax^*\|^2 \\
&\le \|u_n - x^*\|^2.
\end{aligned} \tag{3.4}$$

In addition, it follows from (3.3) that

$$\begin{aligned}
\|u_n - x^*\| &= \|\alpha_n(Qx_n - x^*) + (1 - \alpha_n)(\bar{x}_n - x^*)\| \\
&\le \alpha_n\|Qx_n - x^*\| + (1 - \alpha_n)\|\bar{x}_n - x^*\| \\
&\le \alpha_n(\rho\|x_n - x^*\| + \|Qx^* - x^*\|) + (1 - \alpha_n)\|x_n - x^*\| \\
&= (1 - (1 - \rho)\alpha_n)\|x_n - x^*\| + (1 - \rho)\alpha_n\frac{\|Qx^* - x^*\|}{1 - \rho} \\
&\le \max\{\|x_n - x^*\|, \frac{\|Qx^* - x^*\|}{1 - \rho}\}.
\end{aligned} \tag{3.5}$$

Since $(\gamma_n + \delta_n)k \le \gamma_n$ for all $n \ge 0$, utilizing Lemma 2.5 we obtain from (3.4) and (3.5)

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*) \\
&\quad + (1 - \beta_n - \gamma_n - \delta_n)(u_n - x^*)\| \\
&\le \beta_n\|x_n - x^*\| + \|\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\| \\
&\quad + (1 - \beta_n - \gamma_n - \delta_n)\|u_n - x^*\| \\
&\le \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\|y_n - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n - \delta_n)\|u_n - x^*\| \\
&\le \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\|u_n - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n - \delta_n)\|u_n - x^*\|
\end{aligned} \tag{3.6}$$

$$= \beta_n\|x_n - x^*\| + (1-\beta_n)\|u_n - x^*\|$$
$$\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\max\{\|x_n - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\}$$
$$\leq \max\{\|x_n - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\}.$$

By induction, we conclude that for all $n \geq 0$

$$\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\}.$$

Hence, $\{x_n\}$ is bounded. Since P_C, Q, S, A, B_1 and B_2 are Lipschitz continuous, it is easy to see that $\{u_n\}, \{\tilde{u}_n\}, \{y_n\}, \{\bar{x}_n\}$ and $\{\tilde{x}_n\}$ are bounded, where $\tilde{x}_n = P_C(x_n - \mu_2 B_2 x_n)$ and $\bar{x}_n = P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)]$ for all $n \geq 0$.

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Indeed, define $x_{n+1} = \beta_n x_n + (1-\beta_n)w_n$ for all $n \geq 0$. It follows that

$$\begin{aligned}
&w_{n+1} - w_n \\
&= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\gamma_{n+1}y_{n+1} + \delta_{n+1}Sy_{n+1} + (1-\beta_{n+1}-\gamma_{n+1}-\delta_{n+1})u_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\gamma_n y_n + \delta_n S y_n + (1-\beta_n-\gamma_n-\delta_n)u_n}{1-\beta_n} \\
&= \frac{\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Sy_{n+1} - Sy_n)}{1-\beta_{n+1}} + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})y_n \\
&\quad + (\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n})Sy_n + \frac{1-\beta_{n+1}-\gamma_{n+1}-\delta_{n+1}}{1-\beta_{n+1}}u_{n+1} - \frac{1-\beta_n-\gamma_n-\delta_n}{1-\beta_n}u_n \\
&= \frac{\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Sy_{n+1} - Sy_n)}{1-\beta_{n+1}} + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})y_n \\
&\quad + (\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n})Sy_n + \frac{1-\beta_{n+1}-\gamma_{n+1}-\delta_{n+1}}{1-\beta_{n+1}}(u_{n+1} - u_n) \\
&\quad - (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})u_n - (\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n})u_n.
\end{aligned} \tag{3.7}$$

Since $(\gamma_n + \delta_n)k \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.5 we have

$$\|\gamma_{n+1}(y_{n+1} - y_n) + \delta_{n+1}(Sy_{n+1} - Sy_n)\| \leq (\gamma_{n+1} + \delta_{n+1})\|y_{n+1} - y_n\|. \tag{3.8}$$

Next, we estimate $\|y_{n+1} - y_n\|$. Observe that

$$\begin{aligned}
\|\tilde{u}_{n+1} - \tilde{u}_n\| &= \|P_C(u_{n+1} - \lambda_{n+1}Au_{n+1}) - P_C(u_n - \lambda_n A u_n)\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_n A u_n)\| \\
&= \|(u_{n+1} - u_n) - \lambda_{n+1}(Au_{n+1} - Au_n) + (\lambda_n - \lambda_{n+1})Au_n\| \\
&\leq \|(u_{n+1} - u_n) - \lambda_{n+1}(Au_{n+1} - Au_n)\| + |\lambda_{n+1} - \lambda_n|\|Au_n\| \\
&\leq \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|Au_n\|,
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&\|\bar{x}_{n+1}-\bar{x}_n\|^2 \\
&= \|P_C[P_C(x_{n+1}-\mu_2 B_2 x_{n+1})-\mu_1 B_1 P_C(x_{n+1}-\mu_2 B_2 x_{n+1})] \\
&\quad -P_C[P_C(x_n-\mu_2 B_2 x_n)-\mu_1 B_1 P_C(x_n-\mu_2 B_2 x_n)]\|^2 \\
&\le \|[P_C(x_{n+1}-\mu_2 B_2 x_{n+1})-\mu_1 B_1 P_C(x_{n+1}-\mu_2 B_2 x_{n+1})] \\
&\quad -[P_C(x_n-\mu_2 B_2 x_n)-\mu_1 B_1 P_C(x_n-\mu_2 B_2 x_n)]\|^2 \\
&= \|[P_C(x_{n+1}-\mu_2 B_2 x_{n+1})-P_C(x_n-\mu_2 B_2 x_n)] \\
&\quad -\mu_1[B_1 P_C(x_{n+1}-\mu_2 B_2 x_{n+1})-B_1 P_C(x_n-\mu_2 B_2 x_n)]\|^2 \\
&\le \|P_C(x_{n+1}-\mu_2 B_2 x_{n+1})-P_C(x_n-\mu_2 B_2 x_n)\|^2 \\
&\quad -\mu_1(2\widehat{\beta}_1-\mu_1)\|B_1 P_C(x_{n+1}-\mu_2 B_2 x_{n+1})-B_1 P_C(x_n-\mu_2 B_2 x_n)\|^2 \\
&\le \|P_C(x_{n+1}-\mu_2 B_2 x_{n+1})-P_C(x_n-\mu_2 B_2 x_n)\|^2 \\
&\le \|(x_{n+1}-\mu_2 B_2 x_{n+1})-(x_n-\mu_2 B_2 x_n)\|^2 \\
&= \|(x_{n+1}-x_n)-\mu_2(B_2 x_{n+1}-B_2 x_n)\|^2 \\
&\le \|x_{n+1}-x_n\|^2-\mu_2(2\widehat{\beta}_2-\mu_2)\|B_2 x_{n+1}-B_2 x_n\|^2 \\
&\le \|x_{n+1}-x_n\|^2,
\end{aligned}
\tag{3.10}
$$

and hence

$$
\begin{aligned}
\|u_{n+1}-u_n\| &= \|\bar{x}_{n+1}+\alpha_{n+1}(Qx_{n+1}-\bar{x}_{n+1})-\bar{x}_n-\alpha_n(Qx_n-\bar{x}_n)\| \\
&\le \|\bar{x}_{n+1}-\bar{x}_n\|+\alpha_{n+1}\|Qx_{n+1}-\bar{x}_{n+1}\|+\alpha_n\|Qx_n-\bar{x}_n\| \\
&\le \|x_{n+1}-x_n\|+\alpha_{n+1}\|Qx_{n+1}-\bar{x}_{n+1}\|+\alpha_n\|Qx_n-\bar{x}_n\|.
\end{aligned}
\tag{3.11}
$$

Also, from (3.11) we get

$$
\begin{aligned}
&\|y_{n+1}-y_n\| \\
&= \|P_C(u_{n+1}-\lambda_{n+1}A\tilde{u}_{n+1})-P_C(u_n-\lambda_n A\tilde{u}_n)\| \\
&\le \|(u_{n+1}-\lambda_{n+1}A\tilde{u}_{n+1})-(u_n-\lambda_n A\tilde{u}_n)\| \\
&= \|(u_{n+1}-\lambda_{n+1}Au_{n+1})-(u_n-\lambda_{n+1}Au_n) \\
&\quad +\lambda_{n+1}(Au_{n+1}-Au_n)-\lambda_{n+1}A\tilde{u}_{n+1}+\lambda_n A\tilde{u}_n\| \\
&\le \|(u_{n+1}-\lambda_{n+1}Au_{n+1})-(u_n-\lambda_{n+1}Au_n)\|+\lambda_{n+1}\|Au_{n+1}-Au_n\| \\
&\quad +\|\lambda_{n+1}A\tilde{u}_{n+1}-\lambda_n A\tilde{u}_n\| \\
&\le \|u_{n+1}-u_n\|+\tfrac{\lambda_{n+1}}{\alpha}\|u_{n+1}-u_n\|+\lambda_{n+1}\|A\tilde{u}_{n+1}-A\tilde{u}_n\| \\
&\quad +|\lambda_{n+1}-\lambda_n|\|A\tilde{u}_n\| \\
&\le \|u_{n+1}-u_n\|+\tfrac{\lambda_{n+1}}{\alpha}(\|u_{n+1}-u_n\|+\|\tilde{u}_{n+1}-\tilde{u}_n\|)+|\lambda_{n+1}-\lambda_n|\|A\tilde{u}_n\| \\
&\le \|x_{n+1}-x_n\|+\tfrac{\lambda_{n+1}}{\alpha}(\|u_{n+1}-u_n\|+\|\tilde{u}_{n+1}-\tilde{u}_n\|)+|\lambda_{n+1}-\lambda_n|\|A\tilde{u}_n\| \\
&\quad +\alpha_{n+1}\|Qx_{n+1}-\bar{x}_{n+1}\|+\alpha_n\|Qx_n-\bar{x}_n\|.
\end{aligned}
\tag{3.12}
$$

Hence it follows from (3.7), (3.8) and (2.12) that

$$\begin{aligned}
&\|w_{n+1}-w_n\| \\
\le\ & \tfrac{\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|}{1-\beta_{n+1}}+|\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}-\tfrac{\gamma_n}{1-\beta_n}|\|y_n\| \\
&+|\tfrac{\delta_{n+1}}{1-\beta_{n+1}}-\tfrac{\delta_n}{1-\beta_n}|\|Sy_n\|+\tfrac{1-\beta_{n+1}-\gamma_{n+1}-\delta_{n+1}}{1-\beta_{n+1}}\|u_{n+1}-u_n\| \\
&+|\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}-\tfrac{\gamma_n}{1-\beta_n}|\|u_n\|+|\tfrac{\delta_{n+1}}{1-\beta_{n+1}}-\tfrac{\delta_n}{1-\beta_n}|\|u_n\| \\
\le\ & \tfrac{\gamma_{n+1}+\delta_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\|+|\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}-\tfrac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|u_n\|) \\
&+|\tfrac{\delta_{n+1}}{1-\beta_{n+1}}-\tfrac{\delta_n}{1-\beta_n}|(\|Sy_n\|+\|u_n\|)+\|u_{n+1}-u_n\| \\
\le\ & \|y_{n+1}-y_n\|+|\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}-\tfrac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|u_n\|) \\
&+|\tfrac{\delta_{n+1}}{1-\beta_{n+1}}-\tfrac{\delta_n}{1-\beta_n}|(\|Sy_n\|+\|u_n\|)+\|u_{n+1}-u_n\| \\
\le\ & \|x_{n+1}-x_n\|+\tfrac{\lambda_{n+1}}{\alpha}(\|u_{n+1}-u_n\|+\|\tilde{u}_{n+1}-\tilde{u}_n\|)+|\lambda_{n+1}-\lambda_n|\|A\tilde{u}_n\| \\
&+\alpha_{n+1}\|Qx_{n+1}-\bar{x}_{n+1}\|+\alpha_n\|Qx_n-\bar{x}_n\|+|\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}-\tfrac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|u_n\|) \\
&+|\tfrac{\delta_{n+1}}{1-\beta_{n+1}}-\tfrac{\delta_n}{1-\beta_n}|(\|Sy_n\|+\|u_n\|)+\|u_{n+1}-u_n\|.
\end{aligned}$$

In terms of (3.9) we deduce from condition (v) that $\lim_{n\to\infty}\|\tilde{u}_{n+1}-\tilde{u}_n\|=0$. Since $\{x_n\},\{u_n\}$, $\{\tilde{u}_n\},\{y_n\}$ and $\{\bar{x}_n\}$ are bounded, it follows from conditions (ii), (iv), (v) that

$$\begin{aligned}
&\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|) \\
\le\ & \limsup_{n\to\infty}\{\frac{\lambda_{n+1}}{\alpha}(\|u_{n+1}-u_n\|+\|\tilde{u}_{n+1}-\tilde{u}_n\|)+|\lambda_{n+1}-\lambda_n|\|A\tilde{u}_n\| \\
&+\alpha_{n+1}\|Qx_{n+1}-\bar{x}_{n+1}\|+\alpha_n\|Qx_n-\bar{x}_n\|+|\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}-\tfrac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|u_n\|) \\
&+|\tfrac{\delta_{n+1}}{1-\beta_{n+1}}-\tfrac{\delta_n}{1-\beta_n}|(\|Sy_n\|+\|u_n\|)+\|u_{n+1}-u_n\|\} \\
=\ & 0.
\end{aligned}$$

So, by Lemma 2.2 we obtain $\lim_{n\to\infty}\|w_n-x_n\|=0$. Thus,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.13}$$

Step 3. $\lim_{n\to\infty}\|B_2x_n-B_2x^*\|=0$, $\lim_{n\to\infty}\|B_1\tilde{x}_n-B_1y^*\|=0$ and $\lim_{n\to\infty}\|Au_n-Ax^*\|=0$, where $y^*=P_C(x^*-\mu_2B_2x^*)$.

Indeed, utilizing Lemma 2.5 and the convexity of $\|\cdot\|^2$, we get from (2.8), (3.3) and (3.4)

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&=\|\beta_n(x_n-x^*)+\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)+(1-\beta_n-\gamma_n-\delta_n)(u_n-x^*)\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)\|\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)\|^2\\
&\quad +(1-\beta_n-\gamma_n-\delta_n)\|u_n-x^*\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)\|y_n-x^*\|^2+(1-\beta_n-\gamma_n-\delta_n)\|u_n-x^*\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)[\|u_n-x^*\|^2\\
&\quad -(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2-2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2]+(1-\beta_n-\gamma_n-\delta_n)\|u_n-x^*\|^2\\
&=\beta_n\|x_n-x^*\|^2+(1-\beta_n)\|u_n-x^*\|^2\\
&\quad -(\gamma_n+\delta_n)[(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2+2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2]\\
&=\beta_n\|x_n-x^*\|^2+(1-\beta_n)\|\alpha_n(Qx_n-x^*)\\
&\quad +(1-\alpha_n)(\bar{x}_n-x^*)\|^2-(\gamma_n+\delta_n)[(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2+2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2]\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)[\alpha_n\|Qx_n-x^*\|^2+(1-\alpha_n)\|\bar{x}_n-x^*\|^2]\\
&\quad -(\gamma_n+\delta_n)[(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2+2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2]\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|\bar{x}_n-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2\\
&\quad -(\gamma_n+\delta_n)[(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2+2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2]\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)[\|x_n-x^*\|^2-\mu_2(2\widehat{\beta}_2-\mu_2)\|B_2x_n-B_2x^*\|^2\\
&\quad -\mu_1(2\widehat{\beta}_1-\mu_1)\|B_1\tilde{x}_n-B_1y^*\|^2]+\alpha_n\|Qx_n-x^*\|^2\\
&\quad -(\gamma_n+\delta_n)[(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2+2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2]\\
&=\|x_n-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2-(1-\beta_n)[\mu_2(2\widehat{\beta}_2-\mu_2)\|B_2x_n-B_2x^*\|^2\\
&\quad +\mu_1(2\widehat{\beta}_1-\mu_1)\|B_1\tilde{x}_n-B_1y^*\|^2]-(\gamma_n+\delta_n)\\
&\quad [(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2+2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(\gamma_n+\delta_n)[(1-\tfrac{\lambda_n^2}{\alpha^2})\|u_n-\tilde{u}_n\|^2+2\lambda_n\alpha\|A\tilde{u}_n-Ax^*\|^2]\\
&\quad +(1-\beta_n)[\mu_2(2\widehat{\beta}_2-\mu_2)\|B_2x_n-B_2x^*\|^2+\mu_1(2\widehat{\beta}_1-\mu_1)\|B_1\tilde{x}_n-B_1y^*\|^2]\\
&\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2\\
&\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\alpha_n\|Qx_n-x^*\|^2.
\end{aligned}
$$

Since $\alpha_n\to 0,\ \|x_n-x_{n+1}\|\to 0,\ \liminf_{n\to\infty}(\gamma_n+\delta_n)>0,\ 0<\liminf_{n\to\infty}\lambda_n\le \limsup_{n\to\infty}\lambda_n<\alpha$ and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, it follows that

$\lim_{n\to\infty} \|u_n - \tilde{u}_n\| = 0,$

$$\lim_{n\to\infty} \|A\tilde{u}_n - Ax^*\| = 0, \quad \lim_{n\to\infty} \|B_2 x_n - B_2 x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|B_1 \tilde{x}_n - B_1 y^*\| = 0.$$

Note that A is Lipschitz continuous. Thus, we get that $\lim_{n\to\infty} \|Au_n - Ax^*\| = 0$.

Step 4. $\lim_{n\to\infty} \|Sy_n - y_n\| = 0$.
Indeed, observe that

$$\begin{aligned} \|y_n - \tilde{u}_n\| &= \|P_C(u_n - \lambda_n A\tilde{u}_n) - P_C(u_n - \lambda_n Au_n)\| \\ &\le \|(u_n - \lambda_n A\tilde{u}_n) - (u_n - \lambda_n Au_n)\| \\ &= \lambda_n \|A\tilde{u}_n - Au_n\| \\ &\le \frac{\lambda_n}{\alpha} \|\tilde{u}_n - u_n\|. \end{aligned}$$

This together with $\|\tilde{u}_n - u_n\| \to 0$ implies $\lim_{n\to\infty} \|y_n - \tilde{u}_n\| = 0$ and hence we get that $\lim_{n\to\infty} \|y_n - u_n\| = 0$. By firm nonexpansiveness of P_C, we have

$$\begin{aligned} &\|\tilde{u}_n - x^*\|^2 \\ &= \|P_C(u_n - \lambda_n Au_n) - P_C(x^* - \lambda_n Ax^*)\|^2 \\ &\le \langle (u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*), \tilde{u}_n - x^* \rangle \\ &= \tfrac{1}{2}[\|u_n - x^* - \lambda_n(Au_n - Ax^*)\|^2 + \|\tilde{u}_n - x^*\|^2 \\ &\quad - \|(u_n - x^*) - \lambda_n(Au_n - Ax^*) - (\tilde{u}_n - x^*)\|^2] \\ &\le \tfrac{1}{2}[\|u_n - x^*\|^2 + \|\tilde{u}_n - x^*\|^2 - \|(u_n - \tilde{u}_n) - \lambda_n(Au_n - Ax^*)\|^2] \\ &= \tfrac{1}{2}[\|u_n - x^*\|^2 + \|\tilde{u}_n - x^*\|^2 - \|u_n - \tilde{u}_n\|^2 \\ &\quad + 2\lambda_n \langle u_n - \tilde{u}_n, Au_n - Ax^* \rangle - \lambda_n^2 \|Au_n - Ax^*\|^2] \\ &\le \tfrac{1}{2}[\|u_n - x^*\|^2 + \|\tilde{u}_n - x^*\|^2 + 2\lambda_n \|u_n - \tilde{u}_n\| \|Au_n - Ax^*\|], \end{aligned}$$

that is,

$$\|\tilde{u}_n - x^*\|^2 \le \|u_n - x^*\|^2 + 2\lambda_n \|u_n - \tilde{u}_n\| \|Au_n - Ax^*\|. \tag{3.14}$$

Similarly to the above argument, we obtain

$$\begin{aligned} \|\tilde{x}_n - y^*\|^2 &= \|P_C(x_n - \mu_2 B_2 x_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\ &\le \langle (x_n - \mu_2 B_2 x_n) - (x^* - \mu_2 B_2 x^*), \tilde{x}_n - y^* \rangle \\ &= \frac{1}{2}[\|x_n - x^* - \mu_2(B_2 x_n - B_2 x^*)\|^2 + \|\tilde{x}_n - y^*\|^2 \\ &\quad - \|(x_n - x^*) - \mu_2(B_2 x_n - B_2 x^*) - (\tilde{x}_n - y^*)\|^2] \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}[\|x_n - x^*\|^2 + \|\tilde{x}_n - y^*\|^2 \\
&\quad -\|(x_n - \tilde{x}_n) - \mu_2(B_2x_n - B_2x^*) - (x^* - y^*)\|^2] \\
&= \frac{1}{2}[\|x_n - x^*\|^2 + \|\tilde{x}_n - y^*\|^2 - \|x_n - \tilde{x}_n - (x^* - y^*)\|^2 \\
&\quad +2\mu_2\langle x_n - \tilde{x}_n - (x^* - y^*), B_2x_n - B_2x^*\rangle - \mu_2^2\|B_2x_n - B_2x^*\|^2] \\
&= \frac{1}{2}[\|x_n - x^*\|^2 + \|\tilde{x}_n - y^*\|^2 - \|x_n - \tilde{x}_n - (x^* - y^*)\|^2 \\
&\quad +2\mu_2\|x_n - \tilde{x}_n - (x^* - y^*)\|\|B_2x_n - B_2x^*\|],
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|\tilde{x}_n - y^*\|^2 \leq{}& \|x_n - x^*\|^2 - \|x_n - \tilde{x}_n - (x^* - y^*)\|^2 \\
&+2\mu_2\|x_n - \tilde{x}_n - (x^* - y^*)\|\|B_2x_n - B_2x^*\|.
\end{aligned} \tag{3.15}
$$

Further, similarly to the above argument, we derive

$$
\begin{aligned}
\|\bar{x}_n - x^*\|^2 &= \|P_C(\tilde{x}_n - \mu_1B_1\tilde{x}_n) - P_C(y^* - \mu_1B_1y^*)\|^2 \\
&\leq \langle(\tilde{x}_n - \mu_1B_1\tilde{x}_n) - (y^* - \mu_1B_1y^*), \bar{x}_n - x^*\rangle \\
&= \frac{1}{2}[\|\tilde{x}_n - y^* - \mu_1(B_1\tilde{x}_n - B_1y^*)\|^2 + \|\bar{x}_n - x^*\|^2 \\
&\quad -\|(\tilde{x}_n - y^*) - \mu_1(B_1\tilde{x}_n - B_1y^*) - (\bar{x}_n - x^*)\|^2] \\
&\leq \frac{1}{2}[\|\tilde{x}_n - y^*\|^2 + \|\bar{x}_n - x^*\|^2 - \|(\tilde{x}_n - \bar{x}_n) \\
&\quad -\mu_1(B_1\tilde{x}_n - B_1y^*) + (x^* - y^*)\|^2] \\
&= \frac{1}{2}[\|\tilde{x}_n - y^*\|^2 + \|\bar{x}_n - x^*\|^2 - \|\tilde{x}_n - \bar{x}_n + (x^* - y^*)\|^2 \\
&\quad +2\mu_1\langle\tilde{x}_n - \bar{x}_n + (x^* - y^*), B_1\tilde{x}_n - B_1y^*\rangle - \mu_1^2\|B_1\tilde{x}_n - B_1y^*\|^2] \\
&\leq \frac{1}{2}[\|\tilde{x}_n - y^*\|^2 + \|\bar{x}_n - x^*\|^2 - \|\tilde{x}_n - \bar{x}_n + (x^* - y^*)\|^2 \\
&\quad +2\mu_1\|\tilde{x}_n - \bar{x}_n + (x^* - y^*)\|\|B_1\tilde{x}_n - B_1y^*\|],
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|\bar{x}_n - x^*\|^2 \leq{}& \|\tilde{x}_n - y^*\|^2 - \|\tilde{x}_n - \bar{x}_n + (x^* - y^*)\|^2 \\
&+2\mu_1\|\tilde{x}_n - \bar{x}_n + (x^* - y^*)\|\|B_1\tilde{x}_n - B_1y^*\|.
\end{aligned} \tag{3.16}
$$

Utilizing (3.14), (3.15) and (3.16), we have

$$
\begin{aligned}
&\|y_n-x^*\|^2 = \|\tilde{u}_n-x^*+y_n-\tilde{u}_n\|^2\\
&\le \|\tilde{u}_n-x^*\|^2+2\langle y_n-\tilde{u}_n, y_n-x^*\rangle\\
&\le \|\tilde{u}_n-x^*\|^2+2\|y_n-\tilde{u}_n\|\|y_n-x^*\|\\
&\le \|u_n-x^*\|^2+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|+2\|y_n-\tilde{u}_n\|\|y_n-x^*\|\\
&\le \alpha_n\|Qx_n-x^*\|^2+(1-\alpha_n)\|\bar{x}_n-x^*\|^2+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|\\
&\quad +2\|y_n-\tilde{u}_n\|\|y_n-x^*\|\\
&\le \|\bar{x}_n-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|\\
&\quad +2\|y_n-\tilde{u}_n\|\|y_n-x^*\|\\
&\le \|\tilde{x}_n-y^*\|^2-\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|^2\\
&\quad +2\mu_1\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|\\
&\quad +\alpha_n\|Qx_n-x^*\|^2+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|\\
&\quad +2\|y_n-\tilde{u}_n\|\|y_n-x^*\|\\
&\le \|x_n-x^*\|^2-\|x_n-\tilde{x}_n-(x^*-y^*)\|^2\\
&\quad +2\mu_2\|x_n-\tilde{x}_n-(x^*-y^*)\|\|B_2x_n-B_2x^*\|\\
&\quad -\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|^2+2\mu_1\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|\\
&\quad +\alpha_n\|Qx_n-x^*\|^2+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|+2\|y_n\tilde{u}_n\|\|y_n-x^*\|.
\end{aligned}
\tag{3.17}
$$

Thus from (2.8) and (3.17) it follows that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&= \|\beta_n(x_n-x^*)+\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)\\
&\quad +(1-\beta_n-\gamma_n-\delta_n)(u_n-x^*)\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)\|y_n-x^*\|^2+(1-\beta_n-\gamma_n-\delta_n)\|u_n-x^*\|^2\\
&= \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)\|y_n-x^*\|^2\\
&\quad +(1-\beta_n-\gamma_n-\delta_n)\|y_n-x^*+u_n-y_n\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)\|y_n-x^*\|^2+(1-\beta_n-\gamma_n-\delta_n)[\|y_n-x^*\|^2\\
&\quad +2\langle u_n-y_n, u_n-x^*\rangle]\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2+2\|u_n-y_n\|\|u_n-x^*\|\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)[\|x_n-x^*\|^2-\|x_n-\tilde{x}_n-(x^*-y^*)\|^2\\
&\quad +2\mu_2\|x_n-\tilde{x}_n-(x^*-y^*)\|\|B_2x_n-B_2x^*\|-\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\mu_1\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|+\alpha_n\|Qx_n-x^*\|^2\\
&+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|+2\|y_n-\tilde{u}_n\|\|y_n-x^*\|]+2\|u_n-y_n\|\|u_n-x^*\|\\
\le\ &\|x_n-x^*\|^2-(1-\beta_n)[\|x_n-\tilde{x}_n-(x^*-y^*)\|^2+\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|^2]\\
&+2\mu_2\|x_n-\tilde{x}_n-(x^*-y^*)\|\|B_2x_n-B_2x^*\|\\
&+2\mu_1\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|\\
&+\alpha_n\|Qx_n-x^*\|^2+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|\\
&+2\|y_n-\tilde{u}_n\|\|y_n-x^*\|+2\|u_n-y_n\|\|u_n-x^*\|,
\end{aligned}
$$

which hence implies that

$$
\begin{aligned}
&(1-\beta_n)[\|x_n-\tilde{x}_n-(x^*-y^*)\|^2+\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|^2]\\
\le\ &\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\mu_2\|x_n-\tilde{x}_n-(x^*-y^*)\|\|B_2x_n-B_2x^*\|\\
&+2\mu_1\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|+\alpha_n\|Qx_n-x^*\|^2\\
&+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|+2\|y_n-\tilde{u}_n\|\|y_n-x^*\|+2\|u_n-y_n\|\|u_n-x^*\|\\
\le\ &(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|\\
&+2\mu_2\|x_n-\tilde{x}_n-(x^*-y^*)\|\|B_2x_n-B_2x^*\|\\
&+2\mu_1\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|\|B_1\tilde{x}_n-B_1y^*\|+\alpha_n\|Qx_n-x^*\|^2\\
&+2\lambda_n\|u_n-\tilde{u}_n\|\|Au_n-Ax^*\|+2\|y_n-\tilde{u}_n\|\|y_n-x^*\|+2\|u_n-y_n\|\|u_n-x^*\|.
\end{aligned}
$$

Since $\limsup_{n\to\infty}\beta_n<1,\ 0<\lambda_n\le\alpha,\ \alpha_n\to 0,\ \|Au_n-Ax^*\|\to 0,\ \|B_2x_n-B_2x^*\|\to 0,\ \|B_1\tilde{x}_n-B_1y^*\|\to 0,\ \|y_n-\tilde{u}_n\|\to 0,\ \|y_n-u_n\|\to 0$ and $\|x_{n+1}-x_n\|\to 0$, it follows from the boundedness of $\{x_n\},\{\tilde{x}_n\},\{\bar{x}_n\},\{u_n\},\{\tilde{u}_n\}$ and $\{y_n\}$ that

$$\lim_{n\to\infty}\|x_n-\tilde{x}_n-(x^*-y^*)\|=0\quad\text{and}\quad\lim_{n\to\infty}\|\tilde{x}_n-\bar{x}_n+(x^*-y^*)\|=0.$$

Consequently, it immediately follows that

$$\lim_{n\to\infty}\|x_n-\bar{x}_n\|=0\quad\text{and}\quad\lim_{n\to\infty}\|x_n-u_n\|=0 \tag{3.18}$$

(due to the fact that $\|u_n-x_n\|\le\alpha_n\|Qx_n-x_n\|+(1-\alpha_n)\|\bar{x}_n-x_n\|$). This together with $\|y_n-u_n\|\to 0$ implies that

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Since

$$\|\delta_n(Sy_n-x_n)\|\le\|x_{n+1}-x_n\|+\gamma_n\|y_n-x_n\|+(1-\beta_n-\gamma_n-\delta_n)\|u_n-x_n\|,$$

it follows that

$$\lim_{n\to\infty}\|Sy_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Sy_n - y_n\| = 0.$$

Step 5. $\limsup_{n\to\infty}\langle Q\bar{x} - \bar{x}, x_n - \bar{x}\rangle \leq 0$, where $\bar{x} = P_\Omega Q\bar{x}$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle Q\bar{x} - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle Q\bar{x} - \bar{x}, x_{n_i} - \bar{x}\rangle. \tag{3.19}$$

Also, since H is reflexive and $\{y_n\}$ is bounded, without loss of generality we may assume that $y_{n_i} \rightharpoonup p$ for some $p \in C$. First, it is clear from Lemma 2.3 (ii) that $p \in \mathrm{Fix}(S)$. Now let us show that $p \in \Gamma$. Note that

$$\begin{aligned}\|x_n - G(x_n)\| &= \|x_n - P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)]\| \\ &= \|x_n - \bar{x}_n\| \to 0 \quad (n \to \infty),\end{aligned}$$

where $G : C \to C$ is defined as in Lemma 1.1. According to Lemma 2.3 (ii) we obtain $p \in \Omega$. Further, let us show that $p \in VI(A, C)$. As a matter of fact, since $\|x_n - u_n\| \to 0$, $\|\tilde{u}_n - u_n\| \to 0$ and $\|x_n - y_n\| \to 0$, we deduce that $x_{n_i} \rightharpoonup p$ and $\tilde{u}_{n_i} \rightharpoonup p$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. In this case, the mapping T is maximal monotone and (see [2] for further details)

$$0 \in Tv \text{ if and only if } v \in VI(A, C).$$

Let $\mathrm{Gph}(T)$ be the graph of T and let $(v, w) \in \mathrm{Gph}(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$. On the other hand, from $\tilde{u}_n = P_C(u_n - \lambda_n A u_n)$ and $v \in C$ we have

$$\langle u_n - \lambda_n A u_n - \tilde{u}_n, \tilde{u}_n - v\rangle \geq 0$$

and hence

$$\langle v - \tilde{u}_n, \frac{\tilde{u}_n - u_n}{\lambda_n} + Au_n\rangle \geq 0.$$

From $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$ and $\tilde{u}_{n_i} \in C$ we have

$$\begin{aligned}\langle v - \tilde{u}_{n_i}, w\rangle &\geq \langle v - \tilde{u}_{n_i}, Av\rangle \\ &\geq \langle v - \tilde{u}_{n_i}, Av\rangle - \langle v - \tilde{u}_{n_i}, \frac{\tilde{u}_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\rangle \\ &= \langle v - \tilde{u}_{n_i}, Av - A\tilde{u}_{n_i}\rangle + \langle v - \tilde{u}_{n_i}, A\tilde{u}_{n_i} - Au_{n_i}\rangle - \langle v - \tilde{u}_{n_i}, \frac{\tilde{u}_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle v - \tilde{u}_{n_i}, A\tilde{u}_{n_i} - Au_{n_i}\rangle - \langle v - \tilde{u}_{n_i}, \frac{\tilde{u}_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Hence, we obtain $\langle v-p, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $p \in T^{-1}0$ and hence $p \in VI(A,C)$. Therefore, $p \in \mathrm{Fix}(S) \cap \Gamma \cap VI(A,C) =: \Omega$. Hence it follows from (2.3) and (3.19) that

$$\limsup_{n\to\infty}\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle = \lim_{i\to\infty}\langle Q\bar{x}-\bar{x}, x_{n_i}-\bar{x}\rangle = \langle Q\bar{x}-\bar{x}, p-\bar{x}\rangle \leq 0.$$

Step 6. $\lim_{n\to\infty}\|x_n-\bar{x}\| = 0.$

Indeed, utilizing Lemma 2.1 and Lemma 2.5, we obtain, from (3.3), (3.4) and the convexity of $\|\cdot\|^2$, that

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2\\
&= \|\beta_n(x_n-\bar{x}) + \gamma_n(y_n-\bar{x}) + \delta_n(Sy_n-\bar{x}) + (1-\beta_n-\gamma_n-\delta_n)(u_n-\bar{x})\|^2\\
&\leq \beta_n\|x_n-\bar{x}\|^2 + (\gamma_n+\delta_n)\|\tfrac{1}{\gamma_n+\delta_n}[\gamma_n(y_n-\bar{x}) + \delta_n(Sy_n-\bar{x})]\|^2\\
&\quad + (1-\beta_n-\gamma_n-\delta_n)\|u_n-\bar{x}\|^2\\
&\leq \beta_n\|x_n-\bar{x}\|^2 + (\gamma_n+\delta_n)\|y_n-\bar{x}\|^2 + (1-\beta_n-\gamma_n-\delta_n)\|u_n-\bar{x}\|^2\\
&\leq \beta_n\|x_n-\bar{x}\|^2 + (\gamma_n+\delta_n)\|u_n-\bar{x}\|^2 + (1-\beta_n-\gamma_n-\delta_n)\|u_n-\bar{x}\|\\
&= \beta_n\|x_n-\bar{x}\|^2 + (1-\beta_n)\|u_n-\bar{x}\|^2\\
&\leq \beta_n\|x_n-\bar{x}\|^2 + (1-\beta_n)[(1-\alpha_n)^2\|\bar{x}_n-\bar{x}\|^2 + 2\alpha_n\langle Qx_n-\bar{x}, u_n-\bar{x}\rangle]\\
&\leq \beta_n\|x_n-\bar{x}\|^2 + (1-\beta_n)[(1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle Qx_n-\bar{x}, u_n-\bar{x}\rangle]\\
&= (1-(1-\beta_n)\alpha_n)\|x_n-\bar{x}\|^2 + (1-\beta_n)2\alpha_n\langle Qx_n-\bar{x}, u_n-\bar{x}\rangle\\
&\leq (1-(1-\beta_n)\alpha_n)\|x_n-\bar{x}\|^2 + (1-\beta_n)2\alpha_n[\rho\|x_n-\bar{x}\|^2 + \langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle\\
&\quad + \|Qx_n-\bar{x}\|\|u_n-x_n\|]\\
&\leq [1-(1-2\rho)(1-\beta_n)\alpha_n]\|x_n-\bar{x}\|^2\\
&\quad + (1-\beta_n)2\alpha_n[\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle + \|Qx_n-\bar{x}\|\|u_n-x_n\|]\\
&= [1-(1-2\rho)(1-\beta_n)\alpha_n]\|x_n-\bar{x}\|^2\\
&\quad + (1-2\rho)(1-\beta_n)\alpha_n\tfrac{2[\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle + \|Qx_n-\bar{x}\|\|u_n-x_n\|]}{1-2\rho}.
\end{aligned}$$

Note that $\liminf_{n\to\infty}(1-2\rho)(1-\beta_n) > 0$. It follows that $\sum_{n=0}^{\infty}(1-2\rho)(1-\beta_n)\alpha_n = \infty$. It is clear that

$$\limsup_{n\to\infty}\frac{2[\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle + \|Qx_n-\bar{x}\|\|u_n-x_n\|]}{1-2\rho} \leq 0$$

because $\limsup_{n\to\infty}\langle Q\bar{x}-\bar{x}, x_n-\bar{x}\rangle \leq 0$ and $\lim_{n\to\infty}\|u_n-x_n\| = 0$. Therefore, all conditions of Lemma 2.4 are satisfied. Consequently, we immediately deduce that $\|x_n-\bar{x}\| \to 0$ as $n \to \infty$. This completes the proof. ∎

REFERENCES

1. Y. Yao, Y. C. Liou and S. M. Kang, Approach to common elements of variational inequality problems and fixed point problems via a related extragradient method, *Comput. Math. Appl.*, **59** (2010), 3472-3480.
2. N. Nadezhkina and W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings, *SIAM J. Optim.*, **16** (2006), 1230-1241.
3. H. K. Xu and T. H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, *J. Optim. Theory Appl.*, **119** (2003), 185-201.
4. L. C. Zeng and J. C. Yao, Mixed projection methods for systems of variational inequalities, *J. Global Optim.*, **41** (2008), 465-478.
5. L. C. Ceng and J. C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, *Appl. Math. Comput.*, **190** (2007), 205-215.
6. L. C. Zeng, N. C. Wong and J. C. Yao, Strong convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type, *Taiwanese J. Math.*, **10** (2006), 837-849.
7. H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.*, **61** (2005), 341-350.
8. L. C. Ceng, Q. H. Ansari and J. C. Yao, Relaxed extragradient iterative methods for variational inequalities, *Appl. Math. Comput.*, (2011), doi:10.1016/j.amc.2011.01.061
9. N. Nadezhkina and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **128** (2006), 191-201.
10. L. C. Zeng and J. C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwanese J. Math.*, **10** (2006), 1293-1303.
11. G. M. Korpelevich, An extragradient method for finding saddle points and for other problems, *Ekon. Mate. Metody*, **12** (1976), 747-756.
12. Y. Yao and J. C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.*, **186** (2007), 1551-1558.
13. T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.*, **305** (2005), 227-239.
14. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004), 279-291.
15. L. C. Ceng, C. Y. Wang and J. C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, *Math. Method Oper. Res.*, **67** (2008), 375-390.

16. G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.*, **329** (2007), 336-346.

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